Accurate curved Dirichlet boundaries for fluid flow simulations on the lattice Boltzmann uniform Cartesian grid

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Outline

Lid driven cavity with Palabos on GPUs

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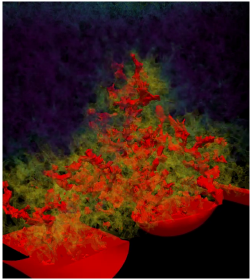
- Lattice Boltzmann method → accurate complex boundaries with a uniform Cartesian mesh.
- What can we do with accurate directional boundary conditions on uniform grids?
  ⇒ High-performance high-fidelity multi-GPU simulations on uniform Cartesian grids!

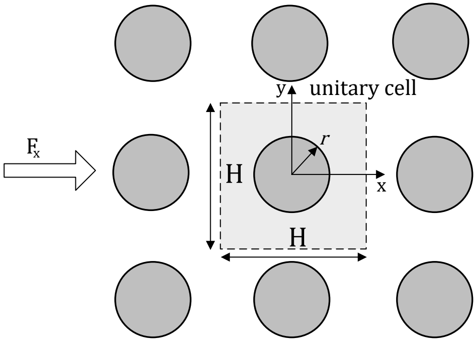
Example: porous media simulations with fully resolved geometries using Palabos (Latt et al., 2021)



https://palabos.unige.ch/class/summer-school/

Academic example for accuracy benchmarking

We perform a test to calculate the accuracy of the permeability for a flow around an array of cylinders.



The (forceless) lattice Boltzmann equation

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BGK-LBE

$$f_i(\boldsymbol{x}+\boldsymbol{c}_i,\ t+1) = \overbrace{f_i\,(\boldsymbol{x},t) - \underbrace{\frac{n_i}{\tau}}_{\hat{n}_i}}^{\hat{f}_i(\boldsymbol{x},t)}$$

$$n_i = f_i - e_i$$

TRT-LBE

$$f_i^{\pm}(\boldsymbol{x}+\boldsymbol{c}_i,\ t+1) = \overbrace{f_i^{\pm}\,(\boldsymbol{x},t) - \underbrace{\frac{n_i^{\pm}}{\tau^{\pm}}}_{\hat{n}_i^{\pm}}}^{\hat{f}_i^{\pm}(\boldsymbol{x},t)}$$

$$\Lambda^{\pm} = \tau^{\pm} - 1/2\,, \quad \Lambda = \Lambda^{+}\Lambda^{-}$$

collide: $f_i(\boldsymbol{x},t) \to \hat{f}_i(\boldsymbol{x},t)$

stream: $\hat{f}_i(\boldsymbol{x},t) \to f_i(\boldsymbol{x}+\boldsymbol{c}_i\Delta t, t+\Delta t)$

e_i = polynomial expansion of Maxwell-Boltzmann distribution

$e_{i,1}$ = linear equilibrium (truncation at the first order)

if D2Q9 $\to i \in \{0,\ldots,8\}$

The Lattice

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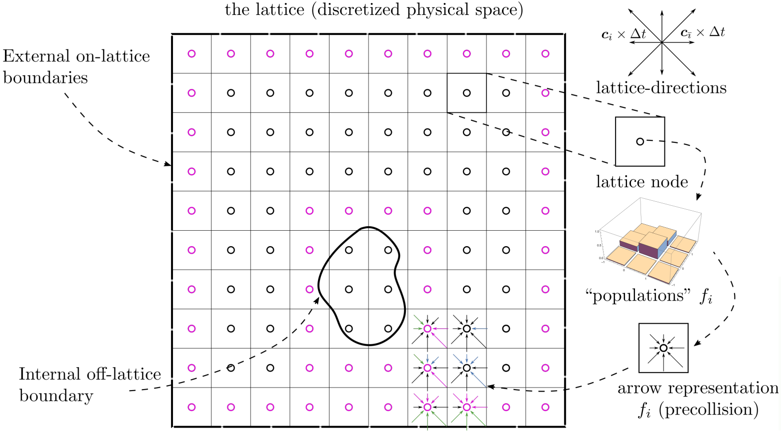
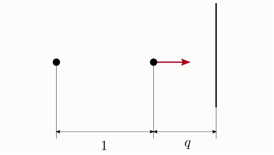
$$f_i(\boldsymbol{x} + \boldsymbol{c}_i,\ t+1) = \overbrace{f_i\,(\boldsymbol{x}, t)}^{\hat{f}_i(x,t)} + \hat{n}_i$$

⚠ the difference between populations have been exaggerated to easy the recognition





The Lattice

$$f_i(\boldsymbol{x}+\boldsymbol{c}_i,\ t+1) = \overbrace{f_i\left(\boldsymbol{x},t\right) + \hat{n}_i}^{\hat{f}_i(\boldsymbol{x},t)}$$

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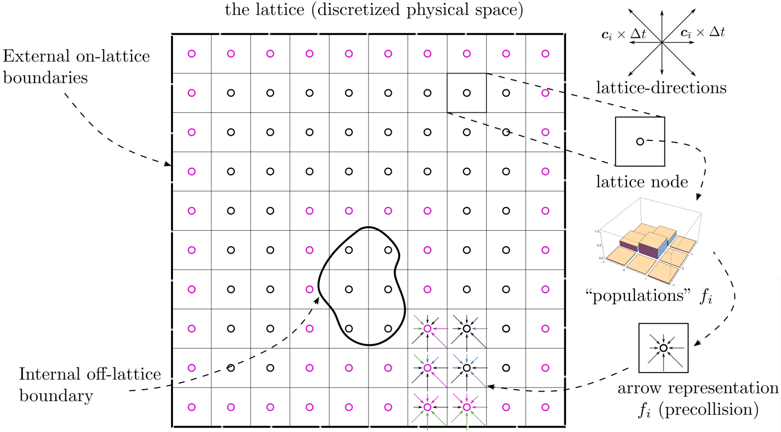

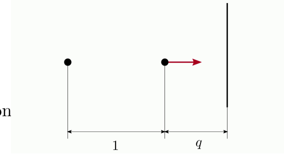

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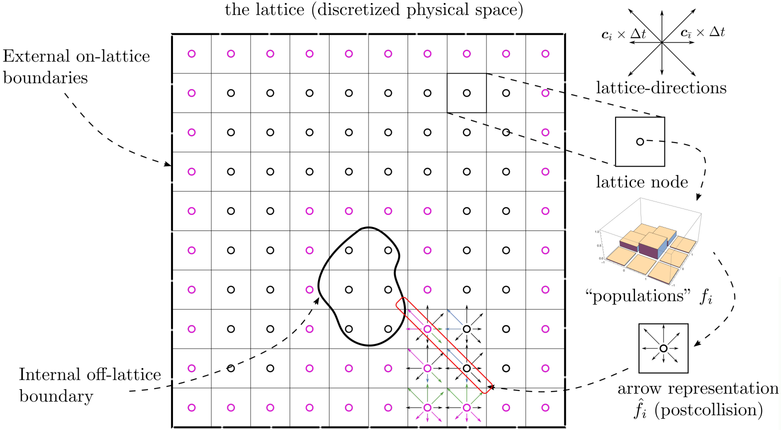

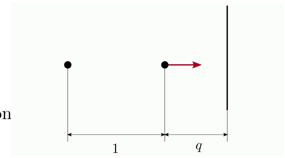

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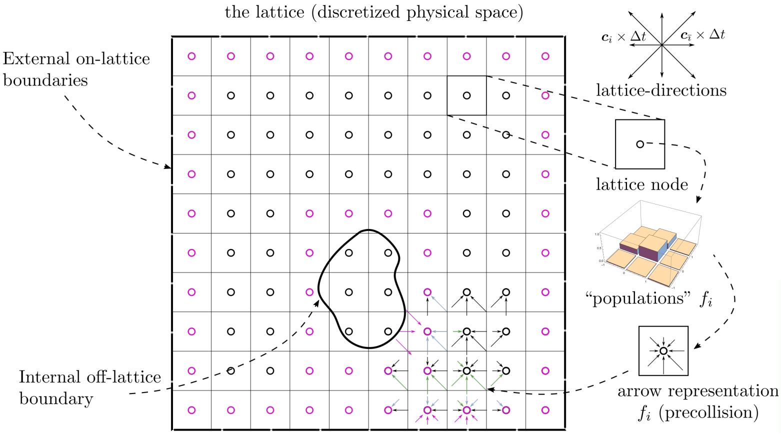

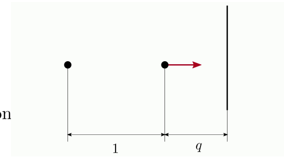

Moments of the distribution function, solvability conditions

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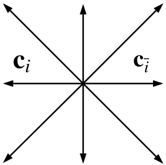
Macroscopic fields (without momentum and mass sources):

$$\rho = f_0^+ + 2\sum_{i=1}^{Q_m/2} f_i^+ \,, \qquad P = c_l^2 \rho \,, \qquad u_\alpha = 2\sum_{i=1}^{Q_m/2} f_i^- c_{i,\alpha}$$



Solvability conditions:

$$\hat{n}_0^+ + 2\sum_{i=1}^{Q_m/2} \hat{n}_i^+ = 0 \ , \qquad 2\sum_{i=1}^{Q_m/2} \hat{n}_i^- c_{i,\alpha} = 0 \ .$$

(Quasi-incompressible) equilibrium

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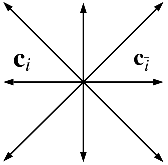
$$e_i = e_i^+ + e_i^-$$

$$e_i^+ = w_i\rho + \underbrace{w_i\rho_0 I_{\mathrm{Ma}^2}\frac{u_{\alpha_1}u_{\alpha_2}(c_{i,\alpha_1}c_{i,\alpha_2} - c_l^2\delta_{\alpha_1\alpha_2})}{2c_l^4} + w_i\rho_0 I_{\mathrm{Ma}^4}\,\mathcal{O}(\mathrm{Ma}^4)}_{e^+_{i,\mathrm{nl}}} + \dots$$

$$e_i^- = \underbrace{w_i\rho_0\frac{c_{i,\alpha_1}u_{\alpha_1}}{c_l^2}}_{e^-_{i,\mathrm{l}}} + \underbrace{w_i\rho_0 I_{\mathrm{Ma}^3}\,\mathcal{O}(\mathrm{Ma}^3)}_{e^-_{i,\mathrm{nl}}} + \dots$$

$$I_{\mathrm{Ma}^j} \in \{0,1\} \quad \forall\, j \in \mathbb{N}^+$$



Stokes flow → the linear equilibrium (truncation) is sufficient
Navier-Stokes flow → we need at least 2nd order expansion.

The values of w_i can be computed in two ways:

- from the Gauss-Hermite quadrature of the polynomial expansion of the Maxwell-Boltzmann
- from isotropy conditions.

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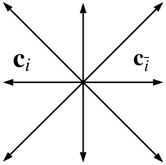

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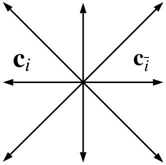

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Solution for the nonequilibrium, Chapman-Enskog + Taylor

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Perturbative expansion of f_i:

$$f_{c,i} = e_i + \epsilon f^{(1)}_{c,i} + \epsilon^2 f^{(2)}_{c,i} = e_i + n_{c,i}$$

Perturbative expansion of the operators, assuming diffusive scaling

$$\partial_t = \epsilon^2 \partial^{(2)}_t \qquad \partial_\alpha = \epsilon \partial^{(1)}_\alpha + \epsilon^2 \partial^{(2)}_\alpha \qquad \forall\, \alpha \in \{x_1, \dots, x_D\}$$

inject into the (directional) Taylor expansion of the TRT-LBE

$$f_{c,i} + (\epsilon c_{i,\alpha}\partial^{(1)}_\alpha + \epsilon^2 c_{i,\alpha}\partial^{(2)}_\alpha + \epsilon^2 \partial^{(2)}_t) f_{c,i} + (\epsilon c_{i,\alpha}\partial^{(1)}_\alpha + \epsilon^2 c_{i,\alpha}\partial^{(2)}_\alpha + \epsilon^2 \partial^{(2)}_t)^2 \frac{f_{c,i}}{2} = f_{c,i} - \frac{n^+_{c,i}}{\tau^+} - \frac{n^-_{c,i}}{\tau^-}.$$

$$\frac{n^\pm_{c,i}}{\tau^\pm} = -c_{i,\alpha}\partial_\alpha e^\mp_i - \partial_t e^\pm_i + \Lambda^\mp (c_{i,\alpha}\partial_\alpha)^2 e^\pm_i\,, \tag{2}$$

- The solution is expressed as equilibrium and its derivative;
- Taking the moments $\Rightarrow$ Navier-Stokes (or Stokes) equations.
- τ^+ relates with viscosity, τ^- is free for athermal incompressible flows

Directional evolution equation at the boundary

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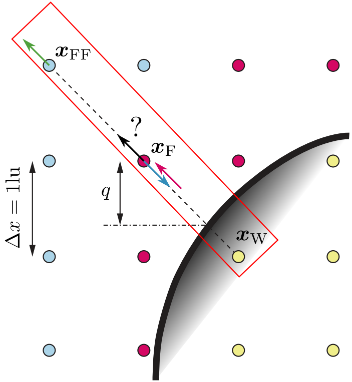
LBE: $f_i(\boldsymbol{x}_{\mathrm{FF}}, t+1) = \hat{f}_i(\boldsymbol{x}_{\mathrm{F}}, t+1)$
We consider compact or local scheme with max 3 populations (LI or ELI)
Advanced accuracy multireflection schemes need more populations

$$f_i(\boldsymbol{x}_{\mathrm{F}}, t+1) = \beta \hat{f}_{\bar{i}}(\boldsymbol{x}_{FF}) + \hat{\alpha}\hat{f}_{\bar{i}}(\boldsymbol{x}_F) + \hat{\beta}\hat{f}_i(\boldsymbol{x}_F) + \mathcal{E}(\boldsymbol{x}_W, \tilde{t}) + \mathcal{N}(\boldsymbol{x}_{\mathrm{F}}, t) \,.$$

$$\mathcal{E}(\boldsymbol{x}_W, \tilde{t}) = -\alpha_\epsilon^+ e_{\bar{i}}^+(\boldsymbol{x}_{\mathrm{F}}, t) - \alpha_\epsilon^- e_{\bar{i}}^-(\boldsymbol{x}_{\mathrm{F}}, t) - \alpha_W^- e_{\bar{i}}^-(\boldsymbol{x}_W, \tilde{t})$$

$$\mathcal{N}(\boldsymbol{x}_{\mathrm{F}}, t) = \hat{K}^+ \hat{n}_{\bar{i}}^+(\boldsymbol{x}_{\mathrm{F}}, t) + \hat{K}^- \hat{n}_{\bar{i}}^-(\boldsymbol{x}_{\mathrm{F}}, t)$$



🕮 LBM directional boundaries: bibliography

Ginzburg and Adler (1994); Ginzburg and d'Humières (2003); Ginzburg et al. (2008); Marson, Thorimbert, et al. (2021); Marson (2022); Ginzburg et al. (2022)

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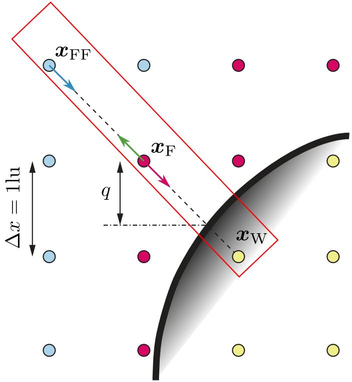

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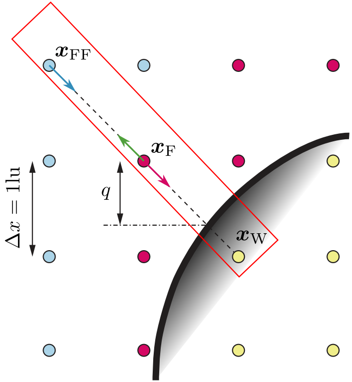

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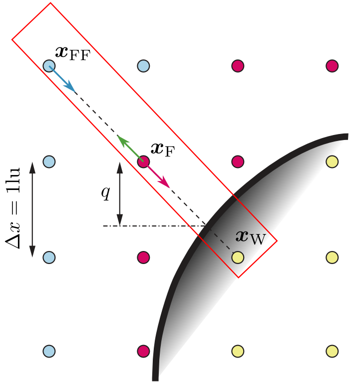

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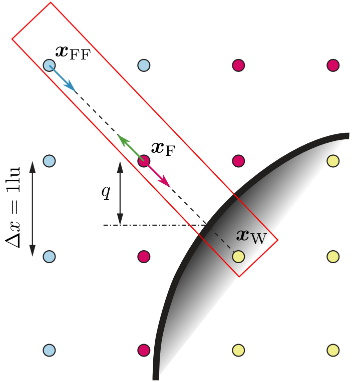

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Uniform formulation: closure

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1. Evolution equation at the boundary

$$\begin{cases} f_i = \beta \hat{f}_{\bar{i}}(x_{FF}) + \hat{\alpha}\hat{f}_{\bar{i}}(x_F) + \hat{\beta}\hat{f}_i(x_F) \\ \quad + \alpha_\epsilon^+ e_{\bar{i}}^+(x_F) \quad + \alpha_\epsilon^- e_{\bar{i}}^-(x_W) + \alpha_W^- e_{\bar{i}}^-(x_W) \\ \quad - \hat{K}_{\bar{i}}^+ \hat{n}_{\bar{i}}^+(x_F) \quad - \hat{K}_{\bar{i}}^- \hat{n}_{\bar{i}}^-(x_F) \end{cases}$$

$$\Longleftrightarrow$$

2. Inject Taylor + CE + '±' decomposition

- $f_i = e_i + n_i$ equilibrium/nonequilibrium
- $f_i = e_i^+ + n_i^+ + e_i^- + n_i^-$
- $n_i = \mathrm{function}(\sum_j \partial^j e_i)$ (Chapman-Enskog)
- Taylor expansion

Why closure description

The eq. representation ⇒ analyze errors introduced by the linear combination.

3. Closure [a]

$$\begin{aligned} \Big[\quad & \alpha^+ e_{\bar{i}}^+ && + \alpha^- e_{\bar{i}}^- \\ & + \beta^+ c_{\bar{i},\alpha}\partial_\alpha e_{\bar{i}}^+ && + \beta^- c_{\bar{i},\alpha}\partial_\alpha e_{\bar{i}}^- \\ & + \gamma^+ (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^+ && + \gamma^- (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^- \Big]_{x_F}^t \\ & = \alpha_\epsilon^+ e_{\bar{i}}^+(x_F) && + \alpha_\epsilon^- e_{\bar{i}}^-(x_F) + \alpha_W^- e_{\bar{i}}^-(x_W) \\ & - \hat{K}_{\bar{i}}^+ \hat{n}_{\bar{i}}^+(x_F) && - \hat{K}_{\bar{i}}^- \hat{n}_{\bar{i}}^-(x_F) \end{aligned}$$

linear vel. + linear press. + parabolic vel. & press. + closure corrections + ghost populations

closure coefficients are linear combinations of the "geometrical" ones.
NB α^- are free prefactors: $\alpha^- \left[f_i^-\right]_{x_F} =$

$$\alpha^- \left[f_i^- + q c_{i,\alpha}\partial_\alpha f_i^- + \frac{q^2}{2}(c_{i,\alpha}\partial_\alpha)^2 f_i^- + \dots \right]_{x_W}$$

[a] Extended starting from (Ginzburg & d'Humières, 2003)

Uniform formulation: closure

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$$\begin{bmatrix} \alpha^+ e_{\bar{i}}^+ & +\alpha^- e_{\bar{i}}^- \\ +\beta^+ c_{\bar{i},\alpha}\partial_\alpha e_{\bar{i}}^+ & +\beta^- c_{\bar{i},\alpha}\partial_\alpha e_{\bar{i}}^- \\ +\gamma^+ (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^+ & +\gamma^- (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^- \end{bmatrix}_{\boldsymbol{x}_F}^t$$

$$= \alpha_\epsilon^+ e_{\bar{i}}^+(\boldsymbol{x}_F) \quad +\alpha_\epsilon^- e_{\bar{i}}^-(\boldsymbol{x}_F) + \alpha_W^- e_{\bar{i}}^-(\boldsymbol{x}_W)$$
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linear vel. + linear press. + parabolic vel. & press. + closure corrections + ghost populations

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[a] Extended starting from (Ginzburg & d'Humières, 2003)

Uniform formulation: closure

UNIVERSITÉ DE GENÈVE

1. Evolution equation at the boundary

$$\begin{cases} f_i = \beta \hat{f}_{\bar{i}}(\boldsymbol{x}_{FF}) + \hat{\alpha} \hat{f}_{\bar{i}}(\boldsymbol{x}_F) + \hat{\beta} \hat{f}_i(\boldsymbol{x}_F) \\ \quad + \alpha_\epsilon^+ e_{\bar{i}}^+(\boldsymbol{x}_F) \quad + \alpha_\epsilon^- e_{\bar{i}}^-(\boldsymbol{x}_W) + \alpha_W^- e_{\bar{i}}^-(\boldsymbol{x}_W) \\ \quad - \hat{K}_{\bar{i}}^+ \hat{n}_{\bar{i}}^+(\boldsymbol{x}_F) \quad - \hat{K}_{\bar{i}}^- \hat{n}_{\bar{i}}^-(\boldsymbol{x}_F) \end{cases} \iff$$

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Why closure description

The eq. representation ⇒ analyze errors introduced by the linear combination.

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$$\begin{aligned} \Big[\quad & \alpha^+ e_{\bar{i}}^+ & & +\alpha^- e_{\bar{i}}^- \\ & + \beta^+ c_{\bar{i},\alpha} \partial_\alpha e_{\bar{i}}^+ & & + \beta^- c_{\bar{i},\alpha} \partial_\alpha e_{\bar{i}}^- \\ & + \gamma^+ (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^+ & & + \gamma^- (c_{\bar{i},\alpha}\partial_\alpha)^2 e_{\bar{i}}^- \Big]_{\boldsymbol{x}_F}^t \\ & = \alpha_\epsilon^+ e_{\bar{i}}^+(\boldsymbol{x}_F) & & + \alpha_\epsilon^- e_{\bar{i}}^-(\boldsymbol{x}_F) + \alpha_W^- e_{\bar{i}}^-(\boldsymbol{x}_W) \\ & - \hat{K}_{\bar{i}}^+ \hat{n}_{\bar{i}}^+(\boldsymbol{x}_F) & & - \hat{K}_{\bar{i}}^- \hat{n}_{\bar{i}}^-(\boldsymbol{x}_F) \end{aligned}$$

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Elementary model for narrow gaps: Stokes channel flows

UNIVERSITÉ DE GENÈVE

Meaning of the closure

- Moments of the CE expanded LBE = Navier-Stokes

$\Rightarrow$ We want that this to hold at boundaries

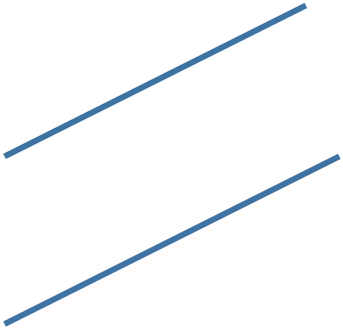
$\Rightarrow$ Taylor-CE-LBE $\rightarrow$ boundary $\Rightarrow$ closure

Analysis of the closure with c/p-flows

- Identify and cancel errors introduced by the boundary
- Errors appear sequentially for increasingly complex flows
- Define elementary c/p-flows to test errors purge
  $\Rightarrow$ cleaning the right error the elementary flow become exact

$\Rightarrow$ accuracy in porous media (Khirevich et al., 2015)

The boundary model is directional $\Rightarrow$ it is valid for any channel inclination and any complex flow.



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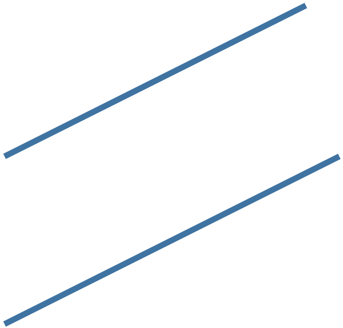

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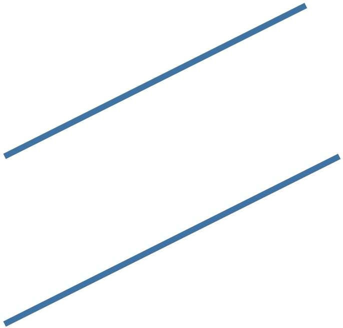

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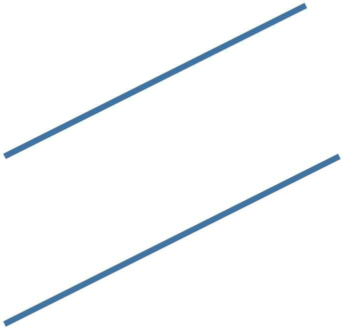

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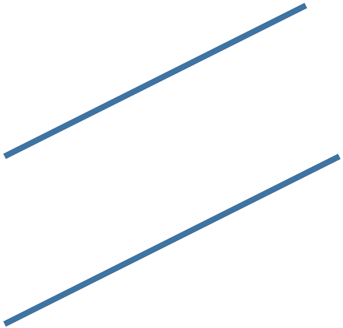

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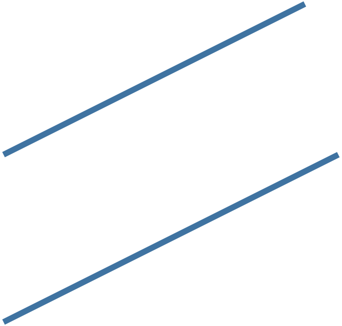

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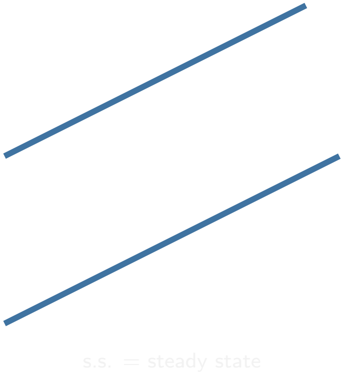

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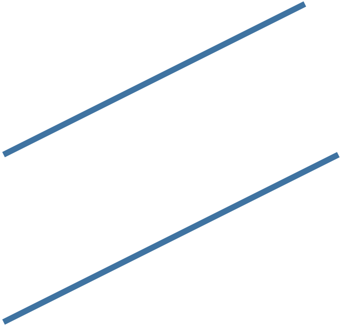

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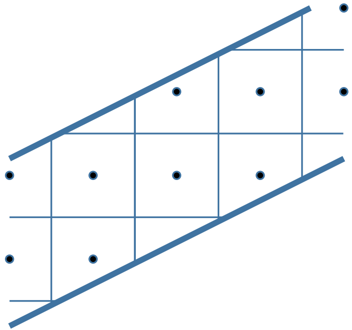

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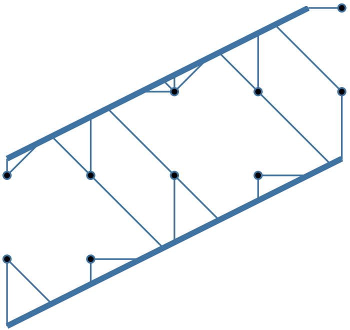

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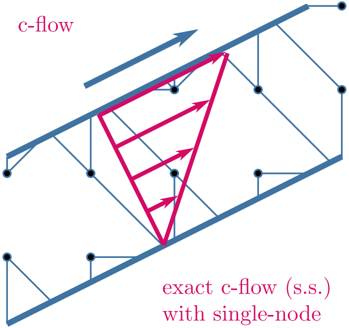

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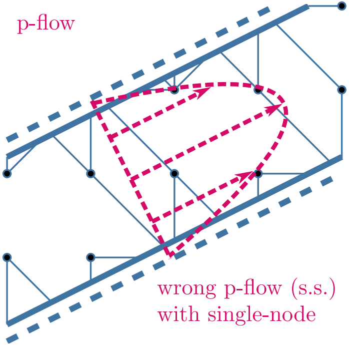

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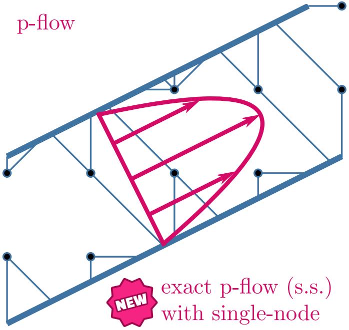

s.s. = steady state

How to cancel errors locally

Linear case:

$$\alpha^+ = -1 + \hat{\alpha} + \beta + \hat{\beta}$$
$$\alpha^- = 1 + \hat{\alpha} + \beta - \hat{\beta}$$
$$\beta^+ = \frac{1}{2}(\hat{\alpha} - \hat{\beta} - \beta - 1) - \alpha^-\Lambda^-$$
$$\gamma^- = -\beta^+\Lambda^+ - \beta$$

$$\alpha^+_{\mathrm{tot}} = \alpha^+ - \alpha^+_\epsilon$$
$$\alpha^-_{\mathrm{tot}} = \alpha^- - \alpha^-_\epsilon - \alpha^-_W$$
$$\beta^+_{\mathrm{tot}} = \beta^+ + \hat{K}^-$$
$$\gamma^-_{\mathrm{tot}} = \underbrace{\gamma^- - \Lambda^+\hat{K}^-}_{\gamma^-_K} - \frac{q^2}{2}\alpha^-_W$$

coefficients (errors) depends on $\Lambda^\pm$!

Standard ($\hat{K}^-_1$) correction[a] $\Rightarrow$ ELI $=$ LI at s.s.

K^-_1 such that $\gamma^-_K \stackrel{\mathrm{def}}{=} \gamma^- - \Lambda^+\hat{K}^- = \alpha^-\Lambda$

New correction $\hat{K}^-_3$ [b] $\Rightarrow$ ELI $=$ LI at s.s.

- K^-_3 such that $\gamma^-_K \stackrel{\mathrm{def}}{=} \gamma^- - \Lambda^+\hat{K}^- = \alpha^-_W \frac{q^2}{2}$
- ` parabolic velocity in Stokes flow!

New correction $\hat{K}^-_4$ [c] $\Rightarrow$ ELI $=$ LI at s.s.

K^-_4 such that $\beta^+_{\mathrm{tot}} = 0$
ϟ linear pressure in Stokes flow!
Resulting single-node schemes are parametrized!

---

[a] 🕮 (Ginzburg et al., 2008)
[b] 🕮 (Marson, 2022; Ginzburg et al., 2022)
[c] 🕮 (Marson, 2022; Ginzburg et al., 2022)

How to cancel errors locally

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Linear case:

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$$\gamma^-_{\mathrm{tot}} = \underbrace{\gamma^- - \Lambda^+\hat K^-}_{\gamma^-_K} - \frac{q^2}{2}\alpha^-_W$$

coefficients (errors) depends on $\Lambda^\pm$!

Standard ($\hat K_1^-$) correction[a] $\Rightarrow$ ELI $=$ LI at s.s.

K_1^- such that $\gamma_K^- \stackrel{\mathrm{def}}{=} \gamma^- - \Lambda^+\hat K^- = \alpha^-\Lambda$

New correction $\hat K_3^-$ [b] $\Rightarrow$ ELI $=$ LI at s.s.

- K_3^- such that $\gamma_K^- \stackrel{\mathrm{def}}{=} \gamma^- - \Lambda^+\hat K^- = \alpha_W^- \frac{q^2}{2}$
- ` parabolic velocity in Stokes flow!

New correction $\hat K_4^-$ [c] $\Rightarrow$ ELI $=$ LI at s.s.

K_4^- such that $\beta^+_{\mathrm{tot}} = 0$

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How to cancel errors locally

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[c] 🕮 (Marson, 2022; Ginzburg et al., 2022)

New schemes: Stokes flow classification

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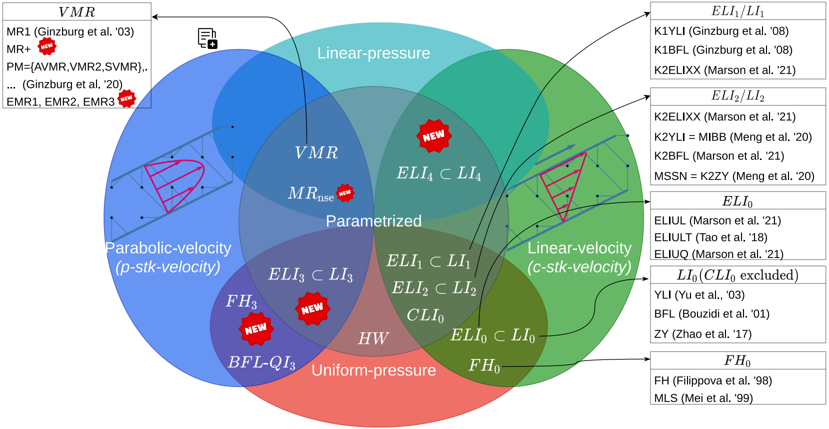


(Filippova & Hänel, 1997; Mei et al., 1999; Yu et al., 2003; Ginzburg & d'Humières, 2003; Ginzburg et al., 2008; Marson, Thorimbert, et al., 2021; Zhao & Yong, 2017; Tao et al., 2018; Ginzburg, 2020; Meng et al., 2020; Bouzidi et al., 2001; Marson, Silva, et al., 2021)

Inclined channels

Scheme or Family	Nodes	*c-stk-flow*	*p-stk-flow*	param.	*c-nse-flow*	*p-nse-flow*
$\mathrm{MR}_{\mathrm{nse}}$	$3 \sim 4$	✓	✓	✓	✓	✓
the parametrized and *p-stk-flow* schemes						
PM={AVMR, EMR}	$2 \sim 3$	✓	✓	✓	✓	×
MR={MR1, $\mathrm{MR1}^{+}$}	$2 \sim 3$	✓	✓	✓	×	×
{IPLI, LI_3}	$1 \sim 2$	✓	✓	✓	×	×
{CELI-IP, ELI_3}	$\boxed{1}$	$\boxed{✓}$	$\boxed{✓}$	$\boxed{✓}$	×	×
the parametrized schemes						
LI_1	$1 \sim 2$	✓	×	✓	×	×
ELI_1	1	✓	×	✓	×	×
LI_4	$1 \sim 2$	✓	×	✓	×	×
ELI_4	1	✓	×	✓	×	×
HW	1	×	×	✓	×	×
non-parametrized schemes						
linear-interpolation based:						
LI_0^{+}	$1 \sim 2$	✓	×	×	×	×
ELI_0	1	✓	×	×	×	×
quadratic-interpolation based:						
$\mathrm{BFL\text{-}QI}_3$	$2 \sim 3$	✓	✓	×	×	×
BFL-QI	$2 \sim 3$	✓	×	×	×	×
equilibrium-interpolation based:						
FH_3	1	✓	✓	×	×	×
FH_0	1	✓	×	×	×	×
MLS_3	2	✓	✓	×	×	×
MLS_0	2	✓	×	×	×	×

Stability optimization

In the bulk

1. collision, with stability condition:

$$\left|1 - 1/\tau^{+}\right| \leq 1$$

2. streaming.

In the boundary

1. modified collision, with stability condition:

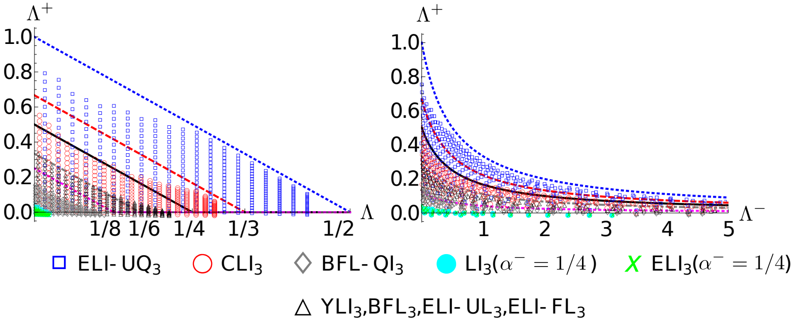
$$\left|1 \mp \beta^{\mp}/\tau^{\pm}\right| \leq 1$$

2. modified streaming.

- accuracy for the steady state solution is the same for all α^{-} (parametrized schemes);
- $\beta^{-} \Rightarrow \alpha^{-}$ controls the stability proprieties of the schemes!
- An optimal stability value for $\alpha^{-}(\Lambda^{\pm})$ exists
- coefficients $\in [-1, 1]$: almost necessary condition, not sufficient with K^{-}

Stability: results

For $\hat{K}_3$ schemes, the steady state accuracy in p-flow is the same, but stability is different for schemes with different α^-



α^- controls the stability proprieties of the schemes!

Stokes flow: circular Couette

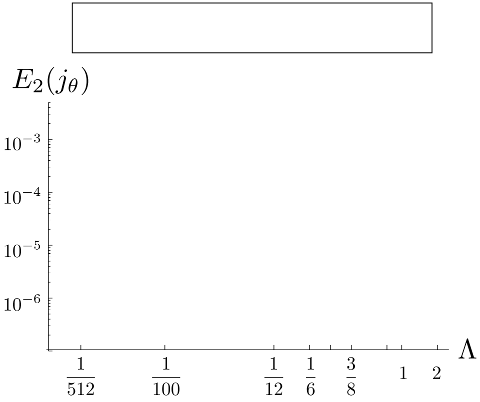
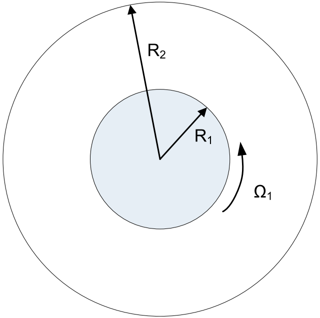


Figure: $R_2 - R_1 = 10\,\mathrm{lu}$



Stokes flow: circular Couette

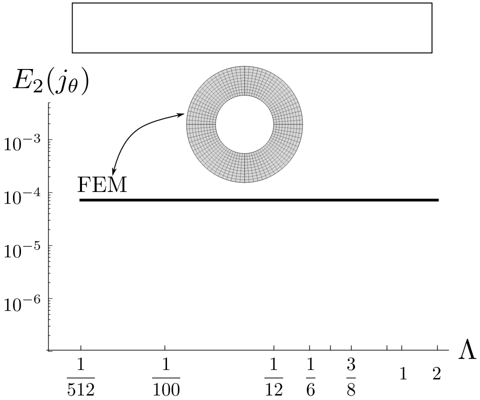
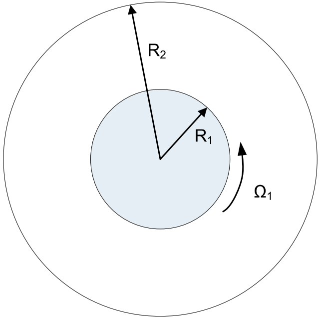


Figure: $R_2 - R_1 = 10\,\mathrm{lu}$



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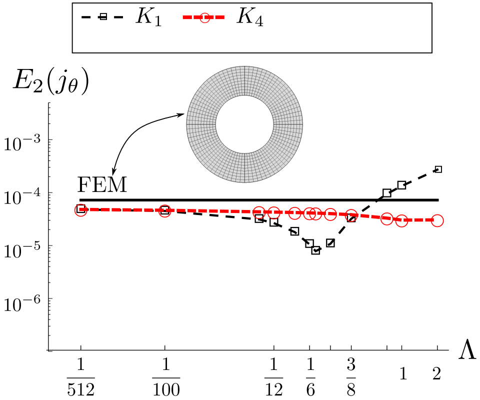

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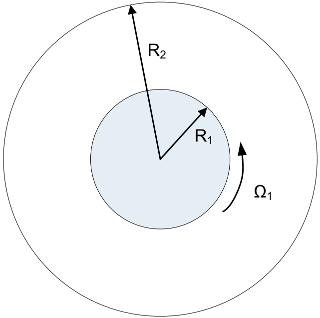

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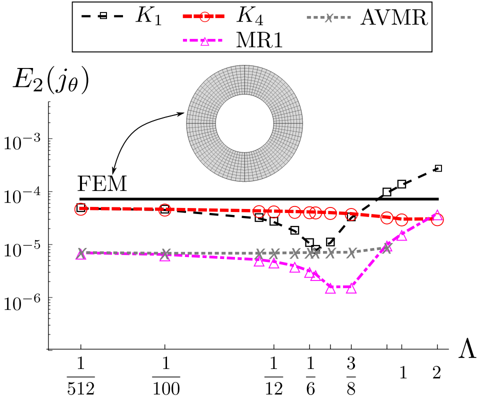
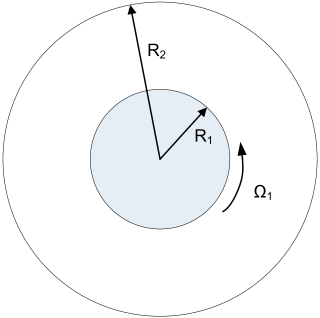


Figure: $R_2 - R_1 = 10\,\mathrm{lu}$



Stokes flow: circular Couette

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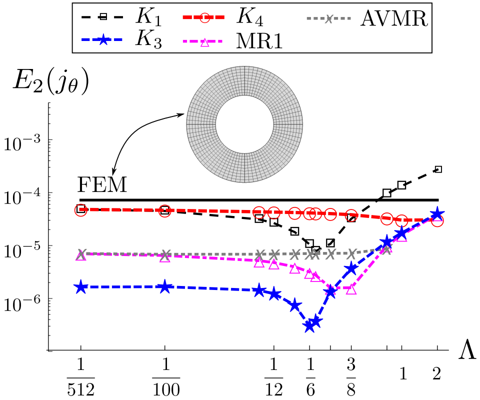

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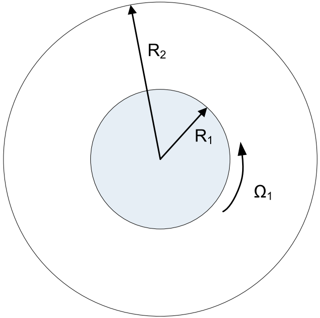

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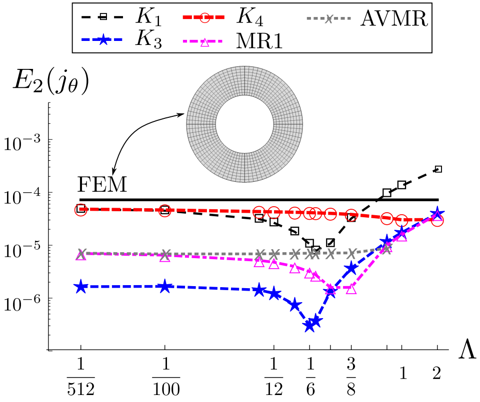

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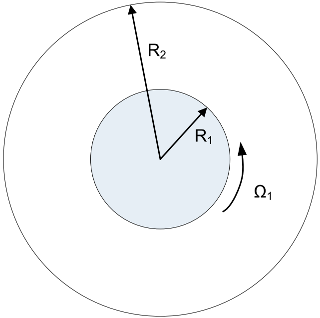


Stokes flow: results in arrays

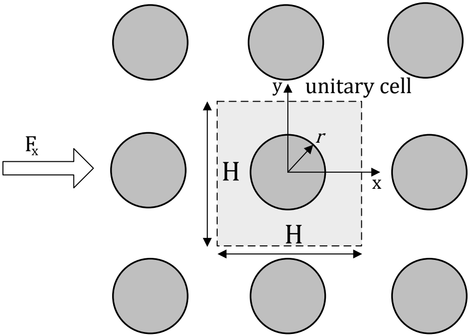


Figure: Schematic representation of the simulation domain for the array of cylinders configuration (Silva, 2018).

Stokes flow: results in arrays

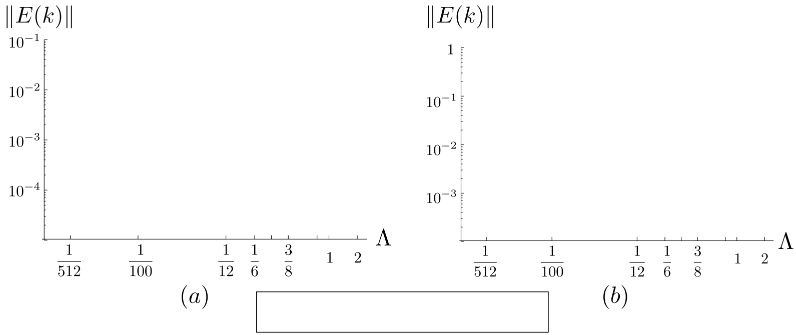
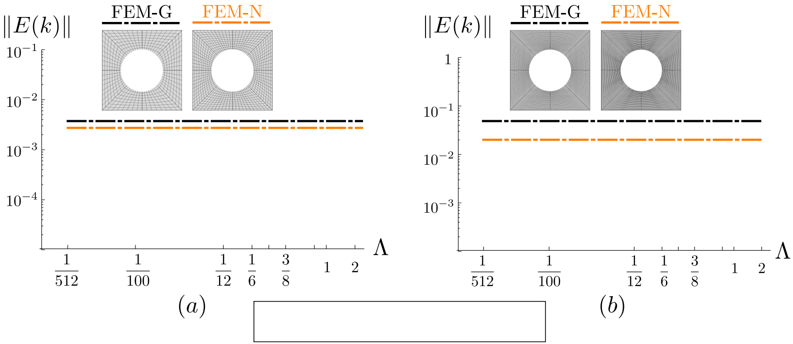


Figure: Permeability estimation error in an array of cylinders using darcy law. (a) and (b): dilute flow with solid fraction $c = 0.2$. (a) cell resolution $H^2 = 33^2\,\mathrm{lu}^2$. (b) cell resolution $H^2 = 99^2\,\mathrm{lu}^2$.

📖 (Ginzburg et al., 2022; Marson, 2022)

Stokes flow: results in arrays



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Stokes flow: results in arrays

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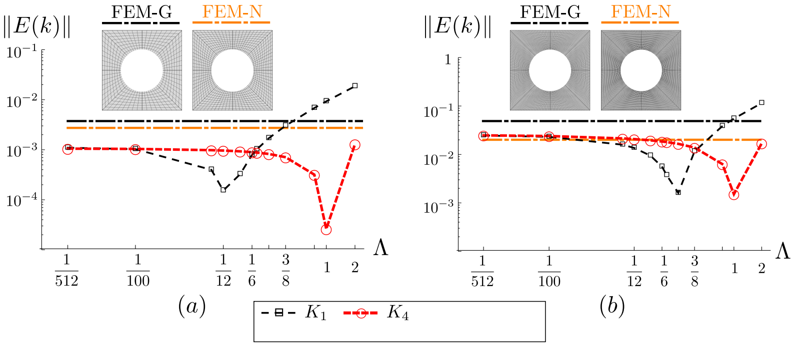

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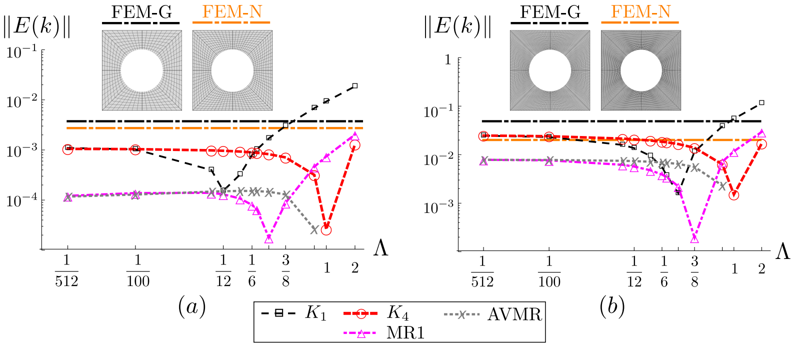
Stokes flow: results in arrays



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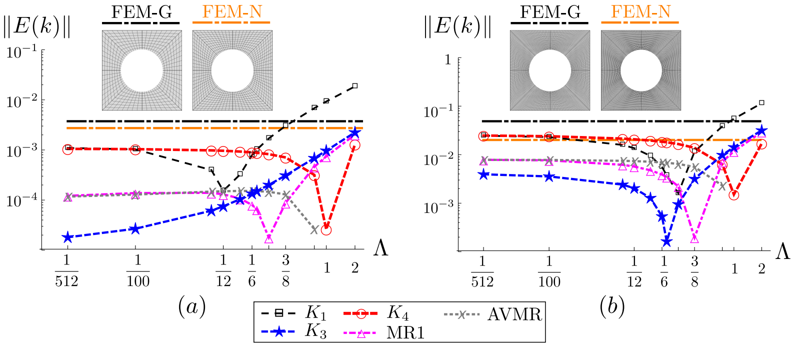
Stokes flow: results in arrays



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(Ginzburg et al., 2022; Marson, 2022)

Finite Re

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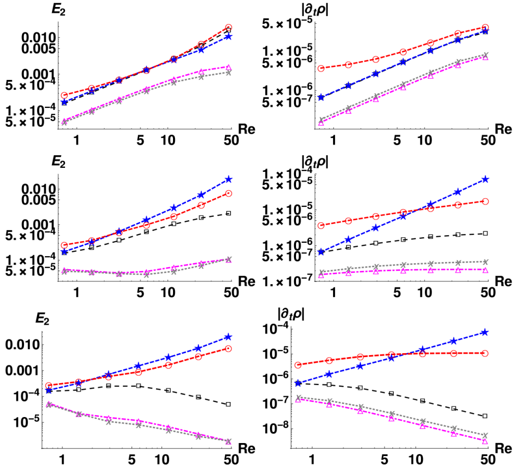
Error as a function of grid Reynolds $\mathrm{Re} = u_{lb}/\nu_{lb}$



- LBE with standard NSE equilibrium → inexact solutions for c/p-nse-flows
- higher order errors appear in the closure
- results are non-parametrized
- find optimal scaling for $\Lambda = \Lambda^+\Lambda^-$

Sandstone porous medium with Palabos on GPUs

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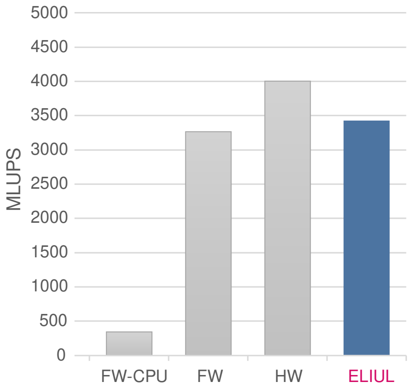

Sandstone porous medium

MLUPS = Million Lattice Updates Per Second

Thank you! Questions?

THANK YOU FOR YOUR ATTENTION! QUESTIONS?

Acknowledgments

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PALABOS

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