ϕ-FEM method and applications

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Outline

1. Motivation and previous works
2. ϕ-FEM method for elasticity problems
   (a) With Dirichlet conditions
   (b) With Neumann conditions
   (c) With mixed conditions
3. Some applications
   (a) Case of fracture problems
   (b) Particulate flows
   (c) Heat problem
4. Summary and outlook

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Motivation

— The standard FEM works under the Ciarlet condition

$$\frac{h_K}{\rho_K} < \gamma$$

— If the mesh contains degenerated cells :

- It is not guaranty that standard FEM converges
- The conditioning number of the FE matrix is bad

Motivation

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$$\frac{h_K}{\rho_K} < \gamma$$

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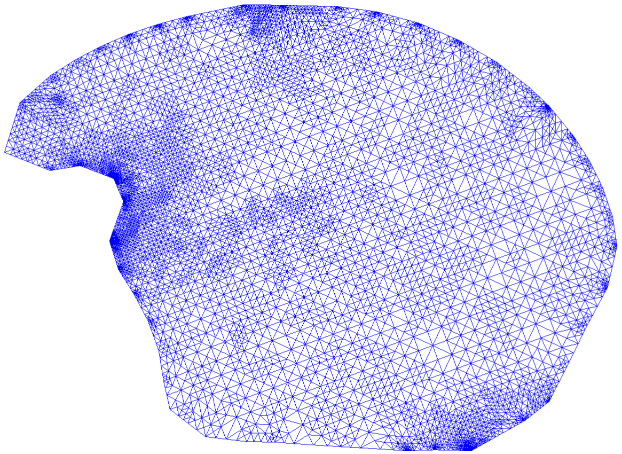
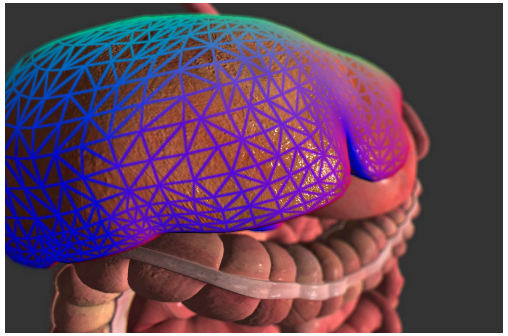
- The conditioning number of the FE matrix is bad

• What can we do on complex geometries ? Meshing soft tissues is a big challenge.
How can we simulate the deformation of soft tissues ?

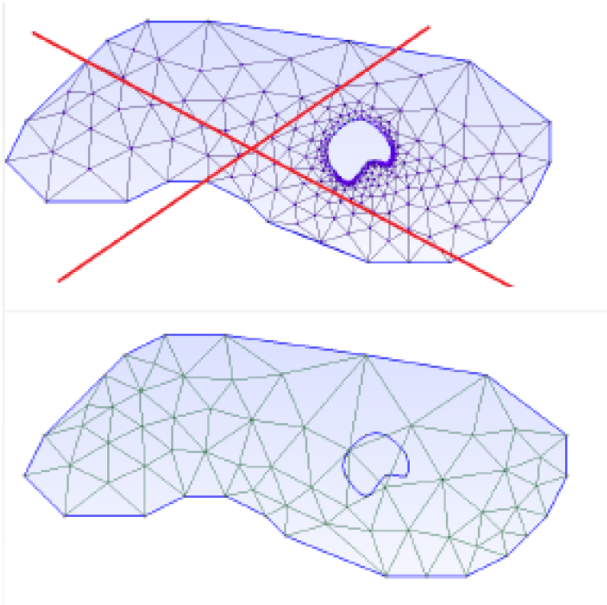


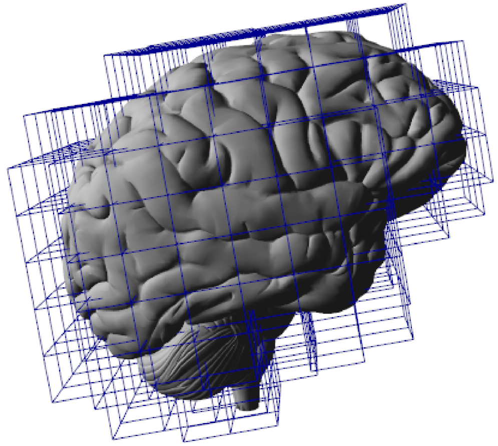


Lleras et al, Applied Mathematical Modelling, 2020

Motivation

Finite elements on non matching grids





- A simpler treatment of complex geometries, cracks, material interfaces, ...
- Inverse problems, shape optimization : geometrical features of a priori unknown shape (domain changing on iterations)
- Fluid-Structure interaction, particulate flows, ... (domain changing in time)

⊕ No need to remesh,
⊕ regular cells to facilitate an efficient matrix-free implementation

⊖ adapt the weak formulation
⊖ Conditioning of the finite element matrix

Previous works on non matching grids

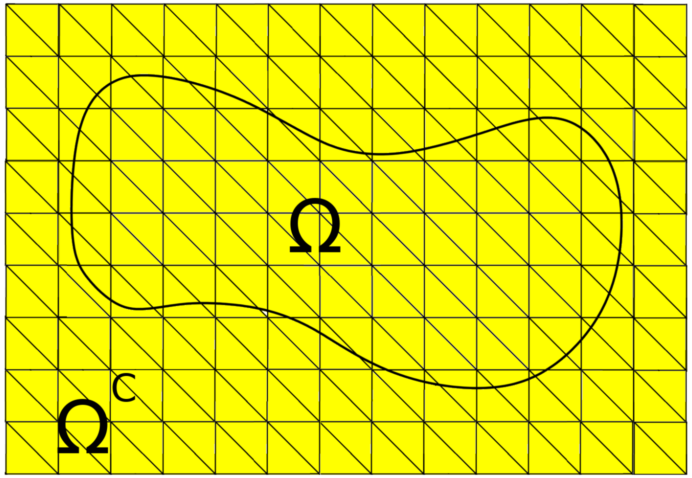
— **Classical fictitious domain methods** *Saul'ev '63 (Dirichlet), Astrakhantsev '78 (Neumann), Glowinski et al. 1990's (several extensions and variants)*

- Extend u to the whole fictitious domain O by the solution of the same governing equation

$$-\Delta u = f \quad \text{in } \Omega$$

$$-\Delta u = f \quad \text{in } \Omega^c = O \setminus \Omega$$

$$u = g \quad \text{on } \Gamma \quad + \quad \text{some boundary conditions on } \partial O$$



Previous works on non matching grids

- **Finite element discretization** with Lagrange multipliers :

Find $u_h \in V_h = \{$ cont. piecewise linear functions on mesh on O$\}$,

$\lambda_h \in M_h = \{$ piecewise constant on a mesh on $\Gamma\}$ such that

$$\int_O \nabla u_h . \nabla v_h + \int_\Gamma \lambda_h v_h = \int_O f v_h \quad \forall v_h \in V_h$$

$$\int_\Gamma \mu_h u_h = \int_\Gamma \mu_h g \quad \forall \mu_h \in M_h$$

⊖ The extension is only $H^{1+\epsilon}$ regular $\Rightarrow$ poor accuracy $O(\sqrt{h})$ (due to the cut triangles)

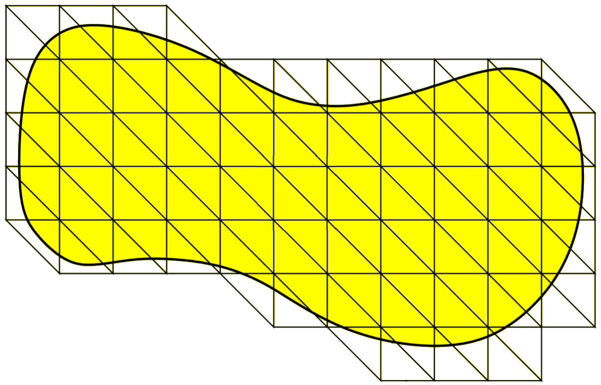
⊖ large FE matrix and bad condition number

⊖ The mesh on Γ must be coarser than the mesh on O in order to verify the inf sup condition.

Previous works on non matching grids

— **CutFEM** *Burman-Hansbo 2010-2014 (and later works)*

- No fictitious extension of the solution
- The finite elements still live on a background simple mesh



- Lagrange multipliers for the boundary conditions (2010)

$$\text{Find } u_h \in V_h = \{ \text{ cont. piecewise linear functions on mesh on } T_h \},$$

$$\lambda_h \in M_h = \{ \text{ piecewise constant on a mesh on } T_h^{\Gamma} \} \text{ such that}$$

$$\int_\Omega \nabla u_h . \nabla v_h + \int_\Gamma \lambda_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_h$$

$$\int_\Gamma \mu_h u_h - \sigma h \sum_{\text{edges cut by } \Gamma} \int [\lambda_h][\mu_h] = \int_\Gamma \mu_h g \quad \forall \mu_h \in M_h$$

Previous works on non matching grids

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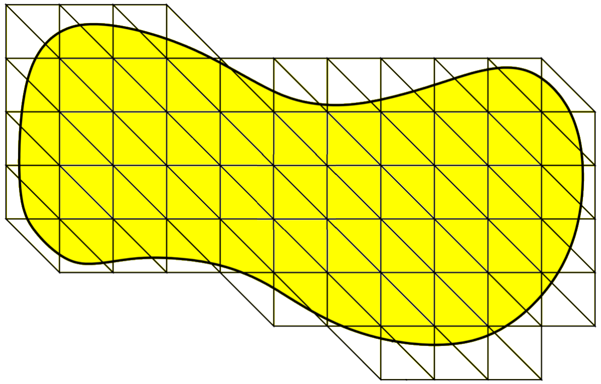


- Nitsche method for the boundary conditions (2012)

$$\text{Find } u_h \in V_h = \{ \text{ cont. piecewise linear functions on mesh on } T_h\}, \quad \forall v_h \in V_h :$$

$$\int_\Omega \nabla u_h . \nabla v_h - \int_\Gamma \frac{\partial u_h}{\partial n} v_h + \int_\Gamma u_h \frac{\partial v_h}{\partial n} + \frac{\gamma}{h} \int_\Gamma u_h v_h + \text{ stab term } = \int_\Omega f v_h + \int_\Gamma g \frac{\partial v_h}{\partial n} + \frac{\gamma}{h} \int_\Gamma g v_h$$

- Appropriate stabilization (Ghost penalty) for the conditioning of the matrix : $\sigma h \sum_{E \in E_h^\Gamma} \int_E \left[\frac{\partial u_h}{\partial n}\right] \left[\frac{\partial v_h}{\partial n}\right]$

⊕ Optimal accuracy
⊖ Not straigtforward to implement : need to evaluate the integrals on cut mesh elements

Previous works on non matching grids

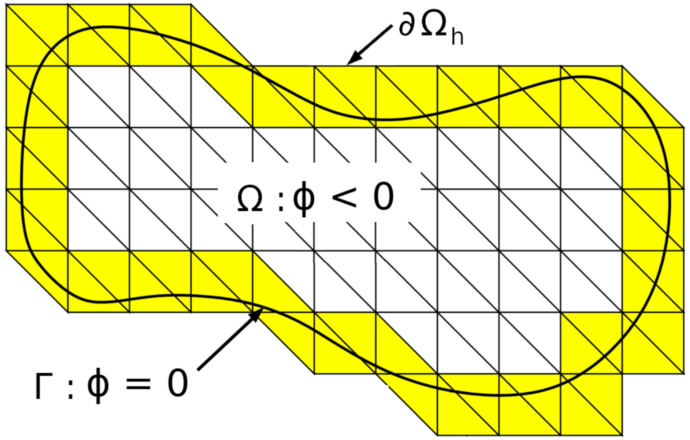
- **XFEM** *Moes-Bechet-Tourbier '06, Haslinger-Renard '09*
  - use of cut shape functions
  - $\oplus$ Good condition number
  - $\ominus$ Non-classical shape functions and discontinuity in the integrals
- **Shifted Boundary Method (SBM)** : *Main-Scovazzi '17, Nouveau and al.*
  - Taylor development near the boundary
  - $\oplus$ Optimal accuracy, no integrals on cut elements, as in ϕ-FEM
  - $\ominus$ Treatment of Neumann conditions

What is the idea of ϕ-FEM ?

Let the domain Ω and its boundary Γ be given by a level-set function ϕ :

$$\Omega := \{\phi < 0\} \text{ and } \Gamma = \{\phi = 0\}$$

Ω_h only slightly larger than Ω.



What is the idea of ϕ-FEM ?

General procedure :

- Extend the governing equations from Ω to Ω_h and write down a non standard variational formulation on the extended domain Ω_h (slightly larger than Ω) without taking into account the boundary conditions on $\partial\Omega$.
- Impose the boundary conditions using appropriate ansatz or additional variables, explicitly involving the level set ϕ which provides the link to the actual boundary.
  For instance, the Dirichlet conditions $\boldsymbol{u} = 0$ on $\partial\Omega$ can be imposed by the ansatz $\boldsymbol{u} = \phi \boldsymbol{w}$.
- Add appropriate stabilization, including the ghost penalty as in CutFEM plus a least square imposition of the governing equation on the mesh cells near the boundary, to guarantee coerciveness/stability on the discrete level.
- The level set is known only approximately, ϕ_h is the Lagrange interpolation of ϕ of order $l \geq k+1$

$$\int_{\Omega_h} \nabla(\phi_h w_h).\nabla(\phi_h q_h) - \int_{\Gamma_h} \frac{\partial}{\partial n}(\phi_h w_h)\phi_h q_h + stab = \int_{\Omega_h} f \phi_h q_h \quad \forall q_h$$

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ϕ-FEM for elasticity problem

We consider the linear elasticity problem for homogeneous and isotropic materials :

$$div\boldsymbol{\sigma}(\boldsymbol{u}) + \boldsymbol{f} = 0$$

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\varepsilon(\boldsymbol{u}) + \lambda(div\boldsymbol{u})I,$$

$$\boldsymbol{u} = \boldsymbol{u}^g \text{ on } \Gamma_D$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{g} \text{ on } \Gamma_N$$

with $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ the strain tensor and the Lamé parameters λ, μ are defined via the Young modulus E and the Poisson coefficient ν by

$$\mu = \frac{E}{2(1+\nu)} \text{ and } \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

ϕ-FEM for elasticity problem : Dirichlet conditions

Direct formulation with Dirichlet conditions :
$\phi_h, \boldsymbol{u}_h^g$ are FE approximations for $\phi, \boldsymbol{u}^g$ on the whole Ω_h and set $\boldsymbol{u}_h = \boldsymbol{u}_h^g + \phi_h \boldsymbol{w}_h$. The direct formulation is : find $\boldsymbol{w}_h \in V_h := \left\{\boldsymbol{v}_h : \Omega_h \to \mathbb{R}^d : \boldsymbol{v}_{h|T} \in \mathbb{P}^k(T)^d \ \ \forall T \in \mathcal{T}_h, \ \boldsymbol{v}_h \text{ continuous on } \Omega_h\right\}$ such that

$$\begin{aligned}\int_{\Omega_h} \boldsymbol{\sigma}(\phi_h \boldsymbol{w}_h) : \nabla(\phi_h \boldsymbol{z}_h) - \int_{\partial\Omega_h} \boldsymbol{\sigma}(\phi_h \boldsymbol{w}_h)\boldsymbol{n} \cdot \phi_h \boldsymbol{z}_h + G_h(\phi_h \boldsymbol{w}_h, \phi_h \boldsymbol{z}_h) + J_h^{lhs}(\phi_h \boldsymbol{w}_h, \phi_h \boldsymbol{z}_h) \\ = \int_{\Omega_h} \boldsymbol{f} \cdot \phi_h \boldsymbol{z}_h - \int_{\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h^g) : \nabla(\phi_h \boldsymbol{z}_h) + \int_{\partial\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h^g)\boldsymbol{n} \cdot \phi_h \boldsymbol{z}_h, \\ + J_h^{rhs}(\phi_h \boldsymbol{z}_h), \quad \forall \boldsymbol{z}_h \in V_h\end{aligned}$$

Here $G_h, J_h^{lhs}, J_h^{rhs}$ stand for the stabilization terms

$$G_h(\boldsymbol{u}, \boldsymbol{v}) := \sigma_D h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}\right] \cdot \left[\boldsymbol{\sigma}(\boldsymbol{v})\boldsymbol{n}\right],$$

$$J_h^{lhs}(\boldsymbol{u}, \boldsymbol{v}) := \sigma_D h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}), \qquad J_h^{rhs}(\boldsymbol{v}) := -\sigma_D h^2 \sum_{T \in \Gamma} \int_T \boldsymbol{f} \cdot \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}).$$

where [.] is the jump on the interface E,

$\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma_h \neq \varnothing\}, \mathcal{F}_h^\Gamma = \{E \text{ such that } \exists T \in \mathcal{T}_h : T \cap \Gamma_h \neq \varnothing \text{ and } E \in \partial T\}$.

ϕ-FEM for elasticity problem : Dirichlet conditions

Dual formulation with Dirichlet conditions

we make the ansatz only locally around Γ, i.e. on Ω_h^Γ : $\quad \boldsymbol{u}_h = \boldsymbol{u}_h^g + \frac{1}{h}\phi_h \boldsymbol{w}_h$

and we impose the relation between $\boldsymbol{u}_h$ and $\boldsymbol{w}_h$ in a least square manner. Find

$\boldsymbol{u}_h \in V_h := \left\{\boldsymbol{v}_h \in (H^1(\Omega_h))^d : \boldsymbol{v}_{h|T} \in \mathbb{P}^k(T)^d \ \ \forall T \in \mathcal{T}_h\right\}$,

$\boldsymbol{w}_h \in Q_{h,D} := \left\{\boldsymbol{w}_h \in (H^1(\Omega_h^{\Gamma_D}))^d : \boldsymbol{w}_{h|T} \in \mathbb{P}^k(T)^d \ \ \forall T \in \mathcal{T}_h^{\Gamma_D}\right\}$ such that

$$\int_{\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h) : \nabla \boldsymbol{v}_h - \int_{\partial\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h)\boldsymbol{n} \cdot \boldsymbol{v}_h + \frac{\gamma}{h^2}\int_{\Omega_h^\Gamma} (\boldsymbol{u}_h - \frac{1}{h}\phi_h \boldsymbol{w}_h) \cdot (\boldsymbol{v}_h - \frac{1}{h}\phi_h \boldsymbol{z}_h) + J_h^{lhs}(\boldsymbol{u}_h, \boldsymbol{v}_h)$$

$$= \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v}_h + \frac{\gamma}{h^2}\int_{\Omega_h^\Gamma} \boldsymbol{u}_h^g \cdot (\boldsymbol{v}_h - \frac{1}{h}\phi_h \boldsymbol{z}_h) + J_h^{rhs}(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \text{ on } V_h, \boldsymbol{z}_h \text{ on } Q_{h,D}$$

J_h^{lhs}, J_h^{rhs} stand for the stabilization terms

$$J_h^{lhs}(\boldsymbol{u}, \boldsymbol{v}) := \sigma h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\boldsymbol{\sigma}(u) \cdot \boldsymbol{n}] \cdot [\boldsymbol{\sigma}(v) \cdot \boldsymbol{n}] + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (div \boldsymbol{\sigma}(\boldsymbol{u})) \cdot (div \boldsymbol{\sigma}(\boldsymbol{v}))$$

$$J_h^{rhs}(\boldsymbol{v}) := -\sigma h^2 \sum_{T \in T_h^\Gamma} \int_T f \cdot (div \boldsymbol{\sigma}(\boldsymbol{v}))$$

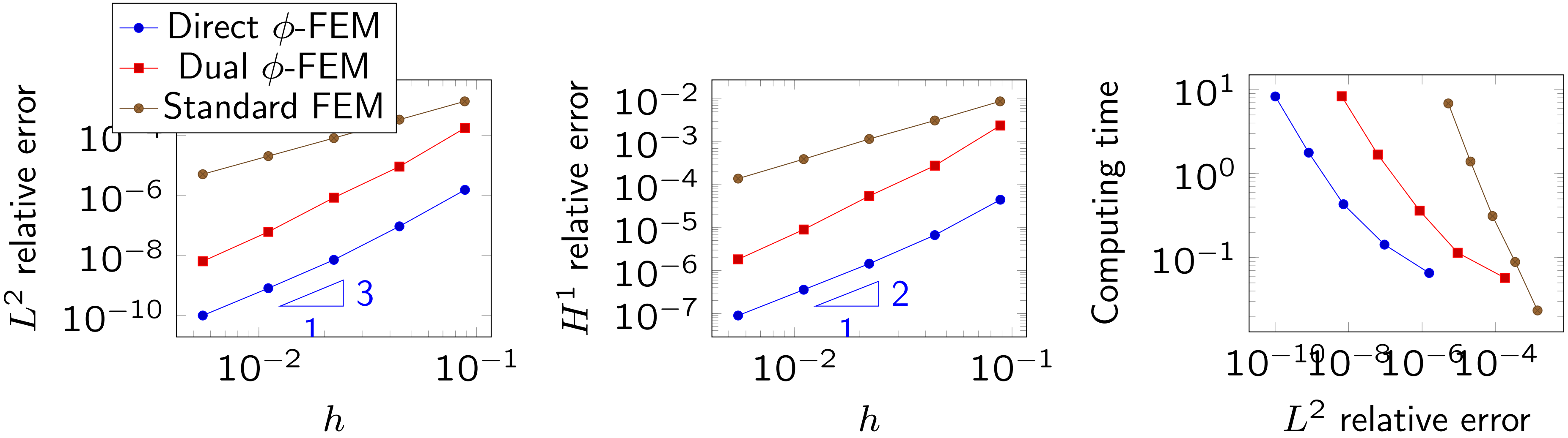
ϕ-FEM for elasticity problem : Dirichlet conditions

$\mathcal{O} = [0,1] \times [0,1]$, $\phi(x,y) = \frac{-1}{8} + (x-0.5)^2 + (y-0.5)^2$.

Parameters : $E = 2$ and $\nu = 0.3$, and $\gamma = \sigma_D = 20.0, k = 2$.

exact solution : $\boldsymbol{u} = \boldsymbol{u}_{ex} := (\sin(x)\exp(y), \sin(y)\exp(x))$

We extend $\boldsymbol{u}^g$ from Γ to Ω_h (direct method) or Ω_h^Γ (dual method) : $\boldsymbol{u}^g = \boldsymbol{u}_{ex}(1+\phi)$



ϕ-FEM for elasticity problem : Neumann conditions

Let us introduce an auxiliary matrix-valued variable $\boldsymbol{y}$ and p a vector-valued auxiliary variable on Ω_h^Γ . The outward-looking unit normal n is given on Γ by $n = \frac{1}{|\nabla\phi|}\nabla\phi$ so the boundary problem is reformulated as the system of 3 equations

$$\begin{aligned} -div\boldsymbol{\sigma}(\boldsymbol{u}) = f, &\quad \text{in } \Omega_h\,, \\ \boldsymbol{y} + \boldsymbol{\sigma}(\boldsymbol{u}) = 0, &\quad \text{in } \Omega_h^\Gamma\,, \\ \boldsymbol{\sigma}(\boldsymbol{u})\cdot\nabla\phi + \boldsymbol{p}\phi = \boldsymbol{g}|\nabla\phi|, &\quad \text{in } \Omega_h^\Gamma\,. \end{aligned}$$

The ϕ-FEM scheme is then obtained : Find $(\boldsymbol{u}_h, \boldsymbol{y}_h, \boldsymbol{p}_h) \in V_h \times Z_h \times Q_{h,N}$ such that

$$\begin{aligned} &\int_{\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h) : \boldsymbol{\nabla}(\boldsymbol{v}_h) + \int_{\Gamma_h} (\boldsymbol{y}_h \cdot \boldsymbol{n})\cdot\boldsymbol{v}_h + \gamma_{div}\int_{\Omega_h^\Gamma} div\boldsymbol{y}_h \cdot div\boldsymbol{z}_h \\ &+\gamma_u \int_{\Omega_h^\Gamma} (\boldsymbol{y}_h + \boldsymbol{\sigma}(\boldsymbol{u}_h))\cdot(\boldsymbol{z}_h + \boldsymbol{\sigma}(\boldsymbol{v}_h)) + \frac{\gamma_p}{h^2}\int_{\Omega_h^\Gamma}\left(\boldsymbol{y}_h\cdot\nabla\phi_h + \frac{1}{h}\boldsymbol{p}_h\phi_h\right)\left(\boldsymbol{z}_h\cdot\nabla\phi_h + \frac{1}{h}\boldsymbol{q}_h\phi_h\right) \\ &+\sigma_p h\int_{\Gamma^i} [\boldsymbol{\sigma}(\boldsymbol{u}_h)\cdot\boldsymbol{n}]\,[\boldsymbol{\sigma}(\boldsymbol{v}_h)\cdot\boldsymbol{n}] = \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v}_h + \gamma_{div}\int_{\Omega_h^\Gamma} \boldsymbol{f}\cdot div\boldsymbol{z}_h \\ &-\frac{\gamma_p}{h^2}\int_{\Omega_h^\Gamma} \boldsymbol{g}|\nabla\phi_h|\left(\boldsymbol{z}_h\cdot\nabla\phi_h + \frac{1}{h}\boldsymbol{q}_h\phi_h\right), \forall\,(\boldsymbol{v}_h, \boldsymbol{z}_h, \boldsymbol{q}_h) \in V_h \times Z_h \times Q_{h,N}. \end{aligned}$$

ϕ-FEM for elasticity problem : Mixed conditions

$\Omega = \{\phi < 0\}$, and assume that the boundary partition into the Dirichlet and Neumann parts is governed by a secondary level set ψ, $\Gamma_D = \Gamma \cap \{\psi < 0\}$.

$$\mathcal{T}_h^{\Gamma_D} := \{T \in \mathcal{T}_h^{\Gamma} : \psi \leqslant 0 \text{ on } T\} \qquad \text{and} \qquad \mathcal{T}_h^{\Gamma_N} := \{T \in \mathcal{T}_h^{\Gamma} : \psi \geqslant 0 \text{ on } T\}\,.$$

we introduce the different finite element spaces :

$$V_h := \left\{\boldsymbol{v}_h \in (H^1(\Omega_h))^d : \boldsymbol{v}_{h|T} \in \mathbb{P}^k(T)^d \;\; \forall T \in \mathcal{T}_h\right\}\,,$$

$$Z_h := \left\{\boldsymbol{z}_h \in (H^1(\Omega_h^{\Gamma_N}))^{(d\times d)} : \boldsymbol{z}_{h|T} \in \mathbb{P}^k(T)^{(d\times d)} \;\; \forall T \in \mathcal{T}_h^{\Gamma_N}\right\}\,,$$

$$Q_{h,N} := \left\{\boldsymbol{q}_h \in (H^1(\Omega_h^{\Gamma_N}))^d : \boldsymbol{q}_{h|T} \in \mathbb{P}^{k-1}(T)^d \;\; \forall T \in \mathcal{T}_h^{\Gamma_N}\right\}\,.$$

and

$$Q_{h,D} := \left\{\boldsymbol{q}_h \in (H^1(\Omega_h^{\Gamma_D}))^d : \boldsymbol{q}_{h|T} \in \mathbb{P}^{k}(T)^d \;\; \forall T \in \mathcal{T}_h^{\Gamma_D}\right\}\,.$$

ϕ-FEM for elasticity problem : Mixed conditions

We get the following scheme : find $\boldsymbol{u}_h \in V_h$, $\boldsymbol{y}_h \in Z_h$, $\boldsymbol{p}_{h,D} \in Q_{h,D}$ and $\boldsymbol{p}_{h,N} \in Q_{h,N}$ such that

$$\int_{\Omega_h} \boldsymbol{\sigma}(\boldsymbol{u}_h) : \boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \int_{\Gamma_{h,N}} (\boldsymbol{y}_h \cdot \boldsymbol{n}) \cdot \boldsymbol{v}_h - \int_{\Gamma_{h,D}} (\boldsymbol{\sigma}(\boldsymbol{u}_h) \cdot \boldsymbol{n}) \cdot \boldsymbol{v}_h + \gamma_{div} \int_{\Omega_h^{\Gamma_N}} div \boldsymbol{y}_h \cdot div \boldsymbol{z}_h$$

$$+\gamma_u \int_{\Omega_h^{\Gamma_N}} (\boldsymbol{y}_h + \boldsymbol{\sigma}(\boldsymbol{u}_h)) \cdot (\boldsymbol{z}_h + \boldsymbol{\sigma}(\boldsymbol{v}_h)) + \frac{\gamma_p}{h^2} \int_{\Omega_h^{\Gamma_N}} \left(\boldsymbol{y}_h \cdot \nabla \phi_h + \frac{1}{h} \boldsymbol{p}_{h,N} \phi_h \right) \left(\boldsymbol{z}_h \cdot \nabla \phi_h + \frac{1}{h} \boldsymbol{q}_{h,N} \phi_h \right)$$

$$+ \frac{\gamma}{h^2} \int_{\Omega_h^{\Gamma_D}} (\boldsymbol{u}_h - \frac{1}{h} \phi_h \boldsymbol{p}_{h,D}) \cdot (\boldsymbol{v}_h - \frac{1}{h} \phi_h \boldsymbol{q}_{h,D}) + G_h^{lhs}(\boldsymbol{u}_h, \boldsymbol{v}_h)$$

$$= \int_{\Omega_h} \boldsymbol{f} \cdot \boldsymbol{v}_h + \gamma_{div} \int_{\Omega_h^{\Gamma_N}} \boldsymbol{f} \cdot div \boldsymbol{z}_h + \frac{\gamma}{h^2} \int_{\Omega_h^D} \boldsymbol{u}_h^g \cdot (\boldsymbol{v}_h - \frac{1}{h} \phi_h \boldsymbol{q}_{h,D})$$

$$- \frac{\gamma_p}{h^2} \int_{\Omega_h^{\Gamma_N}} \boldsymbol{g} \cdot |\nabla \phi_h| (\boldsymbol{z}_h \cdot \nabla \phi_h + \frac{1}{h} \boldsymbol{q}_{h,N} \phi_h) + G_h^{rhs}(v_h), \quad \forall \boldsymbol{v}_h \text{ on } V_h, \boldsymbol{z}_h \text{ on } Z_h, \boldsymbol{q}_{h,D} \text{ on } Q_{h,D}, \boldsymbol{q}_{h,N} \text{ on } Q_{h,N},$$

where G_h^{lhs} and G_h^{rhs} stand for :

$$G_h^{lhs}(\boldsymbol{u}, \boldsymbol{v}) := \sigma h \sum_{E \in \mathcal{F}_h^{\Gamma_N}} \int_E [\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}] \cdot [\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}] + \sigma_D h \sum_{E \in \mathcal{F}_h^{\Gamma_D}} \int_E [\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}] \cdot [\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n}]$$
$$+ \sigma_D h^2 \int_{\Omega_h^{\Gamma_D}} (div \boldsymbol{\sigma}(\boldsymbol{u})) \cdot (div \boldsymbol{\sigma}(\boldsymbol{v})) \text{ and } G_h^{rhs}(\boldsymbol{v}) := -\sigma_D h^2 \int_{\Omega_h^{\Gamma_D}} f \cdot (div \boldsymbol{\sigma}(\boldsymbol{v})) \,.$$

ϕ-FEM for elasticity problem

• ϕ-FEM provides the optimal accuracy with finite elements of any order if the geometry is sufficiently well approximated (via ϕ_h of degree $\geq$ k+1), provided u, f, and ϕ are smooth enough

Theorem
Suppose that $\Omega \subset \Omega_h$ and $\mathbf{f} \in (H^k(\Omega_h))^d$. Let $\mathbf{u} \in (H^{k+2}(\Omega))^d$ be the continuous solution and $(\boldsymbol{u}_h, \boldsymbol{y}_h, \boldsymbol{p}_{h,D}, \boldsymbol{p}_{h,N})$ be the discrete solution. Provided γ_{div}, γ_u, γ_p, γ, σ, σ_D are sufficiently big, it holds

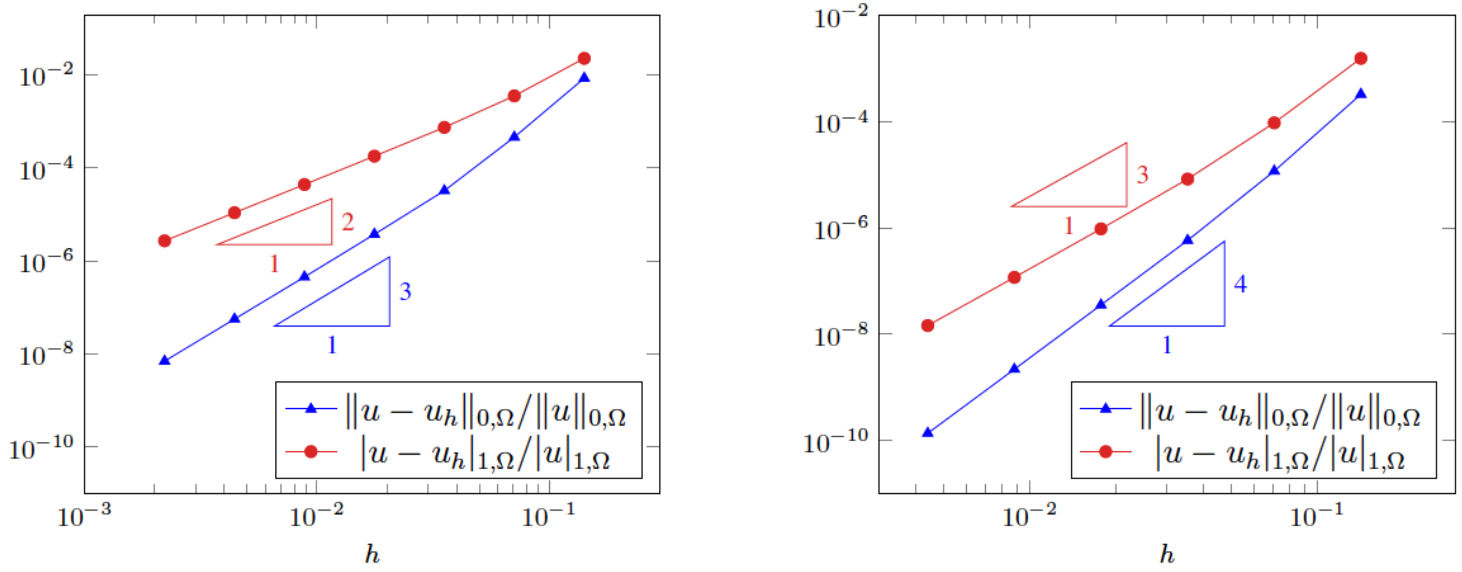
$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} \leq Ch^k(\|\boldsymbol{f}\|_{k,\Omega_h} + \|\boldsymbol{g}\|_{k+1,\Omega_h^\Gamma})$$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \leq Ch^{k+1/2}(\|\boldsymbol{f}\|_{k,\Omega_h} + \|\boldsymbol{g}\|_{k+1,\Omega_h^\Gamma})$$

with $C > 0$ depending on the mesh regularity, on the regularity of ϕ but independent of h, f, and u.

ϕ-FEM for elasticity problem

• ϕ-FEM works high polynomial orders : it suffices to approximate the level set function by piecewise polynomials of the same degree as that used for the primal unknown.

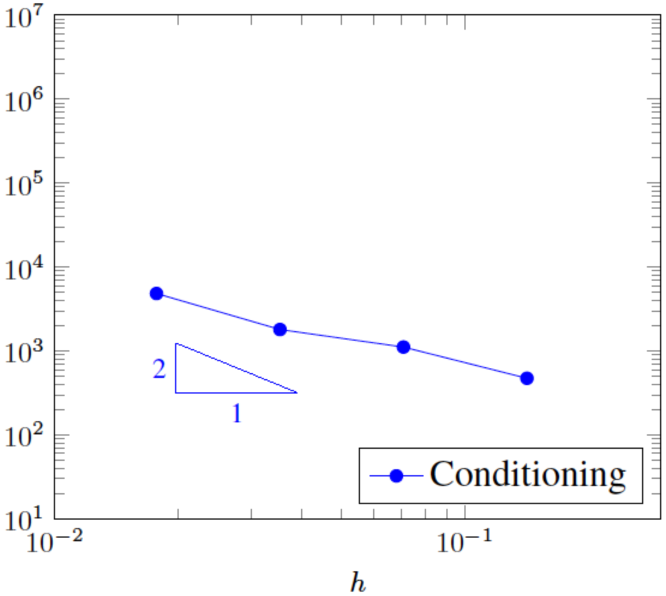


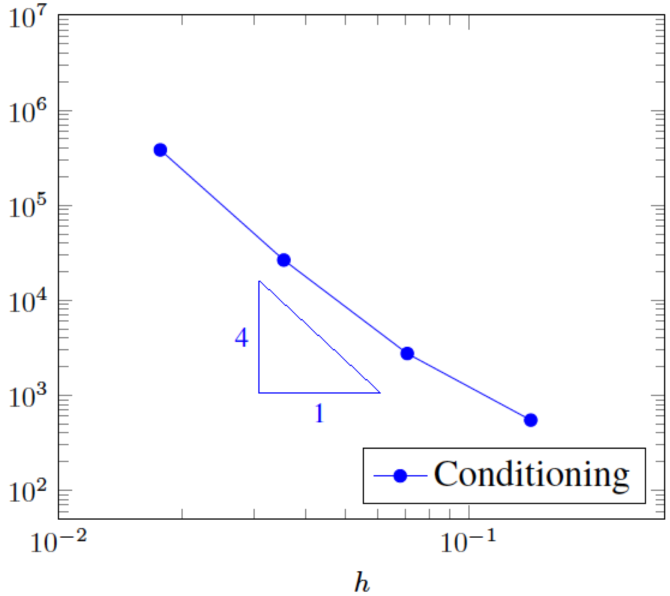
Left : P_2 finite elements ; Right : P_3 finite elements

ϕ-FEM for elasticity problem

- Good conditioning of the matrix : The finite element matrix of ϕ-FEM satisfies

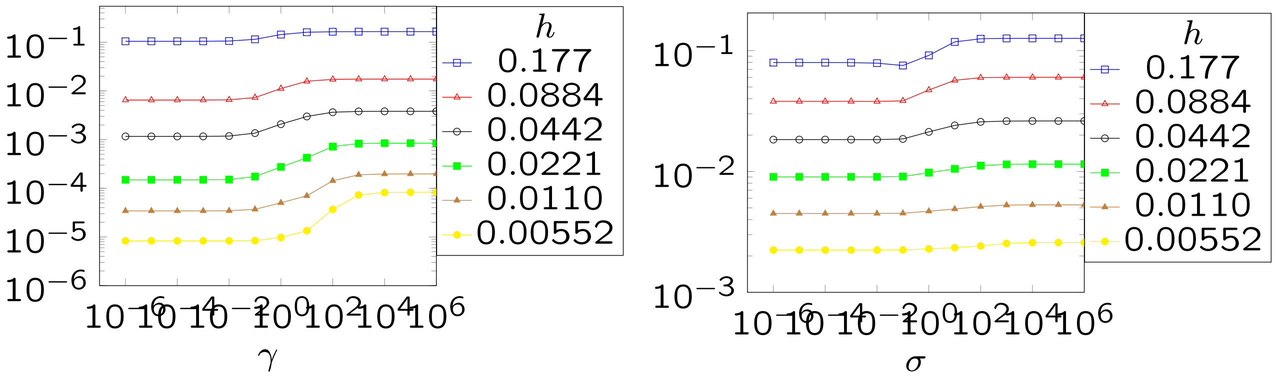
$$\kappa(A) := \|A\|_2 \|A^{-1}\|_2 \leq C h^{-2}$$



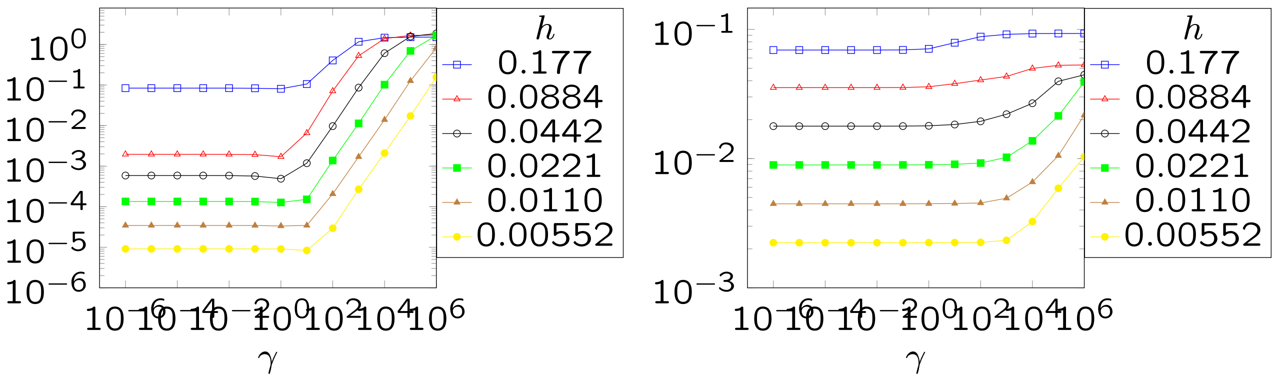


P_1 finite elements. Left : with ghost penalty. Right : without ghost penalty

ϕ-FEM for elasticity problem



Sensitivity of the relative error in ϕ-FEM with respect to σ with $\gamma_u = \gamma_p = \gamma_{div} = \gamma = 10$. Left : L^2 relative error ; Right : H^1 relative error.



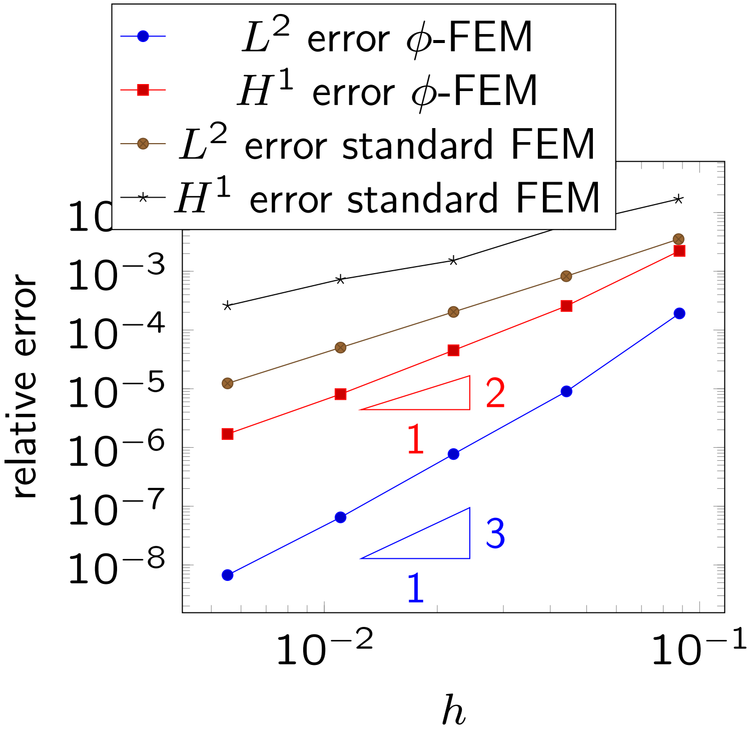
Sensitivity of the relative error in ϕ-FEM with respect to $\gamma_u = \gamma_p = \gamma_{div} = \gamma$ with $\sigma = 0.01$. Left : L^2 relative error ; Right : H^1 relative error.

ϕ-FEM for elasticity problem : Mixed conditions

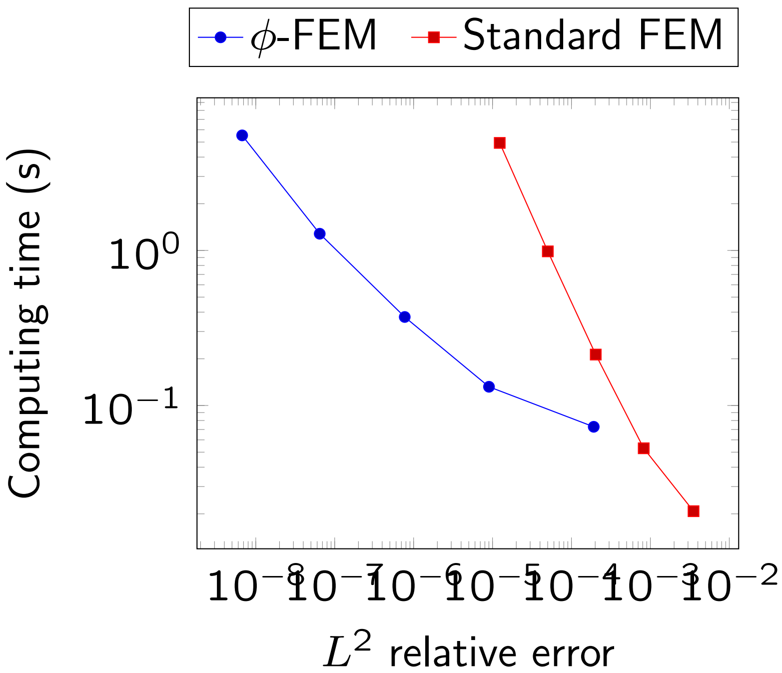
Parameters : $\lambda = 100\ kPa$, $\nu = 0.3$, $\gamma_{div} = \gamma_u = \gamma_p = 1.0$, $\sigma = 0.01$ and $\gamma = \sigma_D = 20.0$.
exact solution : $u_{ex} = (\sin(x) \times \exp(y), \sin(y) \times \exp(x))$
extrapolated boundary conditions

$$\boldsymbol{u}^g = \boldsymbol{u}_{ex} \times (1+\phi), \quad \text{on } \Omega_h^\Gamma \cap \{x \geqslant 0.5\}, \quad \boldsymbol{g} = \boldsymbol{\sigma}(\boldsymbol{u}_{ex}) \times \frac{\nabla\phi}{\|\nabla\phi\|} + \boldsymbol{u}_{ex} \times \phi, \quad \text{on } \Omega_h^\Gamma \cap \{x < 0.5\}$$

where we used $\boldsymbol{u}_{ex} \times \phi$ to add a little perturbation to the exact solution on the boundaries.



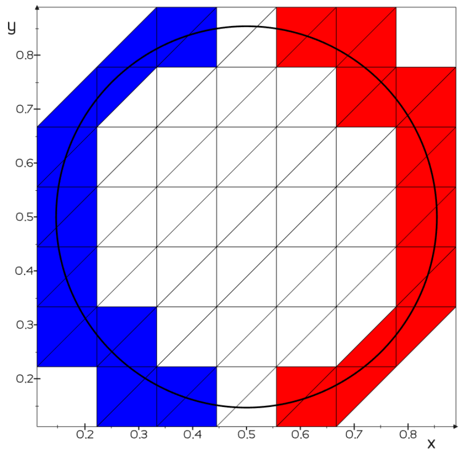
H^1 and L^2 relative errors

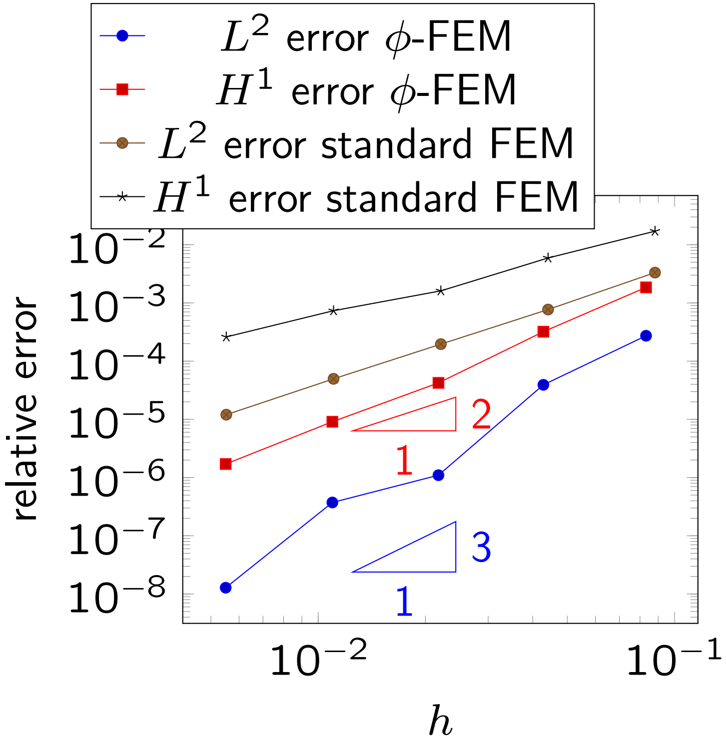


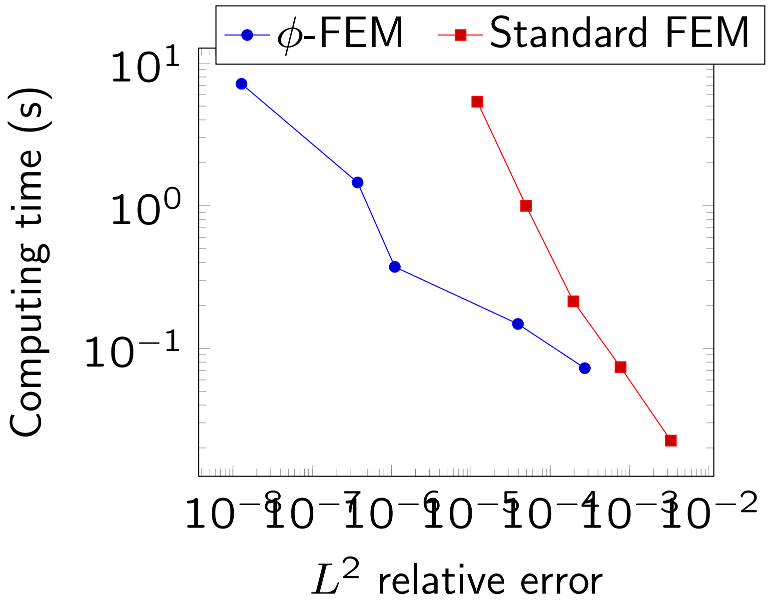
Computing time

ϕ-FEM for elasticity problem : Mixed conditions

meshes not resolving the Dirichlet/Neumann junction :







Outline

1. Motivation and previous works

2. ϕ-FEM method for elasticity problems
   (a) With Dirichlet conditions
   (b) With Neumann conditions
   (c) With mixed conditions

3. Some applications
   (a) Case of fracture problems
   (b) Particulate flows
   (c) Heat problem

4. Summary and outlook

ϕ-FEM for elasticity problem : a fracture problem

We solve a linear elasticity problem, on two materials and we introduce a new boundary condition on the fracture. The considered problem is :

$$\begin{cases} -div\boldsymbol{\sigma}_i(\boldsymbol{u}_i) & = \boldsymbol{f}\,, \text{ on } \Omega_i\,, \\ \boldsymbol{u}_i & = \boldsymbol{u}^g\,, \text{ on } \partial\Omega\,, \\ [\boldsymbol{u}] & = 0\,, \text{ on } \Gamma_{int}\,, \\ [\boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n}] & = 0\,, \text{ on } \Gamma_{int}\,, \\ \boldsymbol{\sigma}(\boldsymbol{u})\cdot\boldsymbol{n} & = \boldsymbol{g}\,, \text{ on } \Gamma_f\,. \end{cases}$$

$\Gamma_f := \Omega \cap \{\phi = 0\} \cap \{\psi < 0\}$ and the remaining (fictitious) part Γ_{int} :
$\Gamma_{int} := \Omega \cap \{\phi = 0\} \cap \{\psi > 0\}$.

We introduce $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{p}$ for the formulation on the interface. To impose the Neumann condition on Γ_f, we also introduce $\boldsymbol{y}_1^N, \boldsymbol{y}_2^N, \boldsymbol{p}_1^N, \boldsymbol{p}_2^N$ for the formulation on the fracture.

Interface :

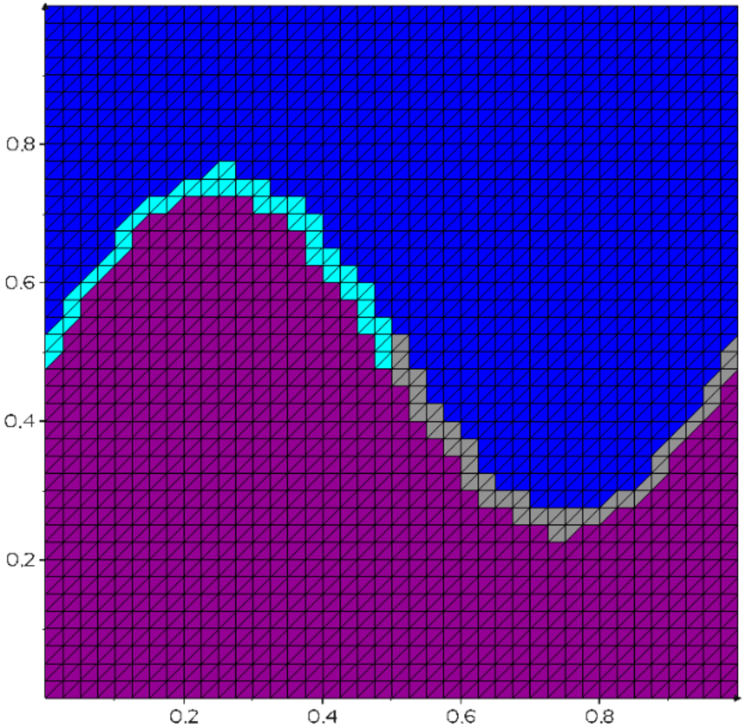
$$\begin{aligned} \boldsymbol{y}_i = -\boldsymbol{\sigma}(\boldsymbol{u}_i)\,, &\quad \text{on } \Omega_h^{\Gamma_{int}}\,, \\ \boldsymbol{u}_1 - \boldsymbol{u}_2 + \boldsymbol{p}\phi = 0\,, &\quad \text{on } \Omega_h^{\Gamma_{int}}\,, \\ \boldsymbol{y}_1\cdot\nabla\phi - \boldsymbol{y}_2\cdot\nabla\phi = 0\,, &\quad \text{on } \Omega_h^{\Gamma_{int}}\,. \end{aligned}$$

Neumann :

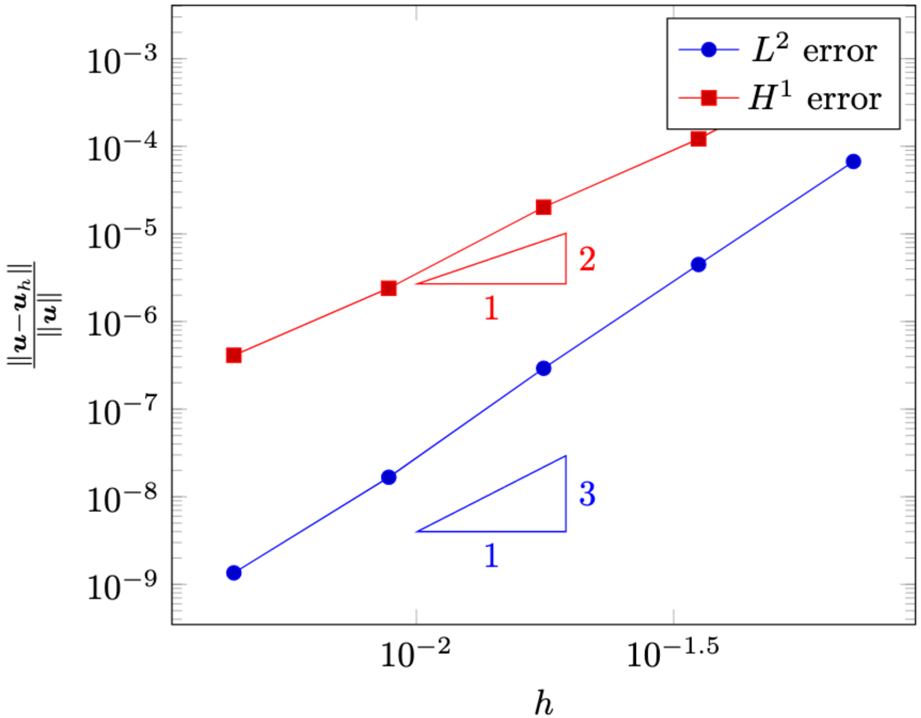
$$\begin{aligned} \boldsymbol{y}_i^N = -\boldsymbol{\sigma}(\boldsymbol{u}_i)\,, &\quad \text{on } \Omega_h^{\Gamma_f}\,, \\ \boldsymbol{y}_i^N\nabla\phi + \boldsymbol{p}_i^N\phi + \boldsymbol{g}|\nabla\phi| = 0\,, &\quad \text{on } \Omega_h^{\Gamma_f}\,. \end{aligned}$$

ϕ-FEM for elasticity problem : a fracture problem

$\phi(x,y) = y - \frac{1}{4}\sin(2\pi x) - \frac{1}{2}$.
exact solution : $u_{ex} = (\sin(x) \times \exp(y), \sin(y) \times \exp(x))$
boundary conditions : $\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{\sigma}(\boldsymbol{u}_{ex}) \times \frac{\nabla\phi}{\|\nabla\phi\|} + \boldsymbol{u}_{ex} \times \phi$, on $\Omega_h^{\Gamma_f} = \Omega_h^{\Gamma} \cap \{x > 0.5\}$.
$\gamma_u = \gamma_p = \gamma_{div} = \gamma_{u,N} = \gamma_{p,N} = \gamma_{div,N} = 1.0$, $\sigma_p = 1.0$ and $\sigma_N = 0.01$.



Left : blue : $\Omega_{h,1}$, purple : $\Omega_{h,2}$, cyan : $\Omega_h^{\Gamma_{int}}$, Gray : $\Omega_h^{\Gamma_f}$.



H^1 and L^2 relative errors with $E_1 = E_2 = 7\ kPa$, $\nu_1 = \nu_2 = 0.3$ and $k = 2$

ϕ-FEM for particulate flows

Let us consider the motion of a fluid around a solid particle in the regime of creeping motion, i.e. all the inertial terms are neglected. The fluid occupies a domain $\Omega \subset \mathbb{R}^d$, the particle occupies a domain $\mathcal{S}$ and Γ_w (the external wall) and $\partial\mathcal{S} = \Gamma$.

Parameters : constant gravitation vector g, the constant fluid density ρ_f, the mass of the particle m, the constant fluid viscosity ν, r the vector from the barycenter of the solid $\mathcal{S}$, n the unit normal looking into the solid.
The unknowns are the fluid velocity u, pressure p, the velocity of the barycenter of the particle U, and the angular velocity of the particle ψ.

Governing equations :

$$\begin{aligned}
-2\nu \operatorname{div} D(u) + \nabla p &= \rho_f g, && \text{in } \Omega\\
\operatorname{div} u &= 0, && \text{in } \Omega\\
u &= U + \psi \times r, && \text{on } \Gamma\\
u &= 0, && \text{on } \Gamma_w\\
-\int_\Gamma (2D(u) - pI)n + mg &= 0\\
\int_\Gamma (2D(u) - pI)n \times r &= 0\\
\int_\Omega p &= 0
\end{aligned}$$

ϕ-FEM for particulate flows

Weak formulation :

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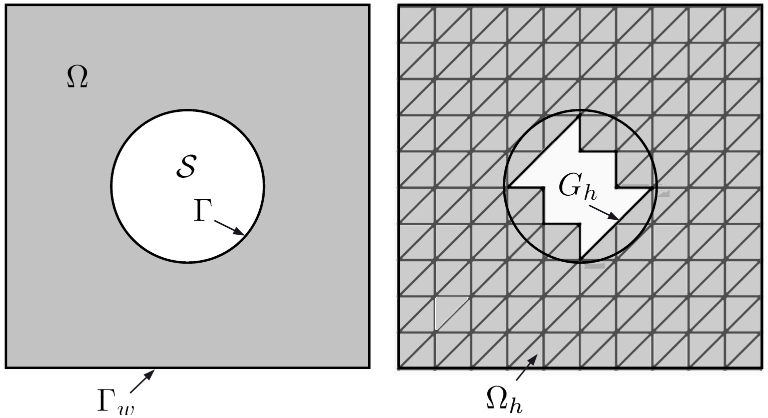


We define the non-conforming active mesh $\mathcal{T}_h$ on $\Omega_h \supset \Omega$ with its internal boundary G_h.

• u and p can be extended from Ω to Ω_h, an integration by parts gives

$$2\nu \int_{\Omega_h} D(u) : D(v) - \int_{\Omega_h} p \operatorname{div} v - \int_{\Omega_h} q \operatorname{div} u - \int_{G_h} (2\nu D(u) - pI)n \cdot v = \int_{\Omega_h} \rho_f g \cdot v.$$

• We make the ansatz $u = \phi w + \chi(U + \psi \times r)$ where ϕ is a level set function for the fluid domain $\Omega = \{\phi < 0\}$, and χ is a sufficiently smooth function on $\mathcal{O}$ such that $\chi = 1$ on the solid $\mathcal{S}$ and $\chi = 0$ on Γ_w.

ϕ-FEM for particulate flows

Weak formulation :

• We introduce $B_h = \Omega_h \setminus \Omega$, i.e. the strip between Γ and G_h, and use the divergence theorem on B_h : find $w : \Omega_h \to \mathbb{R}^d$ vanishing on Γ_w, $U \in \mathbb{R}^d$, $\psi \in \mathbb{R}^{d'}$, and $p : \Omega_h \to \mathbb{R}$ such that

$$2\nu \int_{\Omega_h} D(\phi w + \chi(U + \psi \times r)) : D(\phi s + \chi(V + \omega \times r)) - \int_{G_h} (2\nu D(\phi w + \chi(U + \psi \times r)) - pI) n \cdot \phi s$$
$$- \int_{\Omega_h} p \,\mathrm{div}(\phi s + \chi(V + \omega \times r)) - \int_{\Omega_h} q \,\mathrm{div}(\phi w + \chi(U + \psi \times r))$$
$$= \int_{\Omega_h} \rho_f g \cdot \phi s + \int_{\mathcal{O}} \rho_f g \cdot \chi(V + \omega \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) m g \cdot V$$

for all $s : \Omega_h \to \mathbb{R}^d$ vanishing on Γ_w, $V \in \mathbb{R}^d$, $\omega \in \mathbb{R}^{d'}$, and $q : \Omega_h \to \mathbb{R}$.

• We discretize using the usual finite elements for the trial and test functions, and approximating ϕ, χ by piecewise polynomials.

ϕ-FEM for particulate flows : stabilized scheme

Weak formulation : Setting $u_h = \chi_h(U_h + \psi_h \times r) + \phi_h w_h, \nu = 1$.
Find $u_h \in \mathcal{V}_h^{rbm} = \{\chi_h(V_h + \omega_h \times r) + \phi_h s_h$ with $s_h \in \mathcal{V}_h, V_h \in \mathbb{R}^d, \omega_h \in \mathbb{R}^{d'}\}$ and $p_h \in \mathcal{M}_h = \left\{q_h \in C(\bar{\Omega}) : q_h|_T \in \mathbb{P}^{k-1}(T) \quad \forall T \in \mathcal{T}_h, \quad \int_\Omega q_h = 0\right\}$ such that

$$c_h(u_h, p_h; v_h, q_h) = L_h(v_h, q_h), \quad \forall v_h \in \mathcal{V}_h^{rbm},\ q_h \in \mathcal{M}_h\,,$$

where the bilinear form c_h is given by

$$\begin{aligned} c_h(u_h, p_h; v_h, q_h) = 2\int_{\Omega_h} D(u_h) : D(v_h) - \int_{\partial\Omega_h} (2D(u_h) - p_h I)n \cdot \phi_h s_h \\ - \int_{\Omega_h} q_h \operatorname{div} u_h - \int_{\Omega_h} p_h \operatorname{div} v_h \\ + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (-\Delta u_h + \nabla p_h) \cdot (-\Delta v_h - \nabla q_h) + \sigma \sum_{T \in \mathcal{T}_h^\Gamma} \int_T (\operatorname{div} u_h)(\operatorname{div} v_h) \\ + \sigma_u h \sum_{E \in \mathcal{F}_h^\Gamma} \int_E [\partial_n u_h] \cdot [\partial_n v_h] + \sigma_u h^3 \sum_{E \in \mathcal{F}_h^\Gamma} \int_E \left[\partial_n^2 u_h\right] \cdot \left[\partial_n^2 v_h\right] \end{aligned}$$

and the linear form L_h is given by

$$\begin{aligned} L_h(v_h, q_h) = \int_{\Omega_h} \rho_f g \cdot \phi_h s_h + \int_{\mathcal{O}} \rho_f g \cdot \chi_h(V_h + \omega_h \times r) + \left(1 - \frac{\rho_f}{\rho_s}\right) mg \cdot V_h \\ + \sigma h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \rho_f g \cdot (-\Delta v_h - \nabla q_h). \end{aligned}$$

ϕ-FEM for particulate flows : stabilized scheme

Inf sup condition :
Introduce the norm on $\mathcal{V}_h^{rbm} \times \mathcal{M}_h$

$$|||v_h, q_h|||_h := \left(|v_h|^2_{1,\Omega_h} + \|q_h\|^2_{0,\Omega_h} + h^2 \sum_{T \in \mathcal{T}_h^\Gamma} \| -\Delta v_h + \nabla q_h\|^2_{0,T} + J_u(v_h, v_h) \right)^{1/2} .$$

where J_u is the ghost penalties for the velocity.

Proposition
The following inf-sup condition holds

$$\forall (u_h, p_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h \quad \exists (v_h, q_h) \in \mathcal{V}_h^{rbm} \times \mathcal{M}_h$$

such that

$$\frac{c_h(u_h, p_h; v_h, q_h)}{|||v_h, q_h|||_h} \geqslant \theta \, ||| \, u_h, p_h |||_h$$

with a constant $\theta > 0$ depending only on the mesh regularity.

ϕ-FEM for particulate flows : stabilized scheme

Rates of convergence

Theorem
Let $(u, U, \psi, p) \in H^{k+1}(\Omega)^d \times \mathbb{R}^d \times \mathbb{R}^{d'} \times H^k(\Omega)$ be the solution to the continuous problem and $(w_h, U_h, \psi_h, p_h) \in \mathcal{V}_h \times \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{M}_h$ be the solution to the stabilized scheme. Denoting $u_h := \chi_h(U_h + \psi_h \times r) + \phi_h w_h$, the H^1 a priori error estimate holds for $h \leq h_0$

$$|u - u_h|_{1,\Omega\cap\Omega_h} + |p - p_h|_{0,\Omega\cap\Omega_h} \leq Ch^k(\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega})$$

and the error for the translation and the rotation of the solid is the following :

$$|U - U_h| + |\psi - \psi_h| \leq Ch^k(\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega})$$

with some $C > 0$ and $h_0 > 0$ depending on the maximum of the derivatives of ϕ and χ of order up to $k + 1$, on the mesh regularity, and on the polynomial degree k, but independent of h, f, and u.

Moreover, supposing $\Omega \subset \Omega_h$, the L^2 error of the velocity is :

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2}(\|u\|_{k+1,\Omega} + \|p\|_{k,\Omega})$$

with a constant $C > 0$ of the same type as above.

ϕ-FEM for particulate flows : numerical results

Parameters : $\phi(x,y) = R^2 - (x-0.5)^2 - (y-0.5)^2$, $g = 10$, $\rho_f = 1$, $\rho_s = 2$, $m = \rho_s \pi^2 R^2$, $\sigma = \sigma_u = 20$.

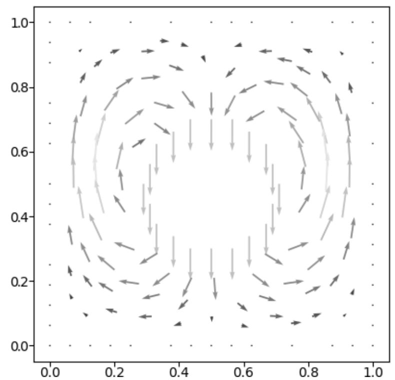
Function χ : we consider the radial polynomial of degree 5 on the interval (r_0, r_1) with $r_0 = 0.21$ and $r_1 = 0.45$ such that $\chi(r_0) = 1$ and $\chi'(r_0) = \chi''(r_0) = \chi(r_1) = \chi'(r_1) = \chi''(r_1) = 0$, that is, for each $r \in (r_0, r_1)$,

$$\chi(r) = 1 + \frac{f(r_0, r_1)}{(r_1 - r_0)^5},$$

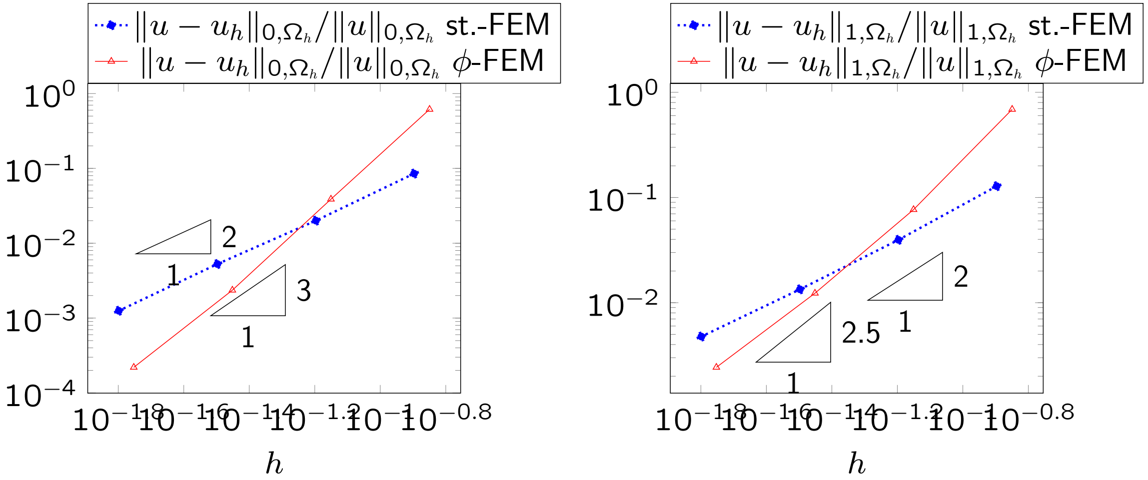
where

$$f(r_0, r_1) = (-6r^5 + 15(r_0 + r_1)r^4 - 10(r_0^2 + 4r_0 r_1 + r_1^2)r^3 + 30 r_0 r_1 (r_0 + r_1) r^2 \\ - 30 r_0^2 r_1^2 r + r_0^3 (r_0^2 - 5 r_1 r_0 + 10 r_1^2)).$$

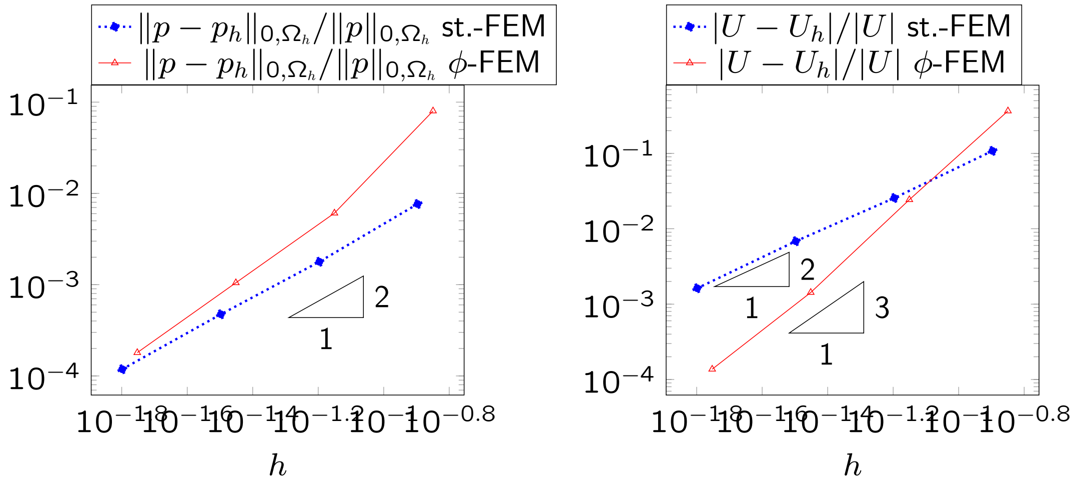
Velocity obtained with the standard Taylor-Hood FEM scheme :



ϕ-FEM for particulate flows : numerical results



L^2 relative error of the velocity (left) and H^1 relative error of the velocity (right) for the standard Taylor-Hood FEM scheme and the ϕ-FEM scheme.



L^2 relative error of the pressure (left) and relative error of the displacement of the solid (right) for the standard Taylor-Hood FEM scheme and the ϕ-FEM scheme.

ϕ-FEM for the heat equation

Governing equations : Let $T > 0$ and $u = u(x,t)$. We consider the Heat-Dirichlet problem

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \Gamma \times (0,T) \\ u(\cdot,0) = u^0 & \text{in } \Omega\,, \end{cases}$$

Let introduce a uniform partition of $[0,T]$ into time steps $0 = t_0 < t_1 < \cdots < t_N = T$ of length Δt for $n = 1, 2, \ldots, N$.

Using an implicit Euler scheme, we get the following scheme in time : find $u^n = \phi w^n$ such that

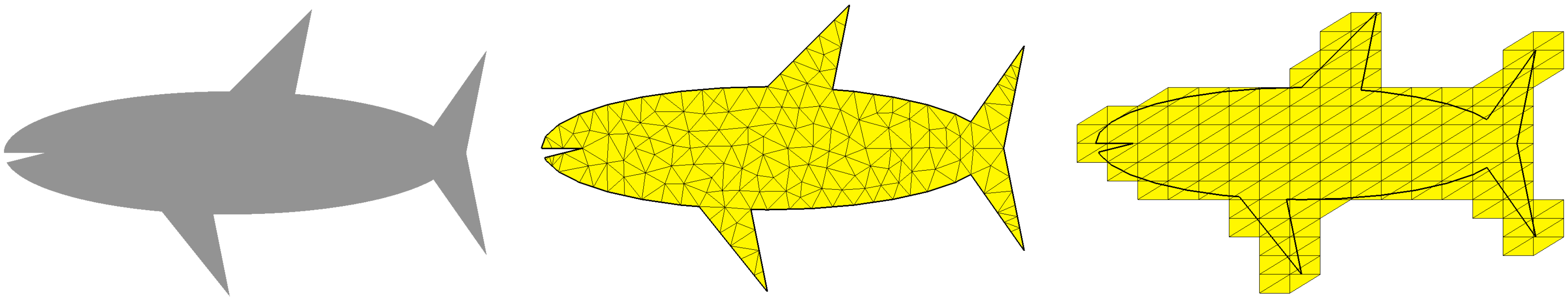
$$\frac{\phi w^{n+1} - \phi w^n}{\Delta t} - \Delta(\phi w^{n+1}) = f^{n+1}\,.$$

We have obtained the ϕ-fem scheme and proved the rate of convergence.

A somewhat unexpected feature of this stabilization is that it works under the constraint on the steps in time and space of the type $\Delta t \geqslant ch^2$. This does not affect the practical interest of the scheme since it is normally intended to be used in the regime $\Delta t \sim h$.

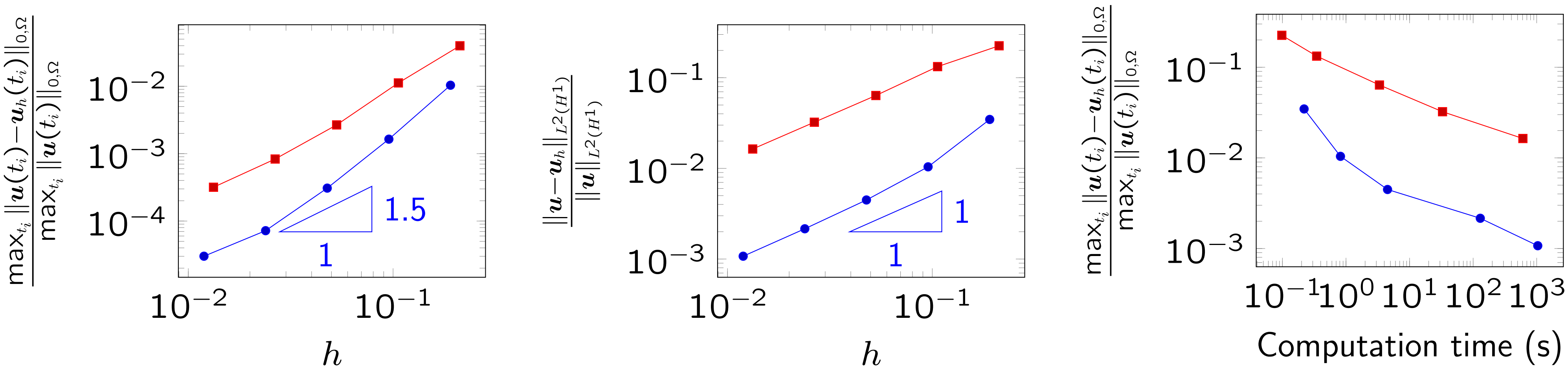
ϕ-FEM for the heat equation

$\phi(x, y) =$ signed distance to the boundary of the domain, $\sigma = 20$, P_1 elements.
exact solution : $u_{ex} = \exp(x)\sin(2\pi y)\sin(t)$
extrapolated boundary conditions $u_h^n = \phi_h w_h^n + I_h u_g(\cdot, t_n)$ where u_g is some lifting of u_D from Γ to Ω_h and I_h stands for an interpolation by finite elements. Here $\boldsymbol{u}^g = \boldsymbol{u}_{ex}(1+\phi)$, on $\Omega_h^{\Gamma_D}$



Left : considered domain. Center : a conforming mesh for the standard FEM. Right : a uniform Cartesian mesh for ϕ-FEM.

ϕ-FEM for the heat equation



Standard FEM (red squares) and ϕ-FEM (blue dots) with $\Delta t = h$. Left : $L^\infty(0, T; L^2(\Omega))$ relative errors against h. Center : $L^2(0, T; H^1(\Omega))$ relative errors against h. Right : $L^\infty(0, T; L^2(\Omega))$ relative errors against the computation time.

Remark : Interesting numerical results on other schemes (Crank-Nicolson or BDF2 time discretizations) are being obtained.

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Conclusion and ongoing works

Results :

— ϕ-FEM has several attractive features :
• Optimal convergence : in the L^2 norm : sub-optimal in theory, optimal in practice
• Discrete problem is well conditioned
• Simple implementation : standard shape functions, all the integrals can be computed by standard quadrature rules on entire mesh cells and on entire boundary facets.
• Formulation available for any order of approximation

— ϕ-FEM works for elasticity problem, a simple fracture problem, Stokes problem and an example of particulate flows, heat equation.

Ongoing works :

— ϕ-fem for fluid structure interaction

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