Macroscopic Geometries and Microstructures

DE LA RECHERCHE À L'INDUSTRIE

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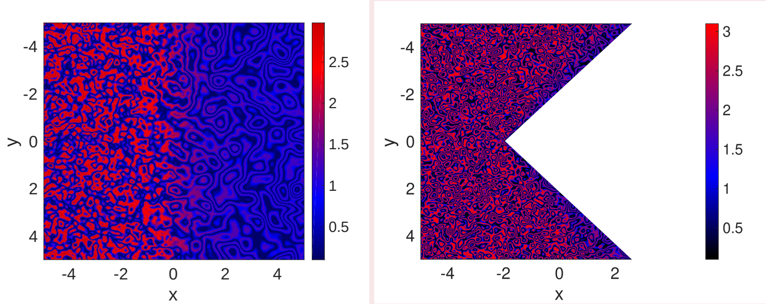
Contents

Homogenization of a linear elliptic equation in divergence form

$$\begin{cases} -\nabla \cdot a_\varepsilon \nabla u^\varepsilon = \nabla \cdot f & \text{in} \quad \Omega, \\ u^\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where ε is a small scale.

Interplay between microstructure and macroscopic geometry

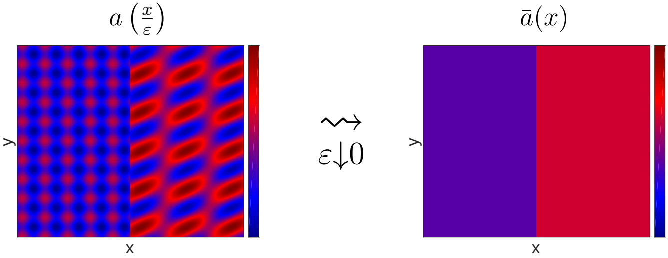


Based on joint works with **Claudia Raithel** and **Mathias Schäffner**.

- Obtain a regularity theory for the oscillating problem, **up to the geometric singularity**,
- Obtain a precise and local approximation of the gradient ∇u^ε (or the flux $a(\cdot/\varepsilon)\nabla u^\varepsilon$) **up to the geometric singularity**.

Remark : *far from* the geometric singularity, the classical theory applies.

$$-\nabla \cdot a_\varepsilon \nabla u^\varepsilon = \nabla \cdot f \quad \underset{\varepsilon\downarrow 0}{\rightsquigarrow} \quad -\nabla \cdot \overline{a}\nabla \overline{u} = \nabla \cdot f .$$



Difficulties for the interface

- **no stationarity** in x_1,
- **non-constant** homogenized matrix $\overline{a}$,
- non-standard **regularity theory** for the homogenized problem.

Difficulties for the corner

- **no stationarity**,
- complex **regularity theory** for the homogenized problem (no bare Lipschitz estimates), **singular** $\overline{a}$-harmonic functions.

Remark : in both cases applying naively the classical two-scale expansion is inefficient.

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Algebraical identity

$$u^{\varepsilon,1} = \overline{u} + \varepsilon\phi(\cdot/\varepsilon)\cdot\nabla\overline{u}, \qquad \text{(Two-scale expansion)}$$

$$-\nabla\cdot a(\cdot/\varepsilon)\nabla\left(u^{\varepsilon} - u^{\varepsilon,1}\right) = \varepsilon\nabla\cdot(a\otimes\phi - \sigma)(\cdot/\varepsilon) : \nabla^2\overline{u}, \qquad \text{(E)}$$

where ϕ, σ are the correctors/flux correctors depending only on a through :

$$\nabla\cdot a\nabla(\phi_i + x_i) = 0 \qquad \text{and} \qquad \nabla\cdot\sigma_i = \overline{a}e_i - a\cdot(e_i + \nabla\phi_i).$$

Role of periodicity and smoothness of the domain

Periodicity is a *practical* assumption to :

- (i) a priori know that $\overline{a}$ is a constant matrix,
- (ii) obtain the identity (E),
- (iii) obtain bounds on ϕ and σ,

Using the **smoothness of the domain** and (i) :

- (iv) $\nabla^2\overline{u}$ is regular.

$\Longrightarrow$ obtain a convergence rate $\left\|\nabla(u^{\varepsilon} - u^{\varepsilon,1})\right\|_{L^\infty} \lesssim \varepsilon\ln(1+\varepsilon^{-1})$. [Avellaneda, Lin, 1987]

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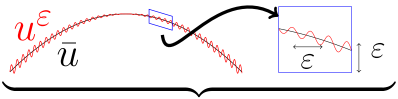
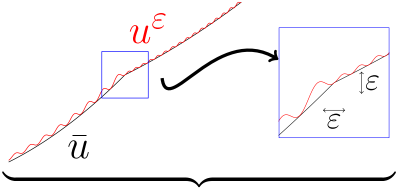


$\bar{a}$ constant

The $\bar{a}$-harmonic functions are affine $x \mapsto x_i$

$\bar{u}(x) \simeq \bar{u}(0) + \partial_i \bar{u}(0) x_i$



$\bar{a}$ piecewise constant

The $\bar{a}$-harmonic functions are piecewise affine $x \mapsto P_i(x)$

$\bar{u}(x) \simeq \bar{u}(0) + \partial_i^+ \bar{u}(0) P_i(x)$

The $\bar{a}$-harmonic functions

The function with **constant flux** P_i, $i \in [\![1, d]\!]$, so that

$$-\nabla \cdot \bar{a} \nabla P_i = 0.$$

Generalizing the two-scale expansion for the interface

The (generalized) correctors (ϕ, σ) are meant to correct the $\overline{a}$-harmonic functions.

Definitions

The correctors are defined by

$$\begin{aligned} -\nabla \cdot a\nabla\left(P_i + \phi_i\right) &= 0 && \text{in} \quad \mathbb{R}^d, \\ \nabla \cdot \sigma_i &= \overline{a}\nabla P_i - a\left(\nabla P_i + \nabla \phi_i\right) && \text{in} \quad \mathbb{R}^d, \end{aligned}$$

and the generalized two-scale expansion by

$$u^{\varepsilon,1} = \overline{u} + \varepsilon\phi\left(\frac{\cdot}{\varepsilon}\right) \cdot \overline{\nabla}\overline{u} \quad \text{for} \quad \overline{\nabla}\overline{u} = (\nabla P)^{-1} \cdot \nabla\overline{u}.$$

Remark : these definitions coincide with the classical ones when $\overline{a}$ is constant.

Algebraic identity

$$-\nabla \cdot a(\cdot/\varepsilon)\nabla\left(u^{\varepsilon} - u^{\varepsilon,1}\right) = \varepsilon\nabla \cdot (a \otimes \phi - \sigma)(\cdot/\varepsilon) : \left(\nabla \otimes \overline{\nabla}\overline{u}\right)^T.$$

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Generalizing the two-scale expansion for the corner

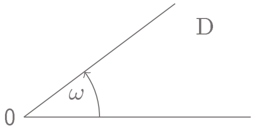


Figure – Geometrical setting

Expansion of the homogeneous solution (case $\overline{a} = \mathrm{I}$)

Using the results of [Dauge & al, 1987], we have

$$\overline{u} = \underbrace{\overline{u}^N_{\mathrm{reg}}}_{\text{regular part}} + \underbrace{\sum_{n=1}^{N} \bar{\gamma}_n \bar{\tau}_n}_{\text{singular part}} \qquad \text{for } \bar{\tau}_n(x) := r^{\frac{n\pi}{\omega}} \sin(\frac{n\pi}{\omega}\theta).$$

Generalizing the two-scale expansion for the corner

Definitions of the two-scale expansion

$$\widetilde{u}_\varepsilon^N := \underbrace{\left(1 + \varepsilon \phi_i^{\mathfrak{D}}\left(\frac{\cdot}{\varepsilon}\right)\partial_i\right)\overline{u}_{\mathrm{reg}}^N}_{\text{standard}} + \underbrace{\sum_{n=1}^{N} \gamma_n \left(\bar{\tau}_n + \varepsilon^{\bar{\rho}_n}\phi_n^{\mathfrak{C}}\left(\frac{\cdot}{\varepsilon}\right)\right)}_{\text{singular}},$$

with usual Dirichlet correctors :

$$\begin{cases} -\nabla \cdot a\nabla(\phi_i^{\mathfrak{D}} + x_i) = 0 & \text{in} \quad \mathrm{D}, \\ \phi_i^{\mathfrak{D}} = 0 & \text{on} \quad \partial\mathrm{D}, \end{cases}$$

and corner correctors $\phi_n^{\mathfrak{C}}$ correcting the singular modes :

$$\begin{cases} -\nabla \cdot a(\nabla\phi_n^{\mathfrak{C}} + \nabla\bar{\tau}_n) = 0 & \text{in} \quad \mathrm{D}, \\ \bar{\tau}_n + \phi_n^{\mathfrak{C}} = 0 & \text{on} \quad \partial\mathrm{D}. \end{cases}$$

Liouville principle [J., Raithel, Schäffner]

The a-harmonic functions and $\bar{a}$-harmonic functions correspond one-to-one.

Assumptions

Philosophy : Assumptions on the coefficient field independent from the geometry and agnostic of its nature (periodic, periodic + defect, stochastic, etc.).

Quantitative homogenization

1. The coefficient field a is elliptic and bounded : $\lambda \leq a \leq 1$
2. There exist a constant matrix $\bar{a}$, and extended correctors ϕ_i and σ_i (that is skew-symmetric), with
$$ae_i = \bar{a}e_i - a\nabla\phi_i + \nabla \cdot \sigma_i.$$
3. The correctors are sublinear ; namely there $\exists \nu \in (0,1]$, with
$$\langle |(\phi,\sigma)(x) - (\phi,\sigma)(y)|^p \rangle^{\frac{1}{p}} \lesssim |x-y|^{1-\nu} \qquad \forall p \geq 1, \forall x, y \in \mathbb{R}^d, |x-y| \geq 1.$$
4. [Optional] a is Hölder continuous

This applies if a is periodic, stationary ergodic with a spectral gap, periodic+defect...

Remark : for usual geometries, ν gives the **error estimate**, namely
$$\|\nabla u^{\varepsilon} - \nabla u^{\varepsilon,1}\|_{\mathrm{L}^p} \lesssim \varepsilon^{\nu}. \tag{1}$$

"Theorem" [J., 2019 ; J. & Raithel, 2021 ; J., Raithel & Schaeffner, 2022]

For interfaces and corners (1) also holds using suitable definitions of correctors and two-scale expansion (up to some logarithmic loss).

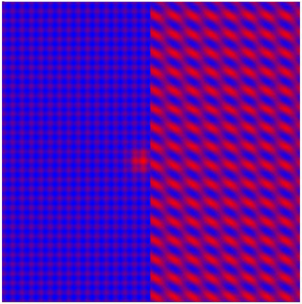
Numerical illustration

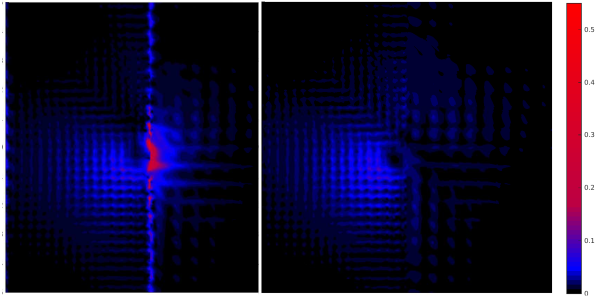
a

error on ∇u^{ε}

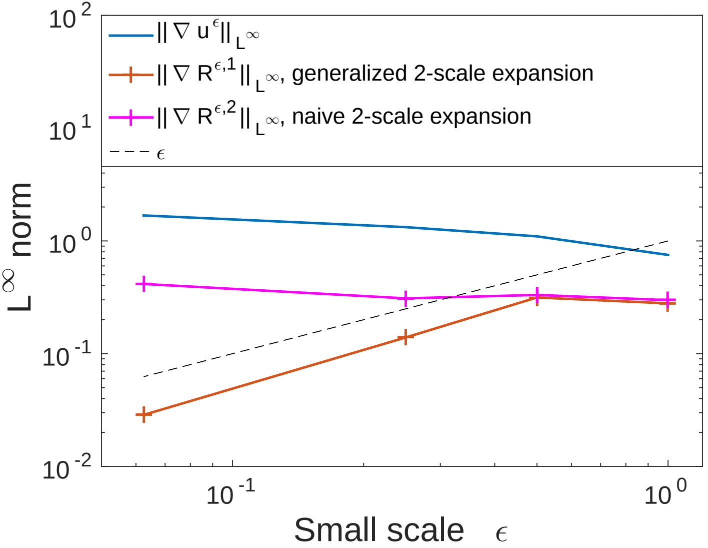
naive expansion

accurate expansion





Numerical illustration (interfaces)



Collaborations :

- **Defects** in a periodic framework [Blanc, J., Le Bris, 2019]
- **Interfaces** in a periodic framework [J., 2019] and in a general framework [J., Raithel, 2021]
- **2D corners** in a general framework [J., Raithel, Schäffner, submitted]
- **Smooth boundary** in a stochastic framework [Bella, Fischer, J., Raithel, in preparation]

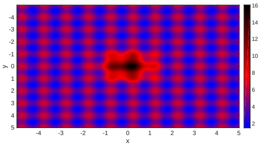


Figure – Defect

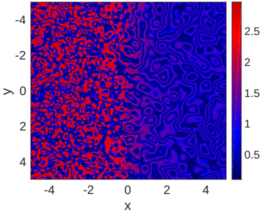


Figure – Interface

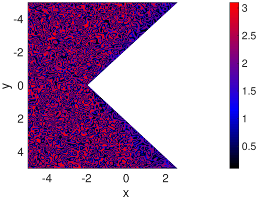


Figure – Corner

$$a(x) := \begin{cases} a_-(x) & \text{if} \quad x_1 < 0, \\ a_+(x) & \text{if} \quad x_1 > 0. \end{cases}$$ with $a_\pm$ satisfying assumptions 1, 2, 3, 4.

Growth rate of correctors

Assume that $d \geq 2$,

$$\langle |(\phi, \sigma)(x) - (\phi, \sigma)(y)|^p \rangle^{\frac{1}{p}} \lesssim |x - y|^{1-\nu} \ln^{\tilde{\nu}}(|x - y| + 2).$$

Error estimate

Assume that $d \geq 3$ and that u^ε and $\overline{u}$ are zero-mean solutions to

$$-\nabla \cdot (a(\cdot/\varepsilon)\nabla u^\varepsilon = -\nabla \cdot \overline{a}\nabla\overline{u} = f \quad \text{in} \quad \mathbb{R}^d$$

where f is supported in $\mathrm{B}(0,1)$. Then, for any $p > d$, there holds

$$\|u^\varepsilon - \overline{u}\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{\nu} \ln^{\tilde{\nu}}\left(\varepsilon^{-1}\right) \|f\|_{\mathrm{L}^p(\mathbb{R}^d)}.$$

$$\left\|\nabla u^\varepsilon - \nabla\overline{u} - \nabla\phi\left(\cdot/\varepsilon\right) \cdot \overline{\nabla}\overline{u}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim \varepsilon^{\nu} \ln^{\tilde{\nu}}\left(\varepsilon^{-1}\right) \|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)}.$$

a itself satisfies the assumtions 1, 2, 3, 4.

Growth rate of correctors

$$\left\langle \left| \phi^{\mathfrak{D}}(x) \right|^p \right\rangle^{\frac{1}{p}} \lesssim |x|^{1-\nu} \ln^{\widetilde{\nu}}(|x|+2),$$

$$\left\langle \left| \phi_n^{\mathfrak{C}}(x) \right|^p \right\rangle^{\frac{1}{p}} \lesssim |x|^{\frac{n\pi}{\omega}-\nu} \ln^{\widetilde{\nu}}(|x|+2).$$

Error estimate

Assume that $d \geq 3$ and that u^ε and $\overline{u}$ are zero-mean solutions to

$$\begin{cases} -\nabla \cdot a(\cdot/\varepsilon)\nabla u^\varepsilon = -\nabla \cdot \overline{a}\nabla\overline{u} = \nabla \cdot f & \text{in} \quad \mathrm{D}, \\ u^\varepsilon = \overline{u} = 0 & \text{on} \quad \partial\mathrm{D}. \end{cases}$$

where f is supported in $\mathrm{B}(0,2)\backslash\mathrm{B}(0,1)$. Then, there holds

$$\langle |(\nabla \widetilde{u}_\varepsilon^2 - \nabla u^\varepsilon)(x)|^p \rangle^{\frac{1}{p}} \lesssim \varepsilon^{\nu} \ln^{\widetilde{\nu}}(\varepsilon^{-1})|x|^{\frac{\pi}{\omega}-1}.$$

A numerical illustration (corners)

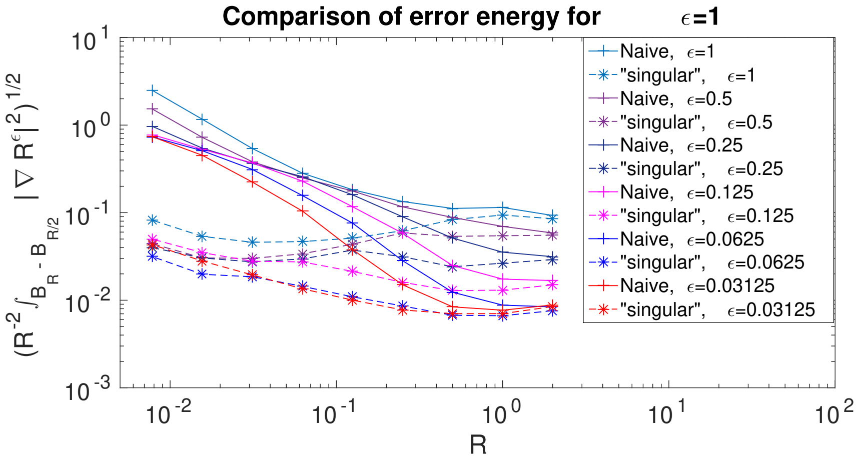


Figure – Measure of the local L^2 error while approximating ∇u^ε either with its classical 2-scale expansion ("naive") or with $\nabla \widetilde{u}^1_\varepsilon$ ("singular").