Global existence of a BV solution to a diagonal hyperbolic system

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Presentation of the problem

We study the existence of solutions $u(t,x) = \big(u^i(t,x)\big)_{i=1,\dots,d}$, where $d \in \mathbb{N}^*$, to

$$\begin{cases} \partial_t u^i(t,x) = \lambda^i(t,x,u(t,x))\partial_x u^i(t,x) & \text{in } (0,T)\times\mathbb{R}, \quad T>0 \\ u^i(0,x) = u_0^i(x) & \text{in } \mathbb{R}. \end{cases} \tag{H}$$

- The velocities λ^i satisfy for every compact $\mathcal{K} \subset \mathbb{R}^d$
$$\lambda^i \in L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}). \tag{K1}$$
- The initial data u_0^i satisfy
$$u_0^i \in BV(\mathbb{R}), \tag{K2}$$
where $BV(\mathbb{R})$ is the space of functions of bounded variations given by
$$BV(\mathbb{R}) = \Big\{ f \in L^1_{loc}(\mathbb{R});\ TV(f) < +\infty \Big\},$$
and equipped with the semi-norm
$$|f|_{BV(\mathbb{R})} = TV(f) = \sup\Big\{ \int_{\mathbb{R}} f(x)\phi'(x)dx;\ \phi \in C_c^1(\mathbb{R}) \text{ and } \|\phi\|_{L^\infty(\mathbb{R})} \le 1 \Big\}.$$

Some tools

- For a locally bounded function f, we denote the upper and lower semi-continuous envelopes respectively by

$$f^\star(X) = \limsup_{Y \to X} f(Y), \quad \text{and} \quad f_\star(X) = \liminf_{Y \to X} f(Y).$$

- For two locally bounded functions $v = (v^i)_{i=1,\dots,d}$ and $u = (u^i)_{i=1,\dots,d}$ on $[0, T) \times \mathbb{R}$ such that $(v^i)_\star \leq (u^i)^\star$ for every $i = 1, \dots, d$, we define the set

$$\mathcal{E}_v^u(t,x) = \prod_{i=1}^{d} \left[(v^i)_\star(t,x), (u^i)^\star(t,x) \right].$$

Useful definition

Ishii, Koike (1991-1992)

Definition

Assume that $\lambda = (\lambda^i)_{i=1,\dots,d}$ is locally bounded on $(0,T) \times \mathbb{R} \times \mathbb{R}^d$ and $u_0 = (u_0^i)_{i=1,\dots,d}$ is locally bounded on $\mathbb{R}$. Let $v = (v^i)_{i=1,\dots,d}$, $u = (u^i)_{i=1,\dots,d}$ be two locally bounded functions on $[0,T) \times \mathbb{R}$ such that $(v^i)_\star \le (u^i)^\star$ for every $i = 1,\dots,d$. We say that u and v are a couple of discontinuous viscosity sub- and super- solutions of (H) if they satisfy the following two conditions

(i)
- $(u^i)^\star(0,x) \le (u_0^i)^\star(x)$, for all $i = 1,\dots,d$ and $x \in \mathbb{R}$
- $(v^i)_\star(0,x) \ge (u_0^i)_\star(x)$, for all $i = 1,\dots,d$ and $x \in \mathbb{R}$.

Useful definition

Ishii, Koike (1991-1992)

Definition

(ii) • Whenever a test function $\phi \in C^1((0,T)\times\mathbb{R})$, $i = 1,\dots,d$ and $(u^i)^\star - \phi$ attains a local maximum at $(t_0,x_0)\in(0,T)\times\mathbb{R}$, then we have

$$\min\Big\{\partial_t\phi(t_0,x_0) - (\lambda^i)^\star(t_0,x_0,r)(\partial_x\phi)^+(t_0,x_0) + (\lambda^i)_\star(t_0,x_0,r)(\partial_x\phi)^-(t_0,x_0) : \\ r\in\mathcal{E}^u_v(t_0,x_0),\ r^i = (u^i)^\star(t_0,x_0)\Big\} \le 0.$$

• Whenever $\phi \in C^1((0,T)\times\mathbb{R})$, $i = 1,\dots,d$ and $(v^i)_\star - \phi$ attains a local minimum at $(t_0,x_0)\in(0,T)\times\mathbb{R}$, then we have

$$\max\Big\{\partial_t\phi(t_0,x_0) - (\lambda^i)_\star(t_0,x_0,r)(\partial_x\phi)^+(t_0,x_0) + (\lambda^i)^\star(t_0,x_0,r)(\partial_x\phi)^-(t_0,x_0) : \\ r\in\mathcal{E}^u_v(t_0,x_0),\ r^i = (v^i)_\star(t_0,x_0)\Big\} \ge 0.$$

Finally, we call a function $w = (w^i)_{i=1,\dots,d}$ a discontinuous viscosity solution of (H) if $w^\star$ and $w_\star$ verify conditions (i) and (ii).

Step 1: Regularization of u_0^i and λ^i

- $u^i_{0,\varepsilon} = u^i_0 \star \rho^1_\varepsilon$ for $= 1, \ldots, d$
- $\lambda^i_\varepsilon = \hat{\lambda}^i \star \rho^{d+2}_\varepsilon$ for $= 1, \ldots, d$, where $\hat{\lambda}^i$ is an extension of λ^i by 0

where ρ^1_ε and ρ^{d+2}_ε are the standard mollifiers in $\mathbb{R}$ and $\mathbb{R}^{d+2}$ respectively.

Step 2: Adding the term $\eta \partial^2_{xx} u^i_{\varepsilon,\eta}$

Thus, we obtain the following parabolic regularized system

$$\begin{cases} \partial_t u^i_{\varepsilon,\eta}(t,x) = \eta \partial^2_{xx} u^i_{\varepsilon,\eta}(t,x) + \lambda^i_\varepsilon(t,x,u_{\varepsilon,\eta}(t,x)) \partial_x u^i_{\varepsilon,\eta}(t,x) & \text{in } (0,T) \times \mathbb{R}, \\ u^i_{\varepsilon,\eta}(0,x) = u^i_{0,\varepsilon}(x) & \text{in } \mathbb{R}. \end{cases} \tag{P}$$

Global solution to (H)

Global existence to a diagonal hyperbolic system for any BV initial data (Nonlinearity 2021)

Theorem (Existence of a weak sense to (H))

Assume that (K1) and (K2) hold. Then, we have

(i) Smooth solution: *there exists a unique smooth solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\ldots,d}$ to system (P), satisfying for all $T > 0$ and $i = 1, \ldots, d$, the following uniform a priori estimates*

$$\left\| u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}$$

$$\left\| u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leq |u^i_0|_{BV(\mathbb{R})}$$

$$\left\| \partial_t u^i_{\varepsilon,\eta} \right\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \left(1 + \left\| \lambda^i \right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)} \right) \left| u^i_0 \right|_{BV(\mathbb{R})}$$

where $\mathcal{K}_0 = \prod_{i=1}^{d} \left[- \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})}, \left\| u^i_0 \right\|_{L^\infty(\mathbb{R})} \right]$.

Global existence to a diagonal hyperbolic system for any BV initial data (Nonlinearity 2021)

Theorem (Existence in a week sense to (H))

(ii) Sub and super solutions: *the upper and lower relaxed semi-limits $\overline{u} = (\overline{u})_{i=1,\dots,d}$ and $\underline{u} = (\underline{u}^i)_{i=1,\dots,d}$ of the solution $u_{\varepsilon,\eta} = (u^i_{\varepsilon,\eta})_{i=1,\dots,d}$ to (P), are a couple of discontinuous viscosity sub- and super- solutions respectively to system (H).*

We define the upper and lower relaxed semi-limits of $u_{\varepsilon,\eta}$ respectively by

$$\overline{u}(t,x) = \limsup_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u_{\varepsilon,\eta}(s,y), \quad \text{and} \quad \underline{u}(t,x) = \liminf_{\substack{(\varepsilon,\eta)\longrightarrow(0,0)\\(s,y)\longrightarrow(t,x)}} u_{\varepsilon,\eta}(s,y).$$

Global existence to a diagonal hyperbolic system for any BV initial data (Nonlinearity 2021)

Theorem (Existence in a weak sense to (H))

(iii) Convergence: *the solutions $u^i_{\varepsilon,\eta}$ of (P) converge, up to the extract of a subsequence, as ε and η tend to zero, for all $i = 1, \ldots, d$, to a function u^i that satisfies the following L^∞ and BV estimates*

$$\left\|u^i\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq \left\|u^i_0\right\|_{L^\infty(\mathbb{R})}$$

$$\left\|u^i\right\|_{L^\infty((0,T);BV(\mathbb{R}))} \leq |u^i_0|_{BV(\mathbb{R})}$$

and the following equality

$$u^i(t,\cdot) = \overline{u}^i(t,\cdot) = \underline{u}^i(t,\cdot)$$

except at most on a countable set in $\mathbb{R}$, for all $t \in [0,T)$.

Global existence to a diagonal hyperbolic system for any BV initial data (Nonlinearity 2021)

Corollary (Non-decreasing viscosity solution to (H))

Assume that (K1) and (K2) are satisfied. Suppose also that u_0^i is non-decreasing for every $i = 1, \dots, d$. Then system (H) has a discontinuous non-decreasing viscosity solution $(u^i)_{i=1,\dots,d}$ satisfying the L^∞ and BV estimates for all $i = 1, \dots, d$.

Our work with respect to the literature

In the case of strictly hyperbolic systems:

- **Bianchini and Bressan (2005):** existence and uniqueness result of a semi-group like solution considering initial data with small total variation.
- **El Hajj and Monneau (2010):** existence and uniqueness of a continuous viscosity solution considering non-decreasing initial data and assuming the velocities are Lipschitz smooth functions.

In the case of not necessarily strictly hyperbolic systems:

- **El Hajj and Monneau (2010):** existence and uniqueness of a Lipschitz continuous viscosity solution assuming non-decreasing initial data with some monotonicity conditions on the velocities.
- **El Hajj, Ibrahim, and Rizik (2021):** existence of a discontinuous viscosity solution considering non-decreasing BV initial data, with a monotonicity condition on the velocities.

Fixed point argument:

- Applied to the integral form of the system

$$u^i_{\varepsilon,\eta}(t,x) = G_\eta(t) \star u^i_0(x) + \int_0^t \Big(G_\eta(t-s) \star \lambda^i_\varepsilon(s,\cdot,u_{\varepsilon,\eta}(s,\cdot)) \partial_x u^i_{\varepsilon,\eta}(s,\cdot) \Big)(x) ds$$

- $G_\eta(t,x) = \dfrac{1}{\sqrt{4\pi t\eta}} e^{-\frac{x^2}{4t\eta}}$
- Obtaining a fixed point in an adapted space X_T.

Uniform a priori estimates

- L^∞ **estimate:** by applying the Maximum principle to (P).
- BV **estimate:** differentiating (P) with respect to x, multiplying by

$$B'_\delta(\partial_x u^i_{\varepsilon,\eta}) \quad \text{where } B_\delta(x) = \sqrt{x^2 + \delta^2},$$

  and then integrating with respect to t.
- **Time derivative estimate:** by duality. We multiply (P) with a test function in $L^1((0,T); W^{1,\infty}(\mathbb{R}))$, and we integrate over $(0,T) \times \mathbb{R}$.

Existence of sub and super solutions to (H)

Using:

- Finite speed propagation property of (P), for $h > 0$

$$\int_{\mathbb{R}} G_\eta(t,y) \min_{|z-(x-y)|\le \Lambda t} u^i_{\varepsilon,\eta}(h,z)dy \le u^i_{\varepsilon,\eta}(t+h,x) \le$$
$$\int_{\mathbb{R}} G_\eta(t,y) \max_{|z-(x-y)|\le \Lambda t} u^i_{\varepsilon,\eta}(h,z)dy, \quad \text{for all} \quad (t,x) \in [0,T-h) \times \mathbb{R},$$

- $\Lambda = \max_{i\in\{1,\dots,d\}} \left\|\lambda^i\right\|_{L^\infty((0,T)\times\mathbb{R}\times\mathcal{K}_0)}$,
- Stability results of viscosity solutions,

we can show that $\overline{u}^i$ and $\underline{u}^i$ satisfy the conditions of viscosity sub- and super-solutions of (H) respectively.

Convergence: using

- Simon's Lemma,
- Uniform a priori estimates,

we can extract a convergent subsequence u^i satisfying the L^∞ and BV estimates.

Existence of nondecreasing solution to (H): using

- Finite speed propagation property of (P)
- Properties of $BV(\mathbb{R})$ functions

we can show that for every $i = 1, \ldots, d$

$$\overline{u}^i = (u^i)^\star \quad \text{and} \quad \underline{u}^i = (u^i)_\star$$

Equality between $\overline{u}^i$ and $\underline{u}^i$:

- Finite speed propagation property of (P)
- Properties of $BV(\mathbb{R})$ functions.

Recent, current and future work

- Proven a similar result to the eikonal system

$$\partial_t u^i(t,x) = \lambda^i(t,x,u(t,x)) \left|\partial_x u^i(t,x)\right|, \quad i = 1, \ldots, d.$$

- Recovered the BV solution of the eikonal system through proposing a convergent numerical scheme.
- Established the existence of a BV^s solution to a particular case of the main diagonal system.

- Prove an almost everywhere uniqueness result of a "piece-wise continuous" solution.
- Establish the existence of a BV^s entropy solution to the diagonal system.

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