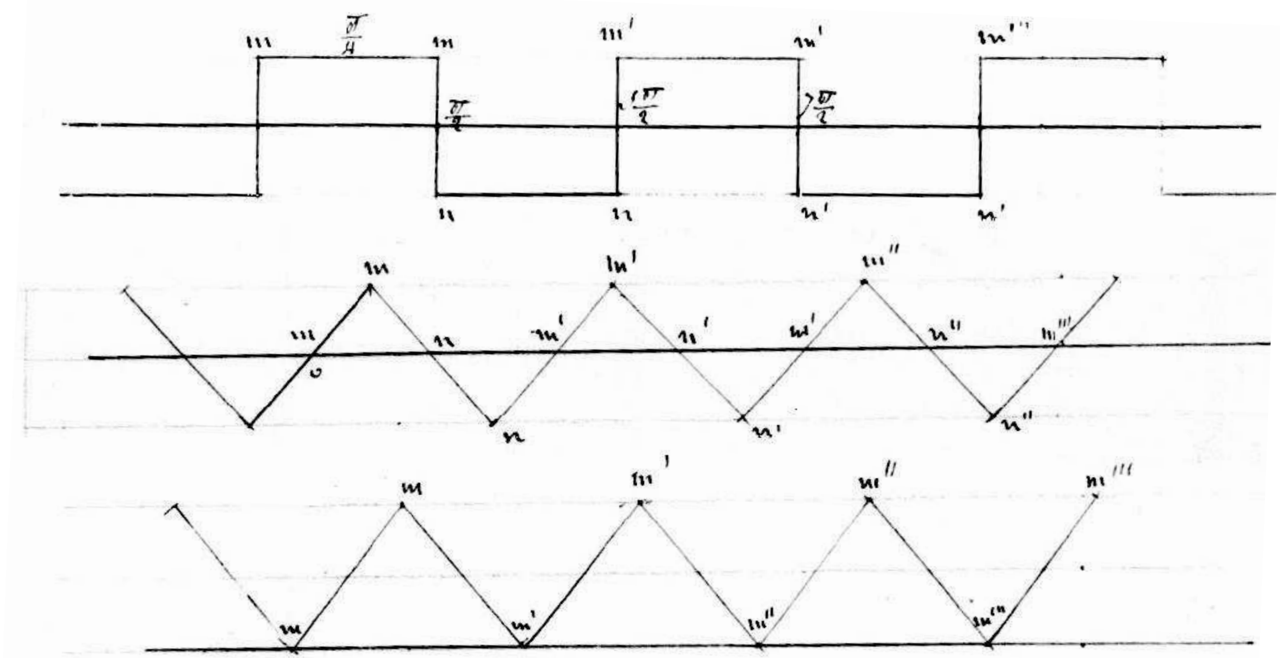
Origin of Fourier Series I

Ph. Henry & G. Wanner, Evian 2022; (Vide MATAPLI 119, Juin 2019)

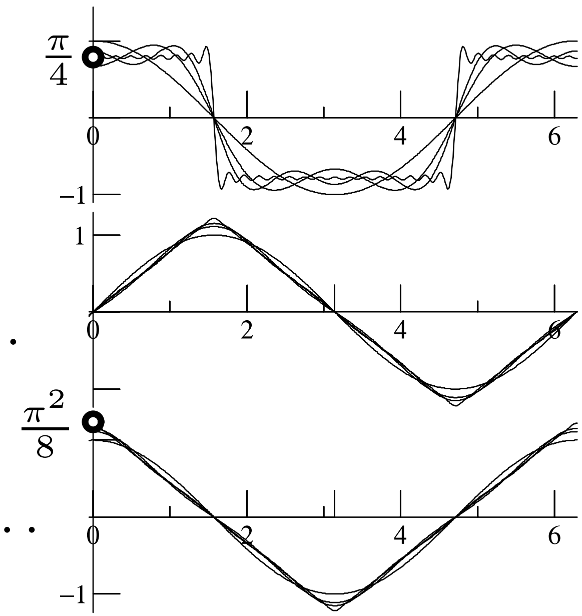


(Fourier, draft from 1805, BNF, Ms. Fr. 22525, fol. 107v)

$$\cos(x) - \tfrac{1}{3}\cos(3x) + \tfrac{1}{5}\cos(5x) \mp \ldots$$

$$\sin(x) - \tfrac{1}{3^2}\sin(3x) + \tfrac{1}{5^2}\sin(5x) \mp \ldots$$

$$\cos(x) + \tfrac{1}{3^2}\cos(3x) + \tfrac{1}{5^2}\cos(5x) + \ldots$$

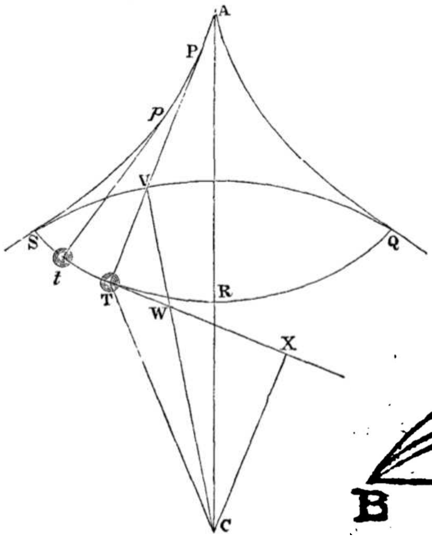


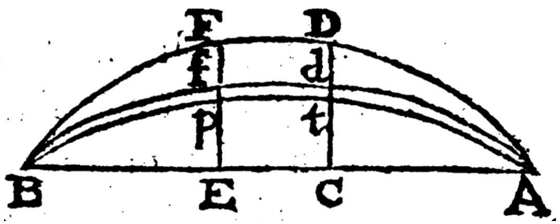
A century earlier ...

Brook Taylor 1713/15

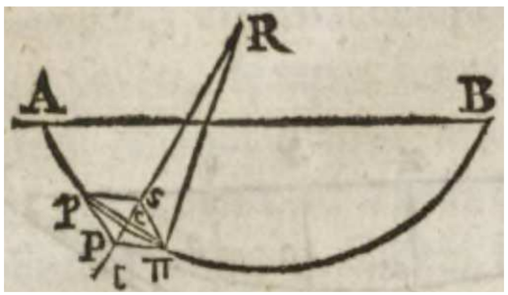
“De motu Nervi tensi”

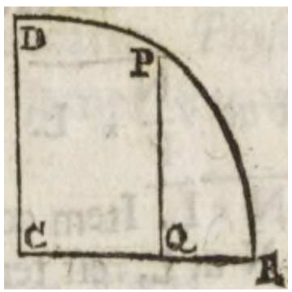
(Principia, Prop.LI,LII)





“curvatura in quovis puncto E
est ut ejusdem distantia ab axe E”





“$CQ = x$ ... arcu DP existente y”

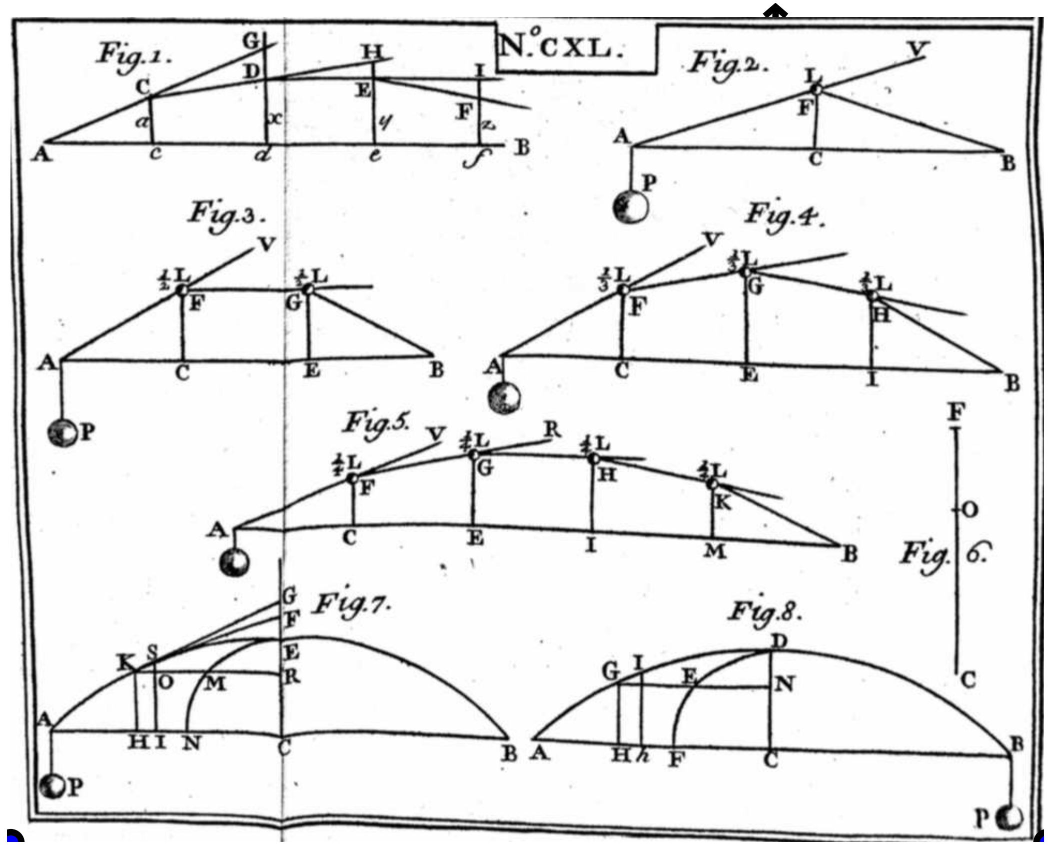
“motus similis erit oscillationi
corporis Funipenduli in Cycloide.”

Johann Bernoulli 1728

“De Chordis Vibrantibus”

“quemadmodum invenit TAYLORUS. Vid. *Meth. Increm.* p. 93”

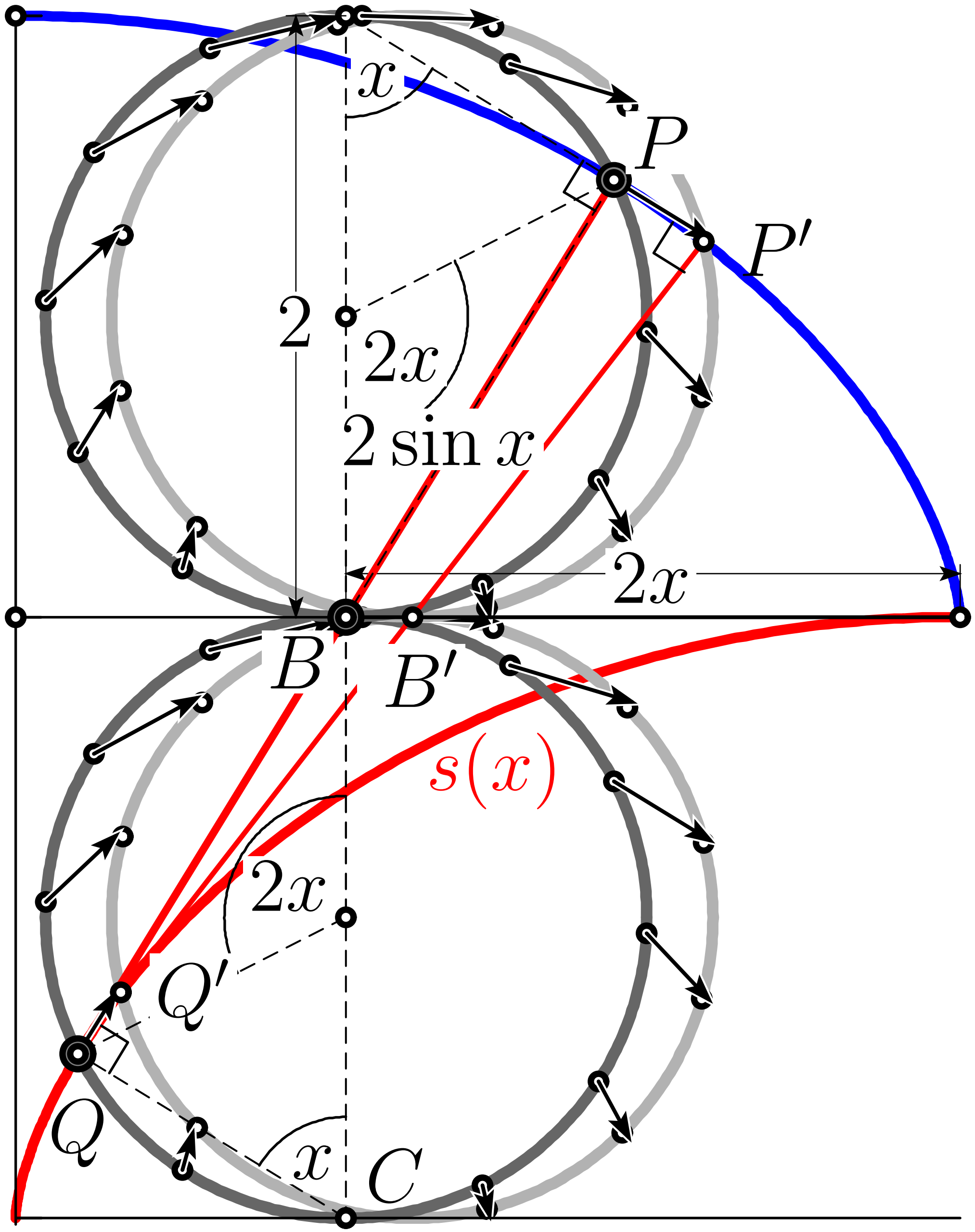
$$= 2x - a - y : x = 2y - x - z : y = 2z - y - t$$

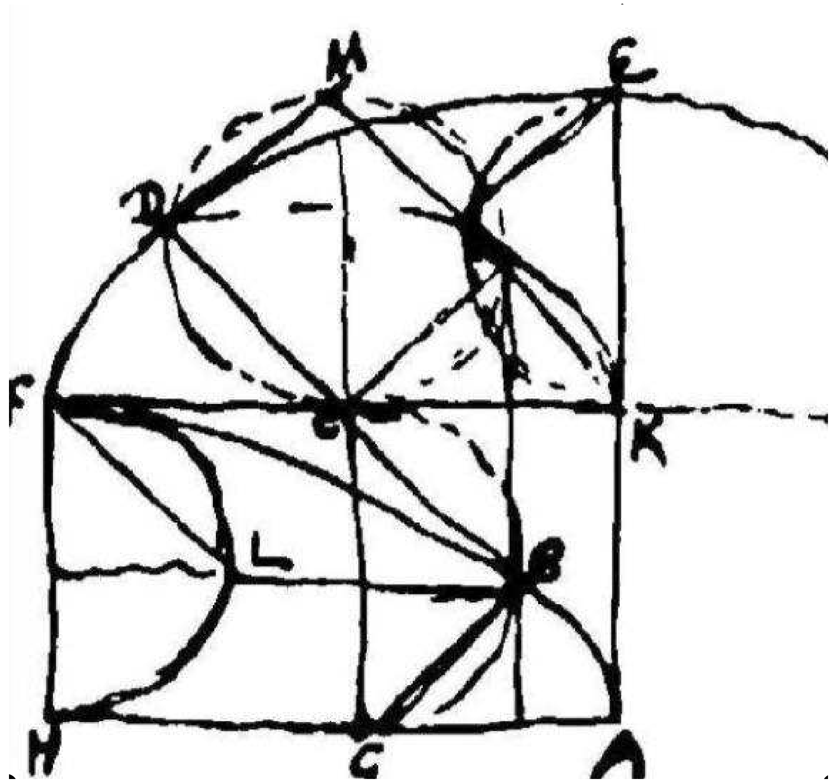


“applicata NG ad arcum DE in ratione const.”

“induere figuram sociæ Trochoidis elongatæ”

Excursion (Huygens 1673)

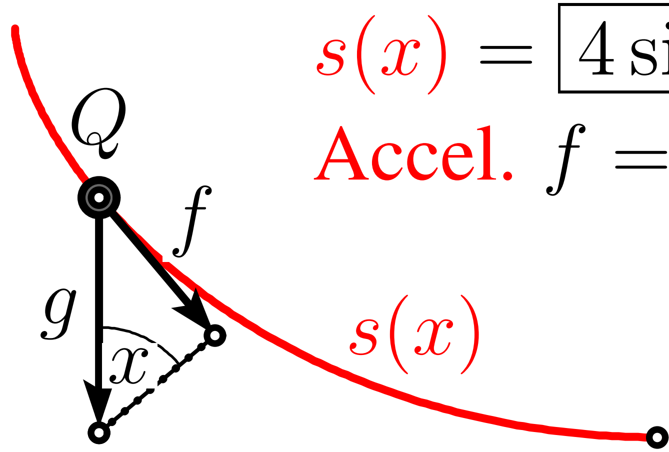




(Huygens 1659)

$s(x) = \boxed{4 \sin x}$

Accel. $f = \boxed{g \sin x}$



Isochronous pendulum.

A century earlier ...

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"De motu Nervi tensi"

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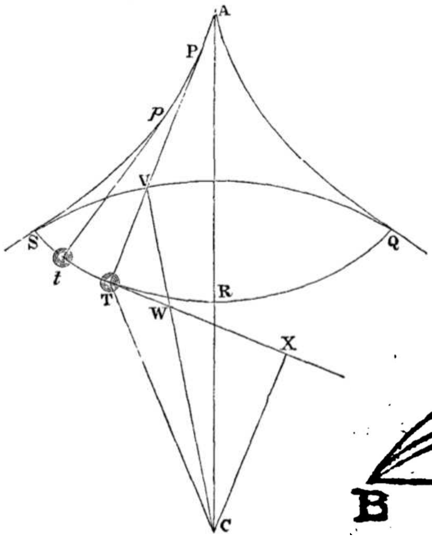

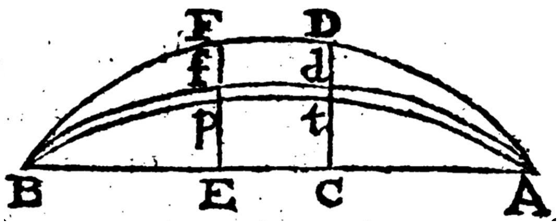

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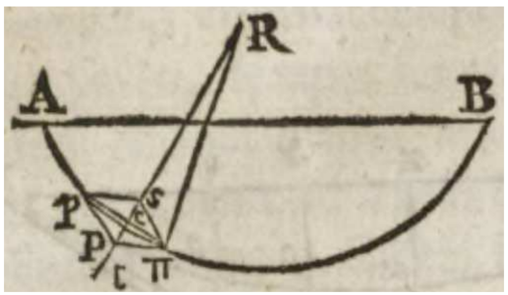

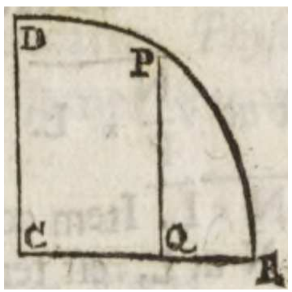

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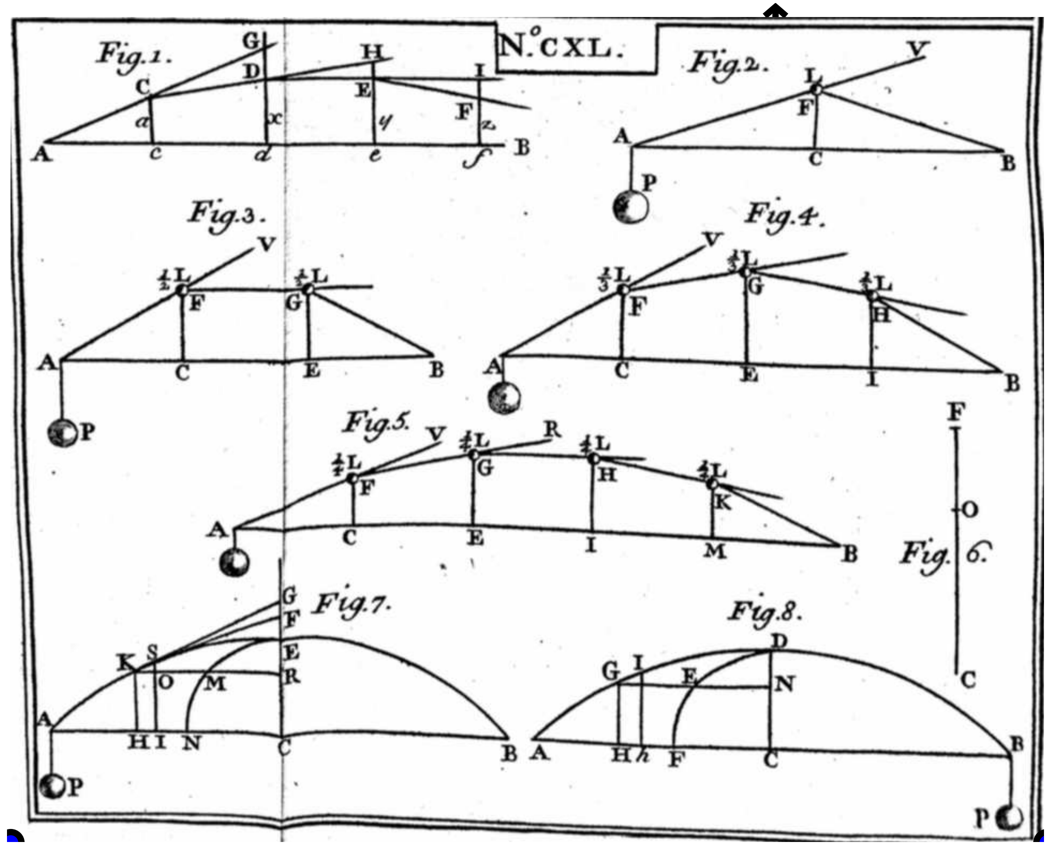


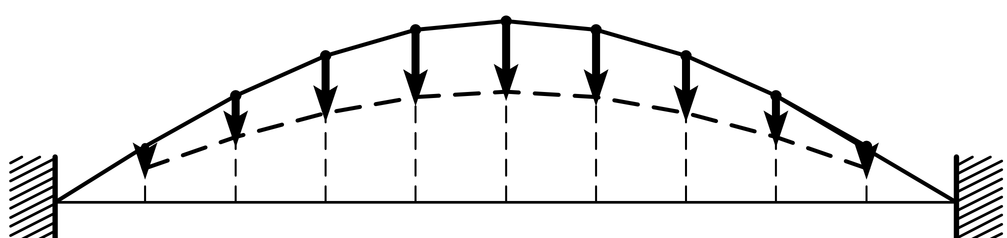
"applicata NG ad arcum DE in ratione const."

"induere figuram sociæ Trochoidis elongatæ"

L. Euler, 2 oct. 1746

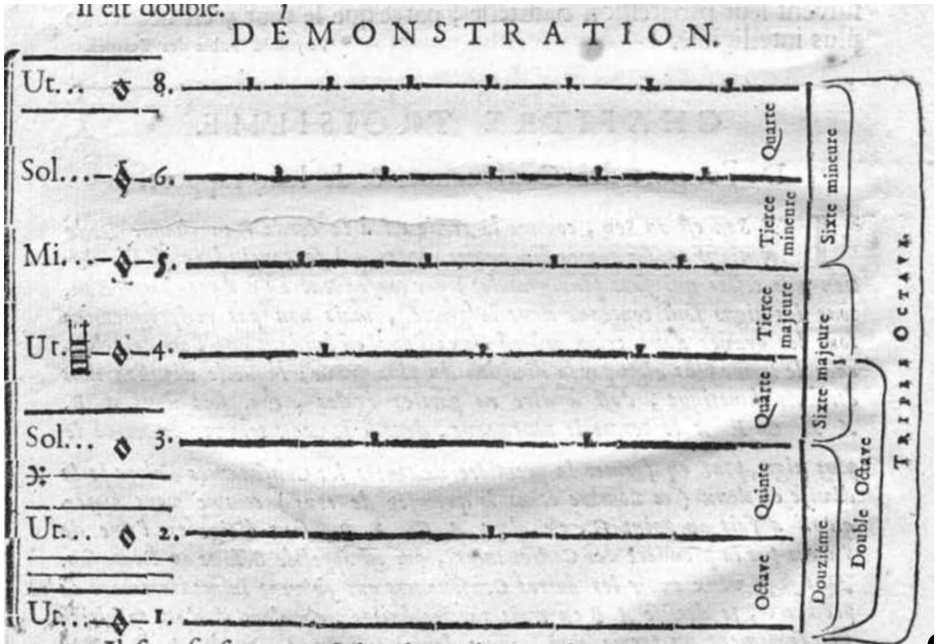
(Letter to D'Alembert; Vanja Hug, Th. Steiner 2014)

"... la trochoide allongée, ou bien la **lineam sinuum** de Leibniz"



J. Bürgi 1584, "Prosthaphaeresin"

Jean-Philippe Rameau 1722: Harmonics:



DÉMONSTRATION.

Ut... 8.

Sol... 6.

Mi... 5.

Ut... 4.

Sol... 3.

Ut... 2.

Ut... 1.

Quarte

Tierce mineure

Tierce majeure

Quarte

Quinte

Octave

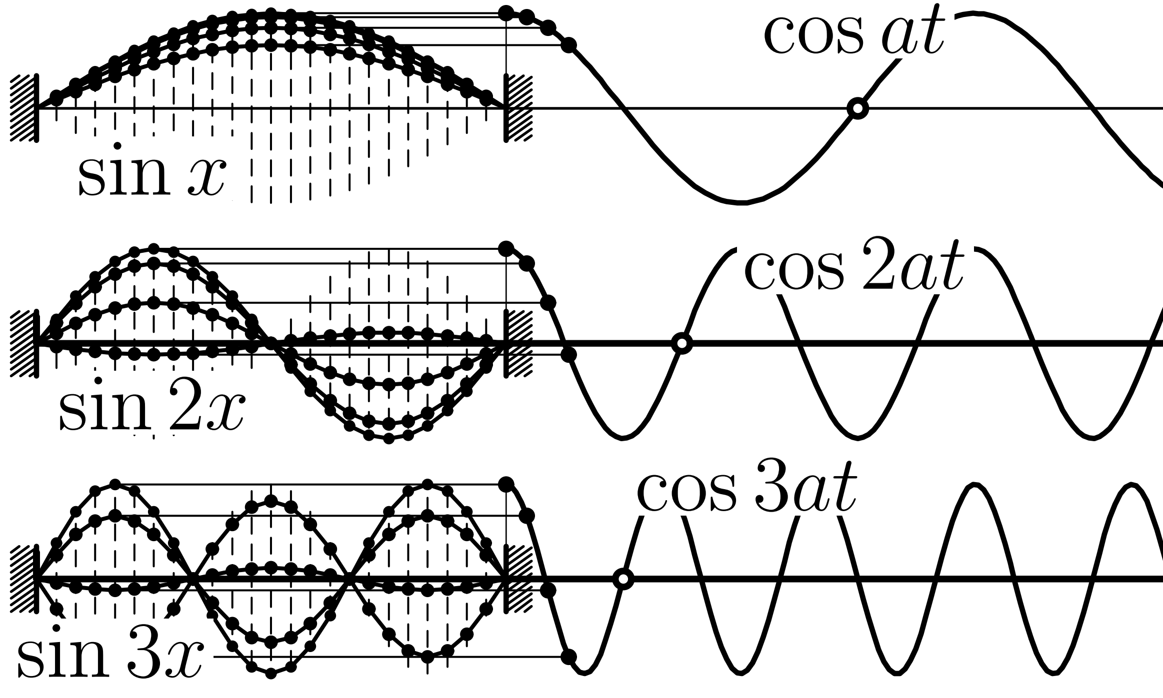
Sixte mineure

Sixte majeure

Douzième

Double Octave

Triple Octave



L. Euler, 2 oct. 1746:

$$y = \alpha \sin\frac{\pi x}{c}\cos\frac{\pi u}{c} + \beta\sin\frac{2\pi x}{c}\cos\frac{2\pi u}{c} + \gamma\sin\frac{3\pi x}{c}\cos\frac{3\pi u}{c} + \delta\sin\frac{4\pi x}{c}\cos\frac{4\pi u}{c} + \text{etc.}$$

$$u = 0 : f(x) = \alpha\sin\frac{\pi x}{c} + \beta\sin\frac{2\pi x}{c} + \gamma\sin\frac{3\pi x}{c} + \delta\sin\frac{4\pi x}{c} + \text{etc.} \Rightarrow \text{ find } \alpha, \beta, \gamma, \delta \ldots$$

J. D'Alembert, 1747

"Preprint" June 1746, pres. 1747, publ. 1749

$$\boxed{\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}}$$

"... il y a une infinité d'autres courbes que la *Compagne de la Cycloide allongée*,..."

$$\boxed{\text{donc } \alpha = \frac{2aF\beta}{p\vartheta^2} = \frac{2ap\dot{m}l\beta}{p\vartheta^2} = \dot{\beta} \cdot \frac{2\dot{a}ml}{\vartheta^2},}$$

"l'equation generale de la courbe" $y = f(x+at) + \varphi(x-at)$

f, φ "des fonctions encore inconnuës".

"ne prenons ici qu'un cas particulier"

$$y = \frac{f(x+at) - f(x-at)}{2}$$

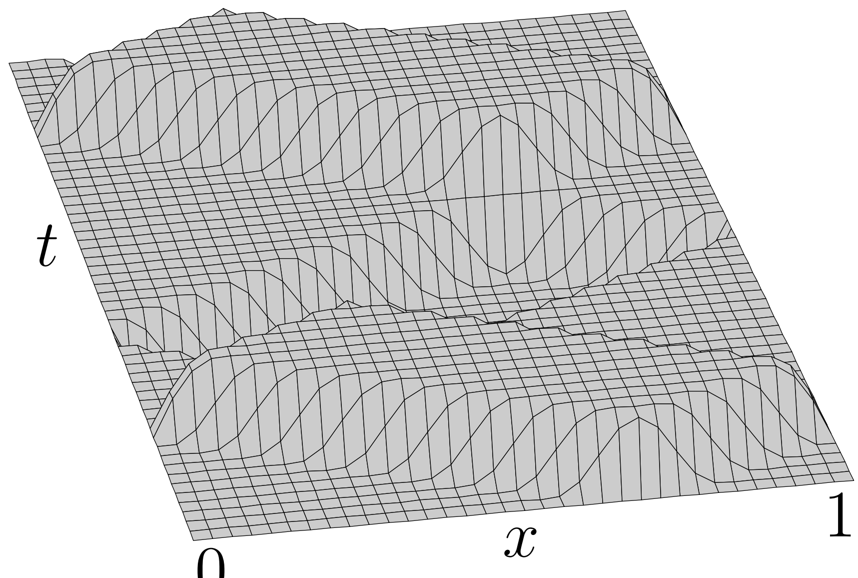
"quand $t = 0$ ...soit etenduë en ligne droite"

$$y|_{t=0} = 0, \quad \frac{\partial y}{\partial t}|_{t=0} = af'(x)$$

"exprime en general la vitesse du point M"

followed by 1 1/2 pages of French text for satisfying

$$y(0) = y(\ell) = 0 \text{ for all } t.$$

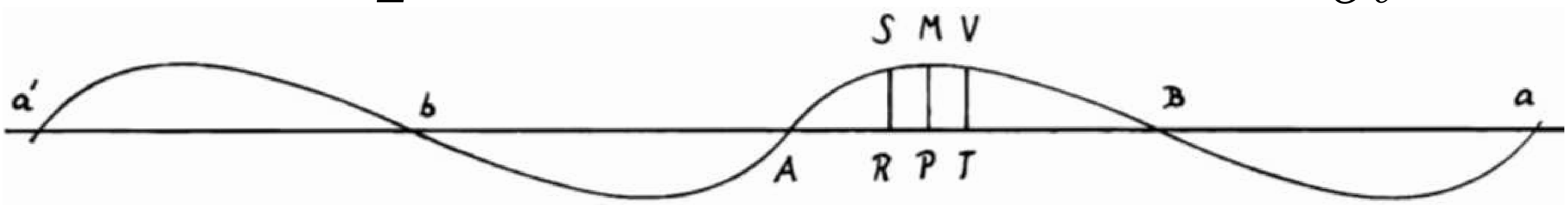


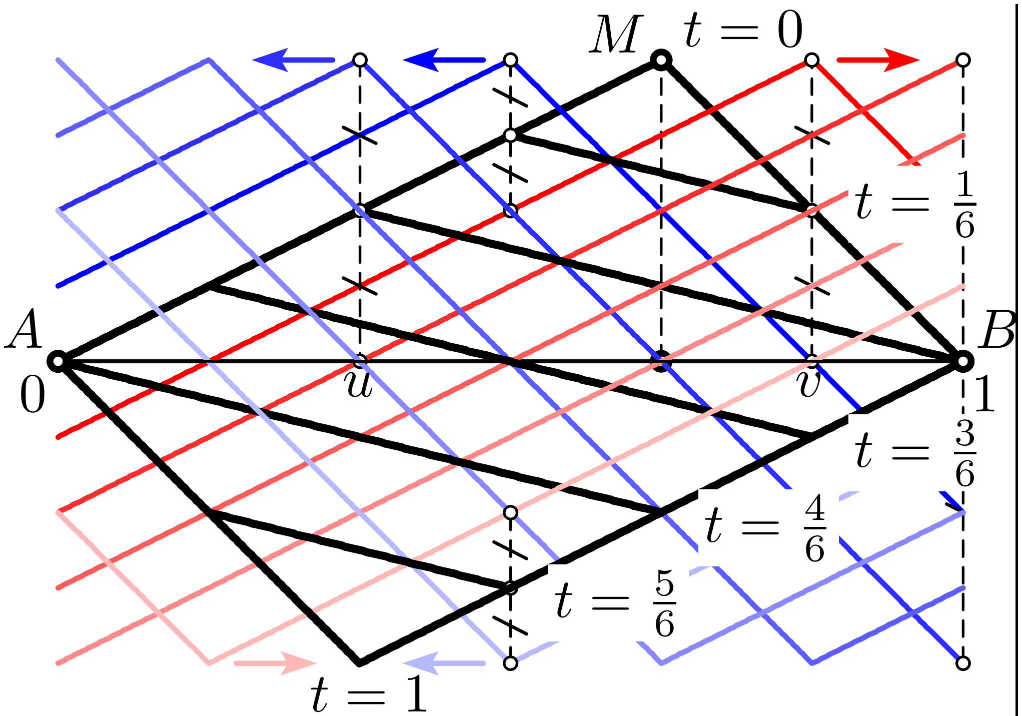
(string of a piano)

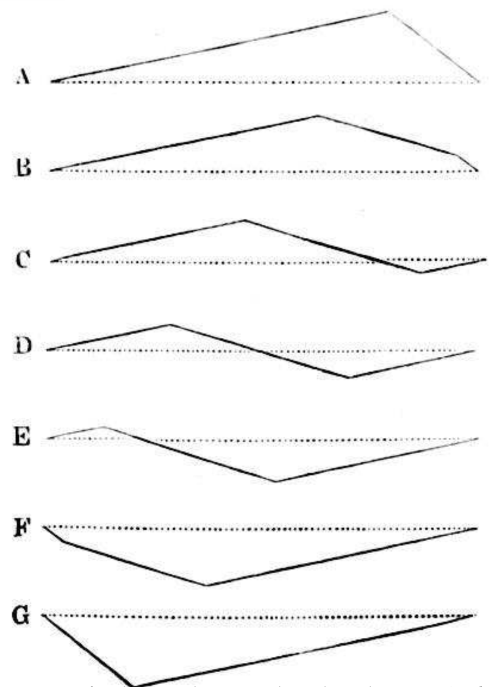
"Eine genaue Analyse der Bewegung der Saite nach dem Anschlag eines Klavierhammers würde ziemlich verwickelt sein." (H.v.Helmholtz, 1862)

Euler's Letter (2 oct 1746) "dont j'ai été tout à fait charmé"...
"Votre solution est aussi parfaite qu'ingenieuse,"...
"Quoique Vous n'en fassies aucune application à des cas part."
"...j'ai remarqué qu'on en peut determiner tres aisement..."
"...donné une figure quelconque en la relachant subitement..."

$$y = \frac{f(x+at)+f(x-at)}{2}, \quad y|_{t=0} = f(x), \quad \frac{\partial y}{\partial t}|_{t=0} = 0$$







(Harpsichord, Helmholtz 1862)

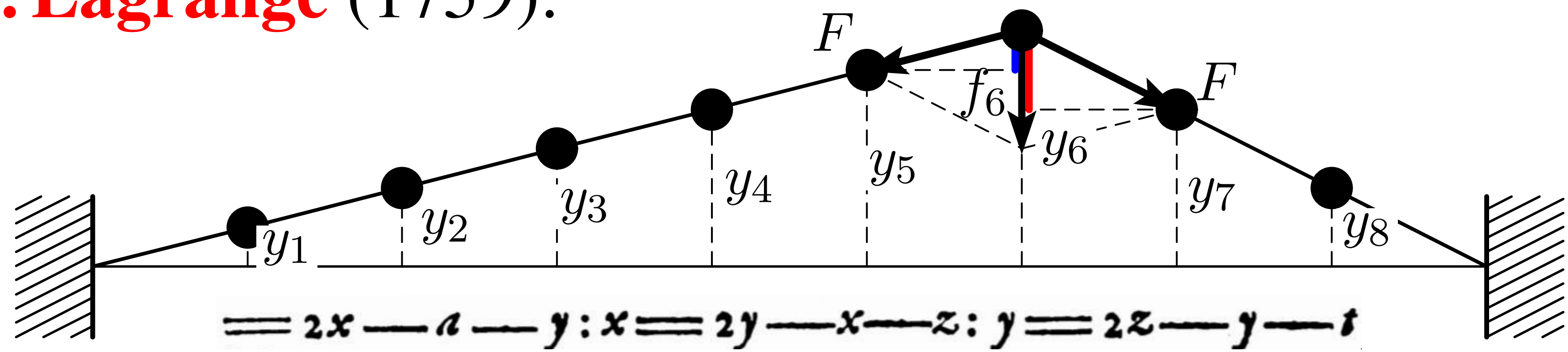
D'Alembert: "... la figure de deux lignes droites qui font un angle entr'elles. Or M. Euler croit-il que dans ce cas sa construction puisse être admise & donne le vrai mouvement de la corde? **Je doute qu'il ait cette prétention**"

Daniel Bernoulli (1748, publ. 1753): "les calculs de D'Alembert et d'Euler relèvent d'une "analyse abstraite" qui, (...) est sujette à nous surprendre plutôt qu'à nous éclairer" — —"**il n'y a que les courbes données par M. *Taylor* qui satisfassent au problème.**"

L. Euler (1753, publ. 1755): "M. *Bernoulli* tire toutes ces excellentes réfléxions ... **& soutient contre M. d'*Alembert* & moi**, que la solution de *Taylor* est suffisante à expliquer tous les mouvemens, dont une corde est susceptible."

B. Riemann (Habil.Th. 1854): Der Streit zwischen *Euler* und *d'Alembert* war indess noch immer unerledigt. Dies veranlasste einen jungen, damals noch wenig bekannten ...

J.-L. Lagrange (1759):



$$\frac{d^2y_1}{dt^2} = K^2(-2y_1 + y_2)$$
$$\frac{d^2y_2}{dt^2} = K^2(y_1 \quad -2y_2 + y_3)$$
$$\dots$$
$$\frac{d^2y_n}{dt^2} = K^2(\qquad y_{n-1} - 2y_n)\,.$$

$$\boxed{y_j = c_j e^{pt}} \quad \Rightarrow$$

$$p^2c_1 = K^2(-2c_1 + c_2)$$
$$p^2c_2 = K^2(c_1 \quad -2c_2 + c_3)$$
$$\dots$$
$$p^2c_n = K^2(\qquad c_{n-1} - 2c_n)\,.$$

$$y_j(t) = \sum_{k=1}^{n} \sin\frac{jk\pi}{n+1}(a_k \cos r_k t + b_k \sin r_k t)\,,$$
$$r_k = 2K \sin\tfrac{\pi k}{2n+2}\,.$$

Problem: Given $y_j(0)$, $\dot{y}_j(0)$, find a_k, b_k $(j, k = 1, .., n)$?

Lagrange's solution:

$$y^{\mathrm{I}} \sin. \frac{\pi}{2m} + y^{\mathrm{II}} \sin. \frac{2\pi}{2m} + y^{\mathrm{III}} \sin. \frac{3\pi}{2m} + \&c. +$$
$$y^{m-1} \sin. (m-1)\frac{\pi}{2m} = S^{\mathrm{I}}$$

$$y^{\mathrm{I}} \sin. \frac{2\pi}{2m} + y^{\mathrm{II}} \sin. \frac{4\pi}{2m} + y^{\mathrm{III}} \sin. \frac{6\pi}{2m} + \&c. +$$
$$y^{m-1} \sin. 2(m-1)\frac{\pi}{2m} = S^{\mathrm{II}}$$

$$y^{\mathrm{I}} \sin. \frac{3\pi}{2m} + y^{\mathrm{II}} \sin. \frac{6\pi}{2m} + y^{\mathrm{III}} \sin. \frac{9\pi}{2m} + \&c. +$$
$$y^{m-1} \sin. 3(m-1)\frac{\pi}{2m} = S^{\mathrm{III}}$$

&c.

$$y^{\mathrm{I}} \sin. (m-1)\frac{\pi}{2m} + y^{\mathrm{II}} \sin. 2(m-1)\frac{\pi}{2m} + \&c. +$$
$$y^{m-1} \sin. (m-1)^2\frac{\pi}{2m} = S^{m-1}$$

dont le nombre ſera $m - 1$.

Il faudroit à préſent, ſelon les règles ordinaires ſubſtituer les valeurs des inconnues $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}$ &c. d'une

(Linear system for
discrete trigonometric interpolation)

"Il faudroit à présent, selon
les règles ordinaires, substituer ..."

" ... tomberait dans des calculs
impraticables ..."

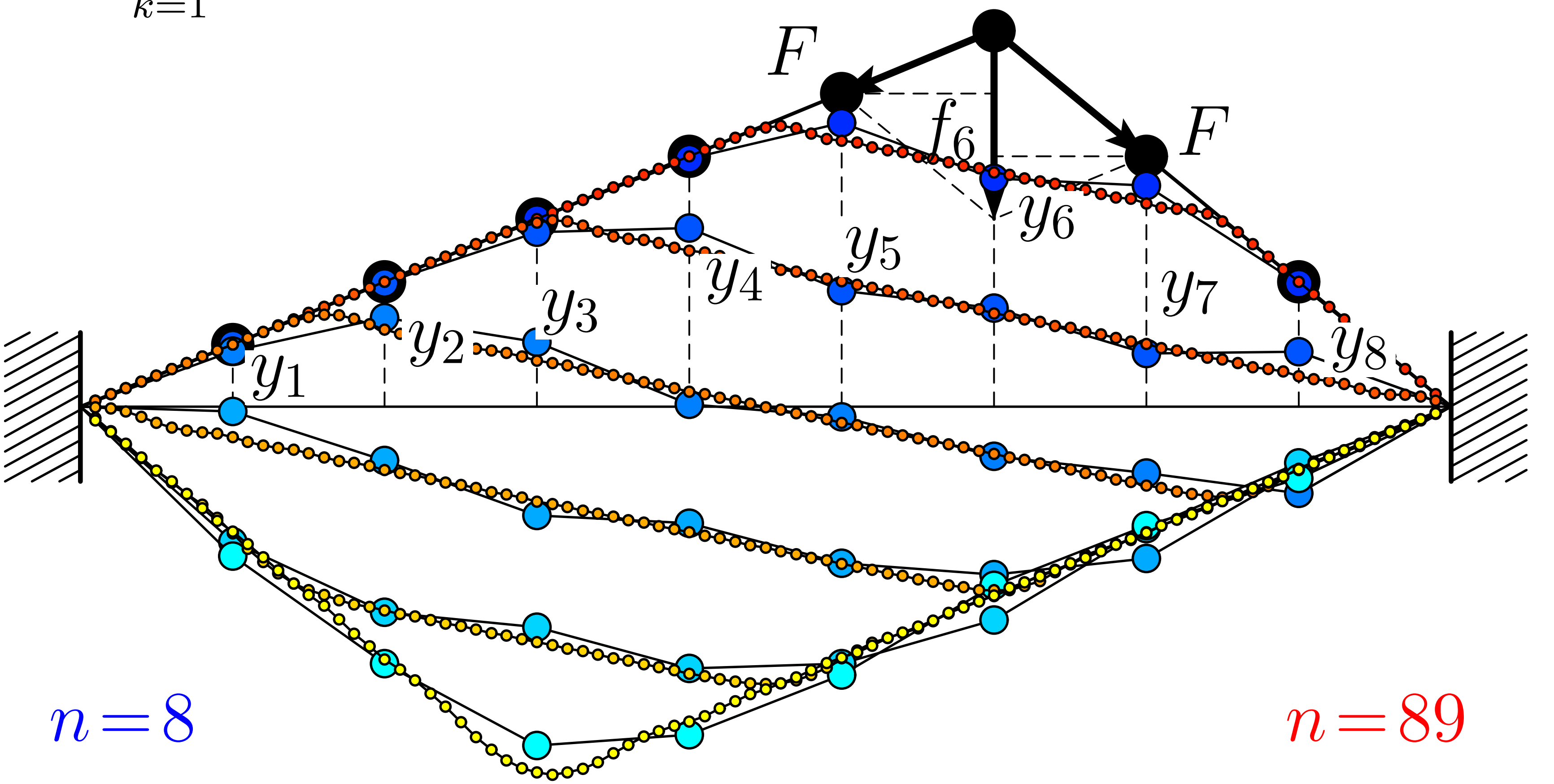
" ... il est donc nécessaire
de suivre une autre route ..."

24. Je multiplie d'abord chacune de ces équations par un des coéficiens indéterminés D^{I}, D^{II}, D^{III}, D^{IV} &c., en ſuppoſant que le premier D^{I} ſoit $= 1$; enſuite je les ajoute toutes enſemble, j'ai

Result:

$$a_k = \frac{2}{n+1} \sum_{j=1}^{n} \sin \frac{kj\pi}{n+1}\, y_j(0), \quad b_k = \frac{1}{r_k} \frac{2}{n+1} \sum_{j=1}^{n} \sin \frac{kj\pi}{n+1}\, \dot{y}_j(0)\,.$$

$$y_j(t) = \sum_{k=1}^{n} \sin \frac{jk\pi}{n+1} (a_k \cos r_k t + b_k \sin r_k t)\,, \quad r_k = 2K \sin \frac{\pi k}{2n+2}\,.$$



"Mit welcher Gewandtheit, mit welchem Aufwande analytischer Kunstgriffe er auch den ersten Theil dieser Untersuchung durchführte, so liess der Uebergang vom Endlichen zum Unendlichen doch viel zu wünschen übrig. . ." (Riemann 1854).

Euler's *Calculus sinuum*. (E246, pres. 1752, publ. 1760).

"Ich arbeite anjetzo an einem Traktat über den calculum differentialem, in welchem ich verschiedene kurieuse Decouverten über die series gemacht habe, wovon ich die Freiheit nehme, Ew. Wohlgebornen einige zu kommunizieren :
I. Sumto in circulo arcu quocunque a, cujus sinus sit $=\alpha$, sinus arcus dupli $=\beta$, sinus arcus tripli $=\gamma$ (...), dico hujus seriei infinitae

$$\frac{1}{2}a+\alpha+\frac{1}{2}\beta+\frac{1}{3}\gamma+\frac{1}{4}\delta+\frac{1}{5}\varepsilon+\text{etc.}$$

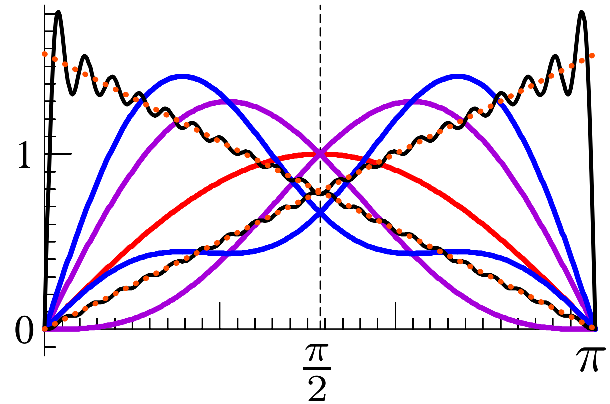
summam semper exprimere longitudinem arcus 90° in eodem circulo [i.e. $\frac{\pi}{2}$]."

(Letter of Euler to Goldbach, june 1744)

$$\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x\mp\ldots=\frac{x}{2}$$

$$\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots=\frac{\pi-x}{2}$$

Very first trigonometric series of history !



How did Euler find them ?

$$ae^{ix}+a^2e^{2ix}+a^3e^{3ix}+a^4e^{4ix}+\ldots=\frac{ae^{ix}}{1-ae^{ix}}=\frac{ae^{ix}-a^2}{1-2a\cos x+a^2},$$

set $a=\mp 1$,

take real parts $\Rightarrow$

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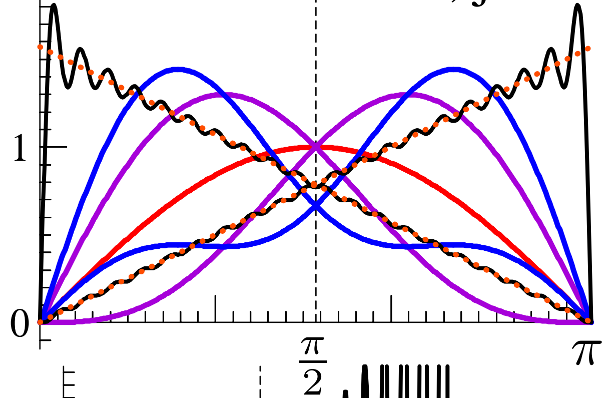
$$\text{fin.}\Phi-\tfrac{1}{2}\text{fin.}2\Phi+\tfrac{1}{3}\text{fin.}3\Phi-\tfrac{1}{4}\text{fin.}4\Phi+\tfrac{1}{5}\text{fin.}5\Phi-\text{etc.}=\tfrac{\Phi}{2}$$

$$\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x\mp\ldots=\frac{x}{2}$$

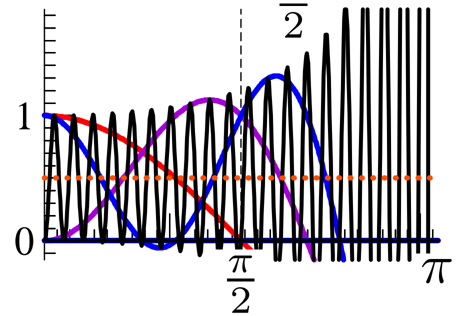
$$\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ldots=\frac{\pi-x}{2}$$



$\downarrow_{\text{int.}}$ $\uparrow^{\text{int.}}$

$$\cos x-\cos 2x+\cos 3x\mp\ldots=\frac{1}{2}$$

$$\cos x+\cos 2x+\cos 3x+\ldots=-\frac{1}{2}$$

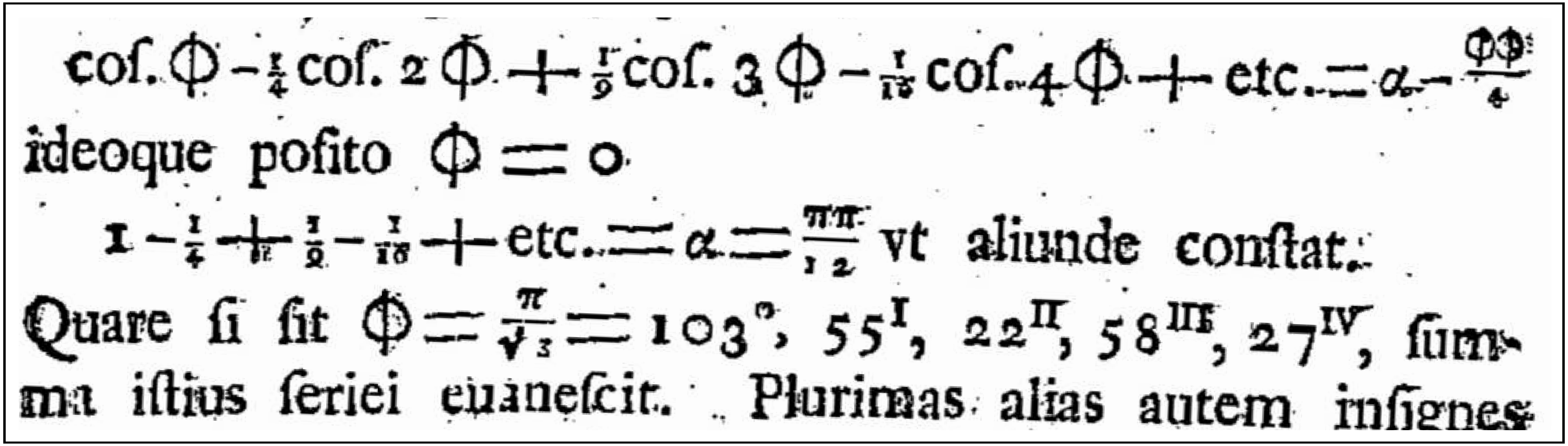


Integrate once more ...

$$\cos x+\tfrac{1}{2^2}\cos 2x+\tfrac{1}{3^2}\cos 3x+\tfrac{1}{4^2}\cos 4x+\ldots=\frac{x^2}{4}-\frac{x\pi}{2}+\frac{\pi^2}{6}$$

Euler's last series from E246

$$\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \frac{1}{4^2}\cos 4x \pm \ldots = \frac{\pi^2}{12} - \frac{x^2}{4}$$

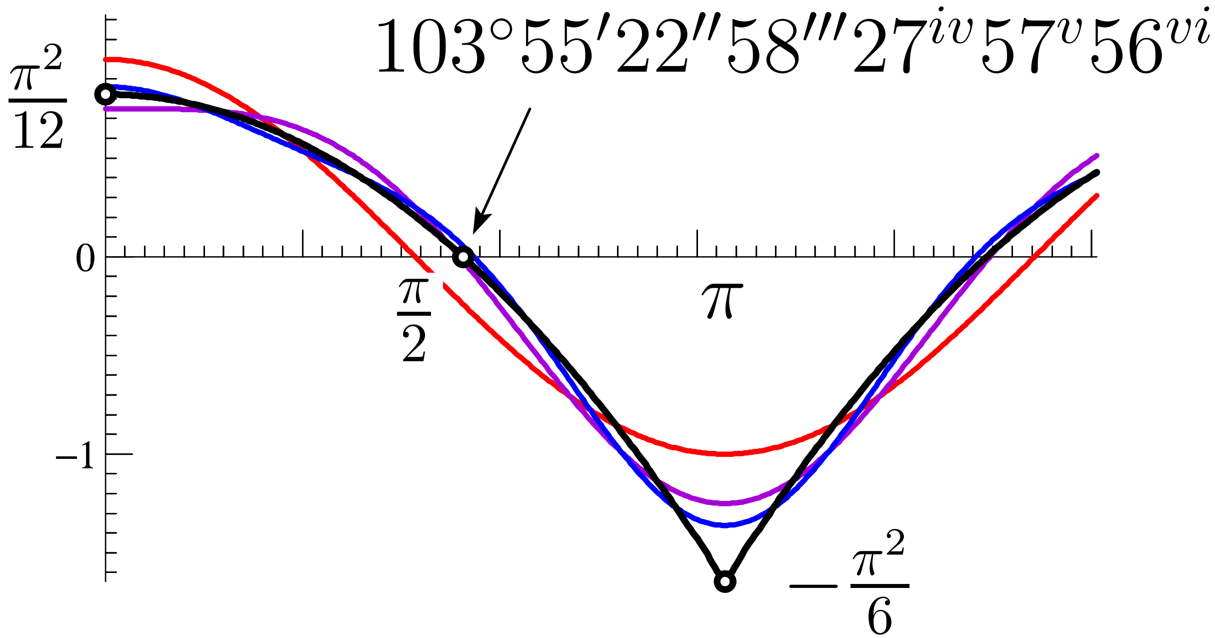


$\mathrm{coſ.}\,\Phi - \frac{1}{4}\,\mathrm{coſ.}\,2\Phi + \frac{1}{9}\,\mathrm{coſ.}\,3\Phi - \frac{1}{16}\,\mathrm{coſ.}\,4\Phi + \mathrm{etc.} = \alpha - \frac{\Phi\Phi}{4}$

ideoque poſito $\Phi = 0$

$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \mathrm{etc.} = \alpha = \frac{\pi\pi}{12}$ vt aliunde conſtat.

Quare ſi ſit $\Phi = \frac{\pi}{\sqrt{3}} = 103^{\circ}, 55^{\mathrm{I}}, 22^{\mathrm{II}}, 58^{\mathrm{III}}, 27^{\mathrm{IV}}$, ſumma iſtius ſeriei euaneſcit. Plurimas alias autem inſignes

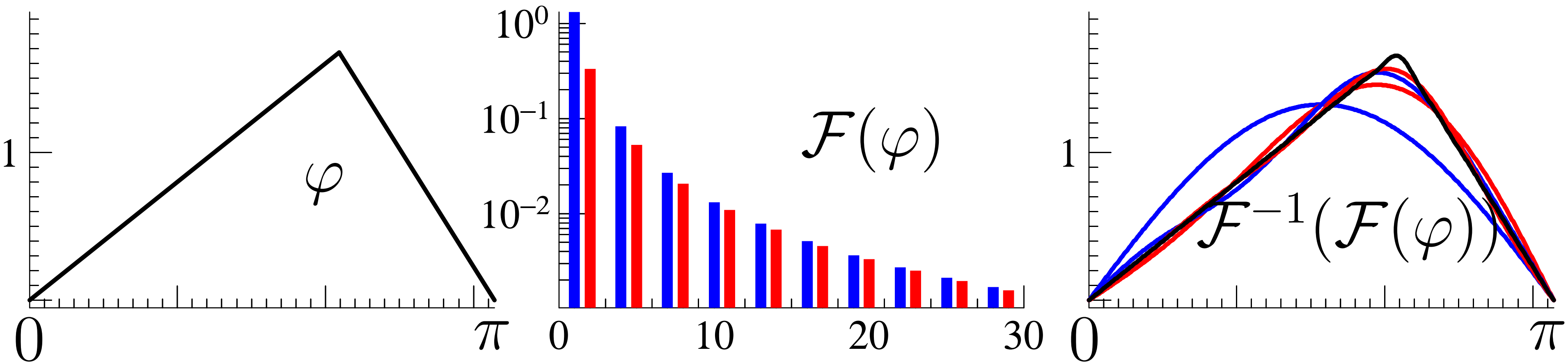


Integral Formula. Given function $f(x)$, find $a_1, a_2, a_3, \ldots$.

After decades of calculations (Lagrange on sound, Euler on planetary motion, E703), 70 years old Euler has one of his GREAT ideas (**E704** from 1777, publ. 1798):

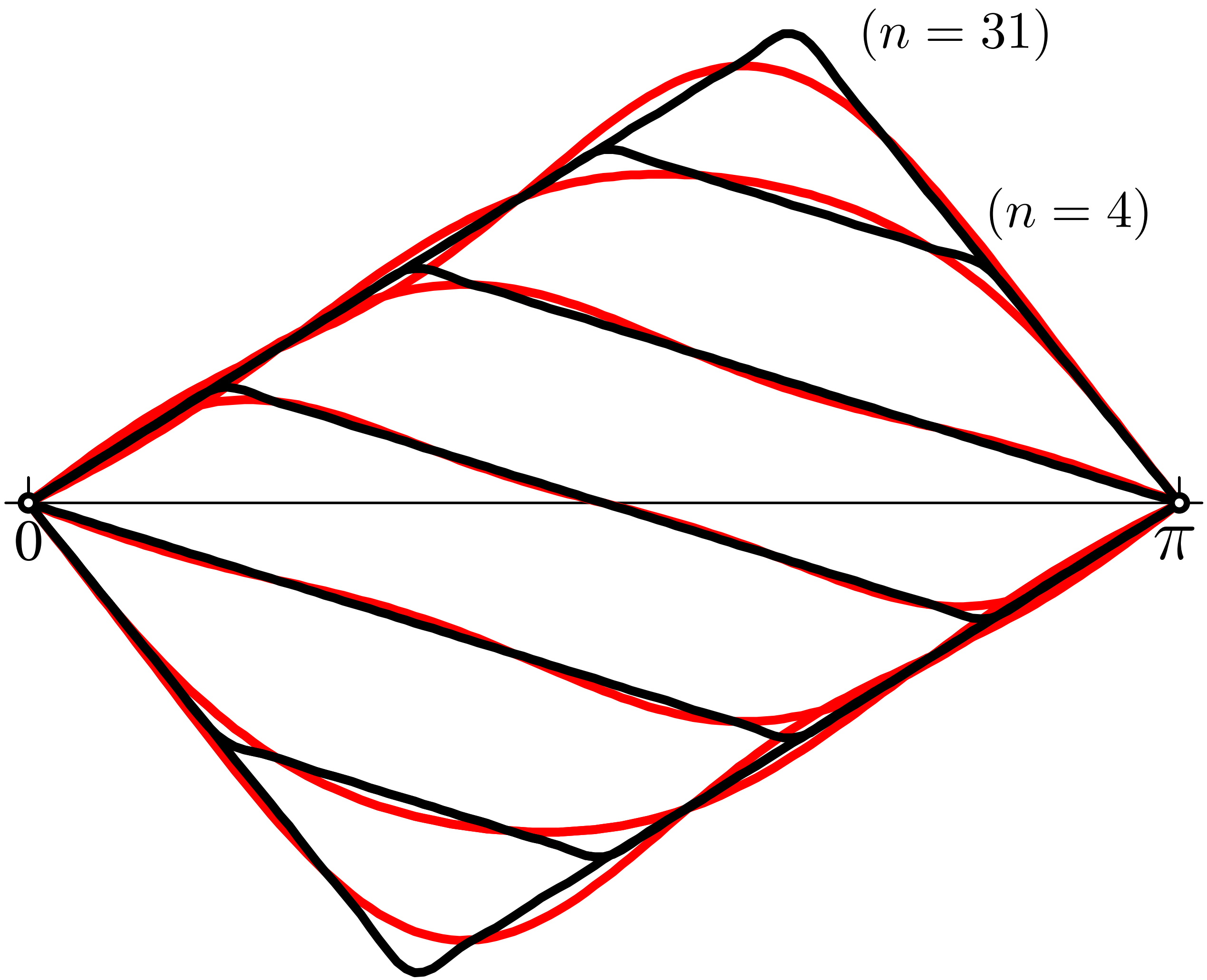
Multiply series $f(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots$ by $\sin kx$, "quod integrale $\int_0^\pi \sin\lambda x \sin kx\, dx$ evanescit ... unicum casum excipiamus, quo $\lambda = k$ ... qui valor ... $\frac{1}{2}\pi$"

$\mathcal{F} : a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \varphi(x)\cdot \sin nx\, dx$	$\mathcal{F}^{-1} : \varphi(x) = \displaystyle\sum_{n=1}^\infty a_n \cdot \sin nx\,.$



Taylor-Rameau-Euler-D.Bernoulli's solution ...

$$y = \alpha \sin \frac{\pi x}{c} \cos \frac{\pi u}{c} + \beta \sin \frac{2\pi x}{c} \cos \frac{2\pi u}{c} + \gamma \sin \frac{3\pi x}{c} \cos \frac{3\pi u}{c} + \delta \sin \frac{4\pi x}{c} \cos \frac{4\pi u}{c} + \text{etc.}$$



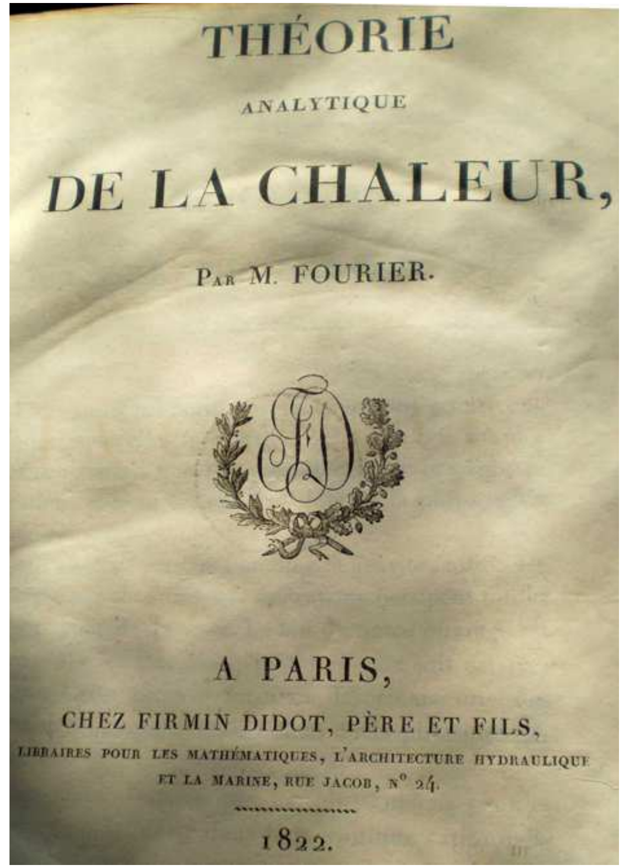
J. Fourier (1768 (!!) – 1830).

“Fouriers *Théorie analytique de la Chaleur* ist die Bibel des mathematischen Physikers.”

(A. Sommerfeld, *Vorl. Th. Phys. VI, Part. Diffgln. der Physik*,1947, very first sentence of this book)

“A PARIS, 1822.”

200 years !!



THÉORIE
ANALYTIQUE
DE LA CHALEUR,
Par M. FOURIER.

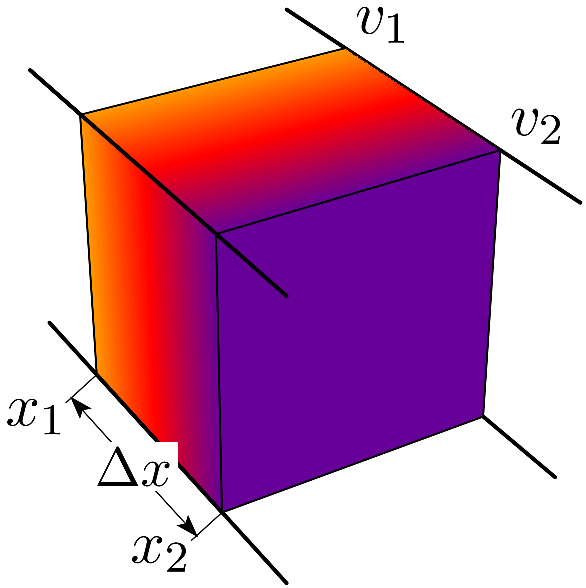
A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

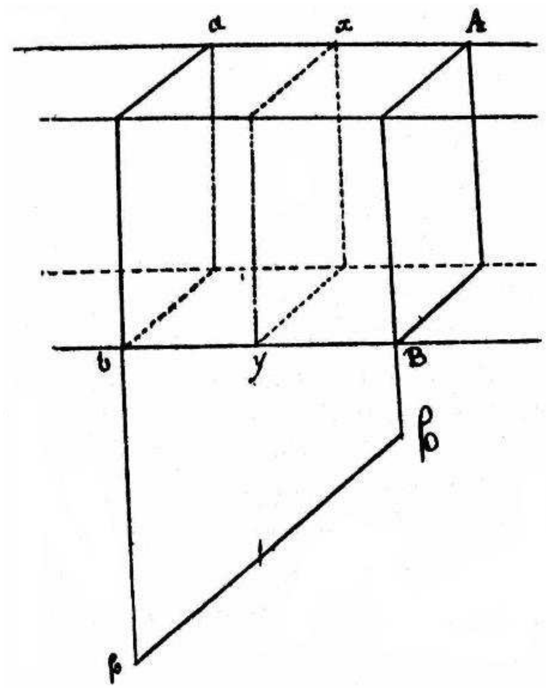
$$\frac{dv}{dt} = \frac{K}{CD}\,\frac{d^2v}{dx^2}$$

Fourier's manuscript from 1807 (publ. I. Grattan-Guinness 1972)

"Son analyse … laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du coté de la rigueur" (Laplace, Lagrange, Legendre; see [Burk1914, p. 957])

Art. 17: ("Températures en équilibre dans un prisme")



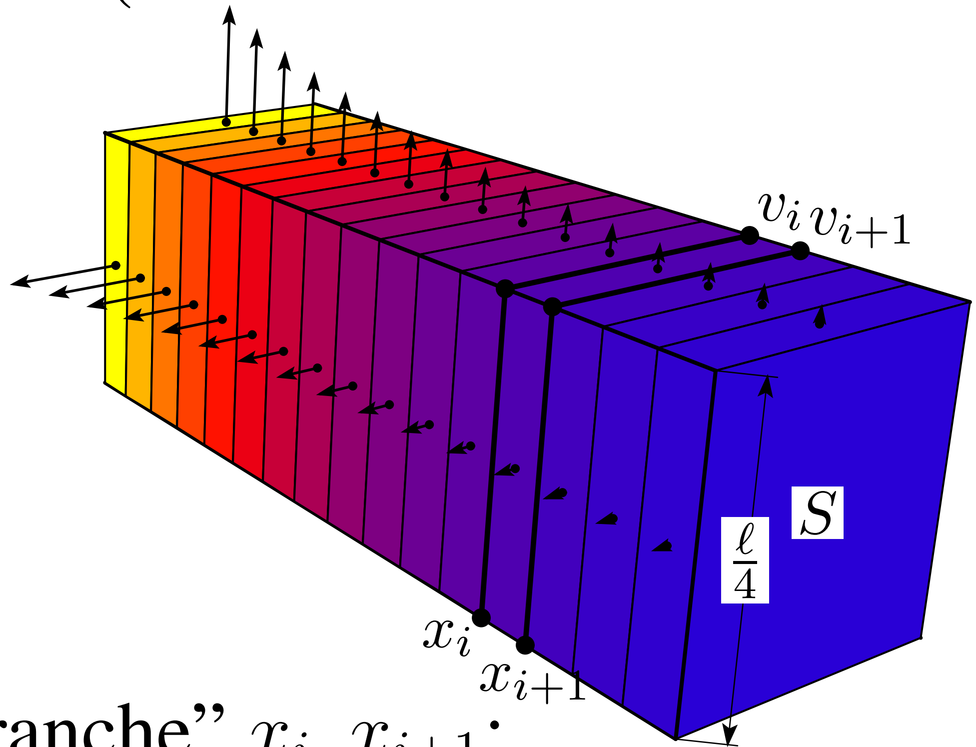


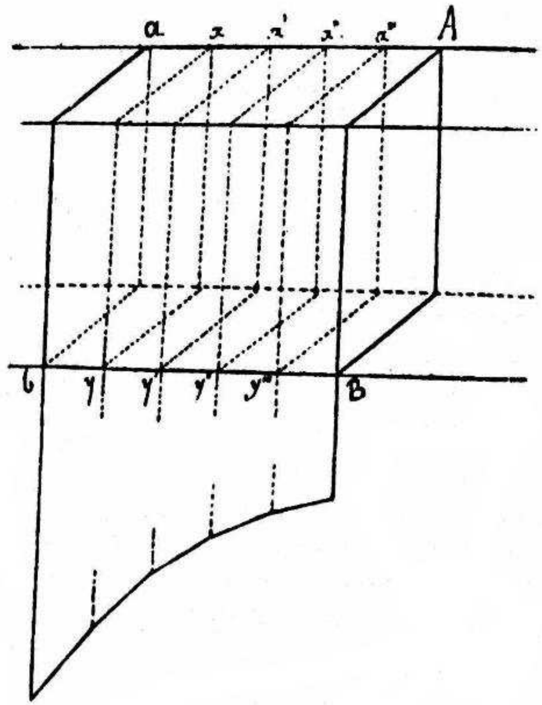
$$v = v_1 + (x - x_1) \cdot \frac{v_2 - v_1}{x_2 - x_1}$$

$$\text{Heat flow:} \quad -K\left[\frac{v_2 - v_1}{x_2 - x_1}\right] \quad \text{or} \quad -K\frac{\Delta v}{\Delta x}$$

($K =$ "conductibilité intérieure")

Art. 19: (“Prisme chauffé à une extrémité et refroidi par l’air”)





in “tranche” x_i, x_{i+1}:

$$\text{Heat}_{(\text{in}-\text{out})} = -SK\left[\frac{dv_i}{dx} - \frac{dv_{i+1}}{dx}\right] = SK\, d\left[\frac{dv}{dx}\right]$$

$$\text{Heat}_{(\text{out}\to\text{air})} = h\ell\, dx \cdot v$$

$$\text{Heat}_{(\text{incr. } v)} = \frac{dv}{dt} \cdot CDS \cdot dx$$

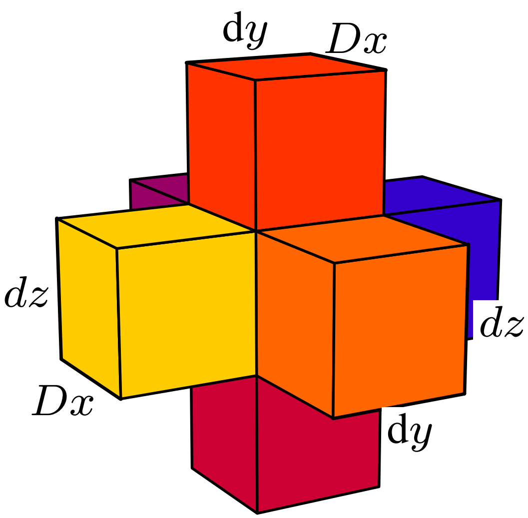
$$\boxed{\frac{\partial v}{\partial t} = \frac{K}{CD}\frac{\partial^2 v}{\partial x^2} - \frac{h\ell}{CDS}v}$$

$$\frac{\partial v}{\partial t} = \frac{K}{C \cdot D}\frac{\partial^2 v}{\partial x^2} - \frac{h\ell}{C \cdot DS} v$$

(in book 1822: Art. 105, p. 102)

In a 3D Solid.

"... on imaginera que le solide est divisé en une infinité de molécules prismatiques égales dont chacune est en contact avec six autres; x, y, z sont les coordonnées de ces molécules et l'on désignera par Dx, $\mathrm{d}y$ et dz les accroissements respectifs ..."
Thus we have for the propagation of heat quantities



in x direction: $K\,\mathrm{d}y\,dz\,D[\frac{Dv}{Dx}]$

in y direction: $K\,Dx\,dz\,\mathrm{d}[\frac{\mathrm{d}v}{\mathrm{d}y}]$

in z direction: $K\,Dx\,\mathrm{d}y\,d[\frac{dv}{dz}]$.

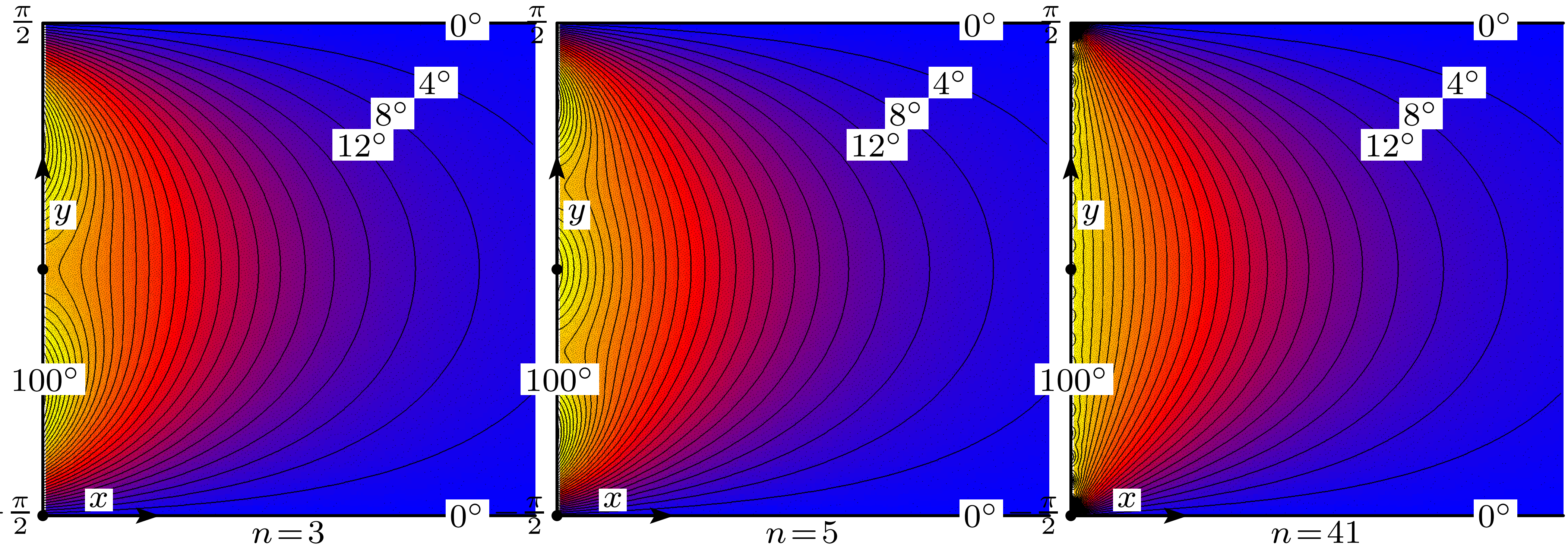
Dividing by $Dx\,\mathrm{d}y\,dz$ we get, in modern notation,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \quad \text{for equilibrium, and} \quad \frac{\partial v}{\partial t} = \frac{K}{CD}\Big[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\Big]$$

for the heat flow.

Method of separation of variabes (Art. 32-33, [1822] p. 159f)

"examen attentif et détaillé" d'une question "des plus curieuses et des plus simples que l'on puisse se proposer dans cette matière" ... la "plus propre qu'aucune autre à faire connaître les éléments de la méthode:" Prisme $0 \leq x < \infty, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, -\infty < z < \infty$



$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (\text{set } v = \varphi(x) \cdot \psi(y)) \qquad \underbrace{\frac{\varphi''}{\varphi}}_{m^2} + \underbrace{\frac{\psi''}{\psi}}_{-m^2} = 0$$

$$\psi(y) = \cos my \ (m = 1, 3, 5, \dots)$$

$$\varphi(x) = e^{-mx}, \quad \cancel{e^{+mx}} \qquad \Rightarrow z = \frac{4}{\pi}\left(e^{-x}\cos y - \frac{1}{3}e^{-3x}\cos 3y + \frac{1}{5}e^{-5x}\cos 5y \mp \dots\right)$$

$$1 = \frac{4}{\pi}\left(\cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y \mp \dots\right)$$

Fourier's first proof of $1 = \frac{4}{\pi}(\cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y \mp \ldots)$:

Put $a\cos y + b\cos 3y + c\cos 5y + d\cos 7y + \&c. = 1$ and compare Taylor coefficients at $y = 0$:

$$\begin{array}{l}
a + b + c + d + e + f + g = 1 \\
a + 3^2 b + 5^2 c + 7^2 d + 9^2 e + 11^2 f + 13^2 g = 0 \\
a + 3^4 b + 5^4 c + 7^4 d + 9^4 e + 11^4 f + 13^4 g = 0 \\
a + 3^6 b + 5^6 c + 7^6 d + 9^6 e + 11^6 f + 13^6 g = 0 \\
a + 3^8 b + 5^8 c + 7^8 d + 9^8 e + 11^8 f + 13^8 g = 0 \\
a + 3^{10} b + 5^{10} c + 7^{10} d + 9^{10} e + 11^{10} f + 13^{10} g = 0 \\
a + 3^{12} b + 5^{12} c + 7^{12} d + 9^{12} e + 11^{12} f + 13^{12} g = 0
\end{array}$$

... three pages of eliminations, first for g, then f ,... finally a: "la valeur de a qui correspond à un nombre infini d'équations"

$$a = \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdots = \frac{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\ldots}{2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\ldots} = \frac{4}{\pi}$$

"suivant le théorème de Wallis". After 4 other pages $b = -\frac{a}{3}$, $c = \frac{a}{5}$ etc.

Later (1822, Art. 189) he found "le moyen le plus simple d'obtenir cette équation"

$$\arctan e^{iy} + \arctan e^{-iy} = \arctan \frac{e^{iy} + e^{-iy}}{1 - e^{iy}e^{-iy}} = \arctan \infty = \frac{\pi}{2}\,.$$

Fourier's discovery of the integral formula: (Art. 51f)

$$\varphi(x) = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + a_4 \sin 4x + a_5 \sin 5x + \&c.$$

Taylor series analysis ...

$$\boxed{\begin{aligned}
A &= a + 2\,b + 3\,c + 4\,d + 5\,e + \text{etc.}\\
B &= a + 2^3 b + 3^3 c + 4^3 d + 5^3 e + \text{etc.}\\
C &= a + 2^5 b + 3^5 c + 4^5 d + 5^5 e + \text{etc.}\\
D &= a + 2^7 b + 3^7 c + 4^7 d + 5^7 e + \text{etc.}\\
E &= a + 2^9 b + 3^9 c + 4^9 d + 5^9 e + \text{etc.}\\
&\text{etc.}
\end{aligned}}$$

$\Rightarrow$ 26 pages of calculations ([1822], pp 210–235), ending up in

$$s + \frac{1}{n^2}\frac{d^2 s}{dx^2} = \varphi(x) \quad \overset{\text{var. of const.}}{\Longrightarrow} \quad \boxed{\int_0^\pi \varphi(x)\cdot \sin nx\, dx}$$

THAT'S THE ANSWER !!!

Finally ([1822], Art. 221, p. 235) Fourier gives ("On peut aussi vérifier l'équation précédente...") the usual proof by orthogonality.

End of part I, thank you.