Interface equilibrium Stokes problem with unfitted hybrid high order method

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Motivation

High-fidelity **simulation of immiscible droplets in a carrier fluid subject to external shear**

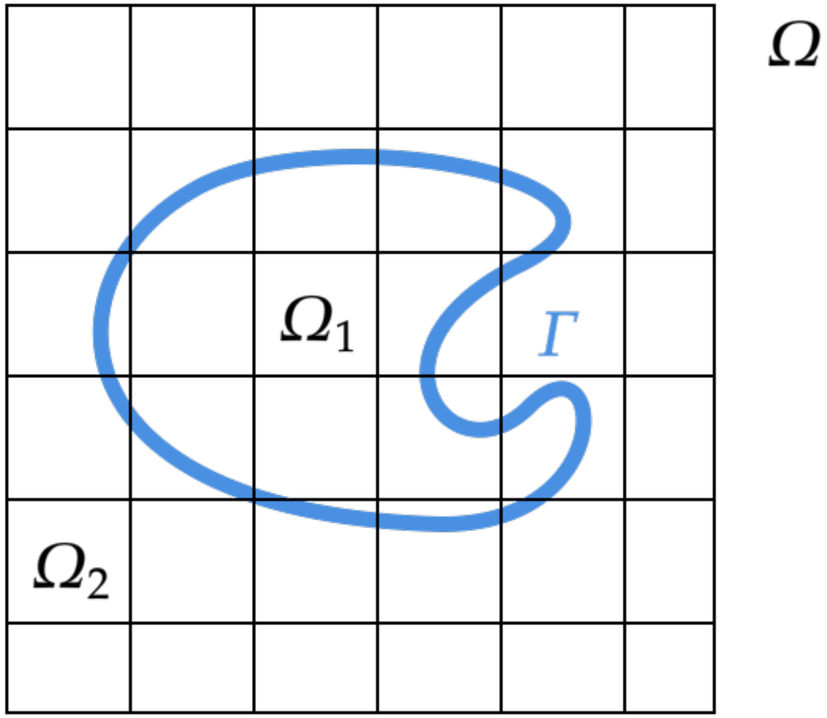
DROPLET INTERFACE

Unknown **Evolving** **Disappearing**

Initial Problem

2D interface Stokes problem

- 1 droplet considered
- Surface tension effects at the interface



Numerical challenges

High-fidelity **simulation of immiscible droplets in a carrier fluid subject to external shear**

DROPLET INTERFACE

Unknown | **Evolving** | **Disappearing**

Strategy:

- High-order approximation **Hybrid High-Order (HHO) method**
- Unfitted meshes **The mesh is independent from the droplet interface**

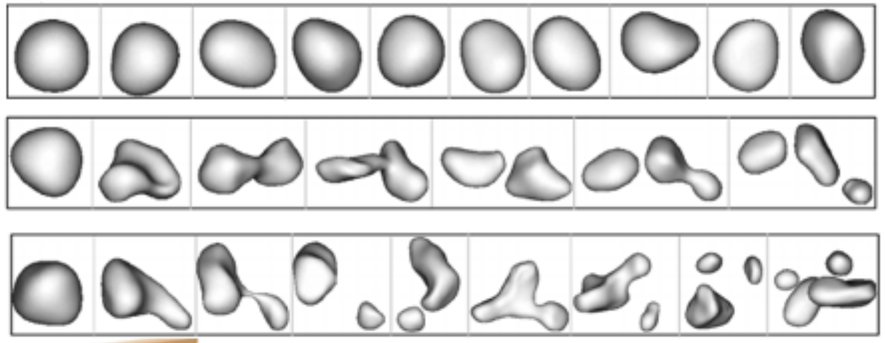
Fitted meshes issues

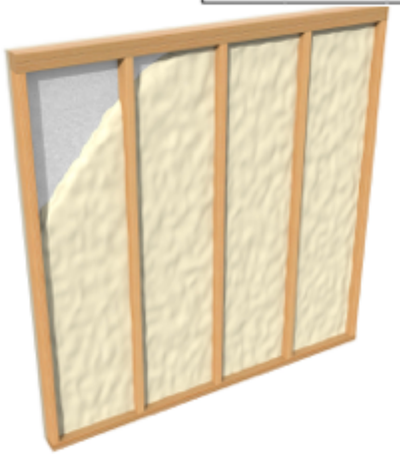
- **Computational cost** for generating high-order meshes
- **Geometrical errors** limiting the accuracy

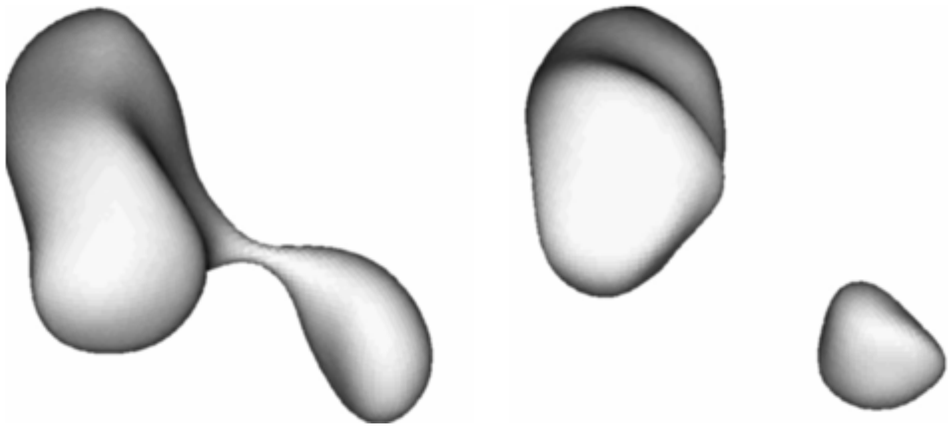
Physical inspiration

Foams for building insulation

- Less energy consumption
- Less gas emission
- Not flammable







Insulation quality ⟷ **Foam porosity**

Foam porosity is affected by:

- Transport
- Setting-up

Outline

- ❑ **Unfitted HHO method**
- ❑ **Interface Stokes problem**
  - Equilibrium droplet in shear flow
- ❑ **Unknown equilibrium interface**
  - Moving interface: **level-set scheme**

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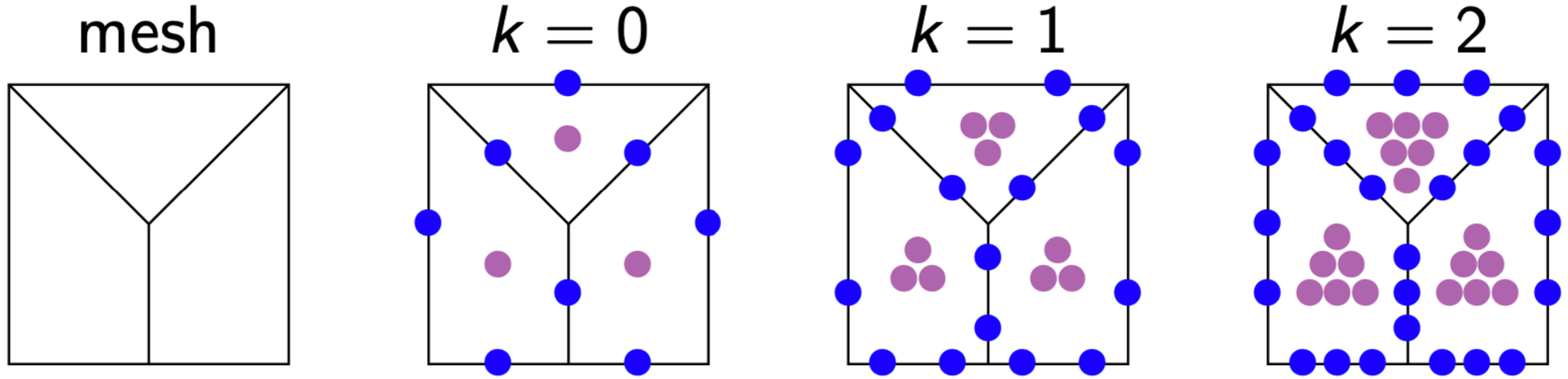
Hybrid methods

- one unknown is a polynomial defined in each cell
- one unknown is a polynomial defined on each face



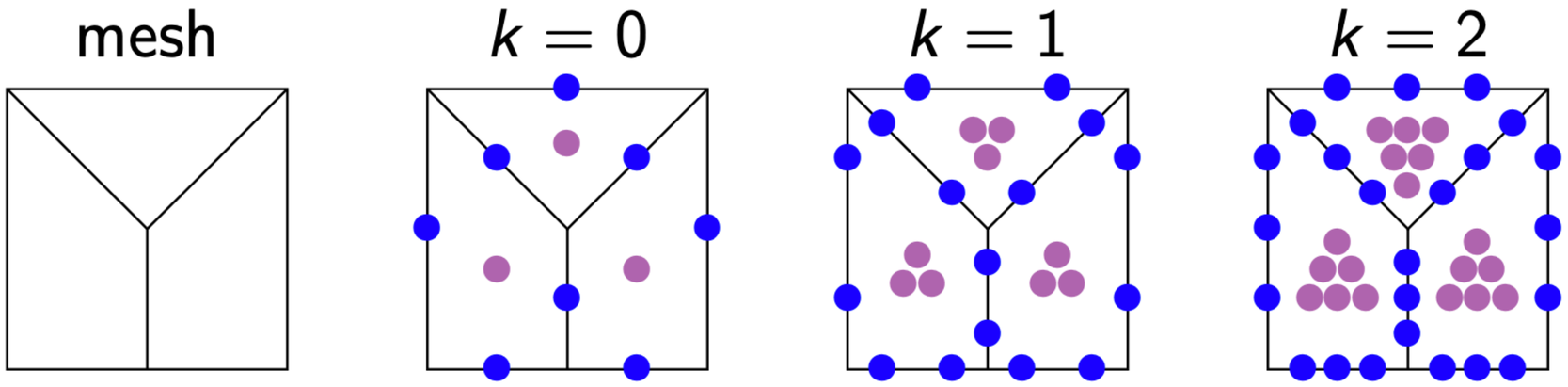
Bridge to Hybridizable DG in [2]

[1] D.A. Di Pietro, A. Ern, and S. Lemaire. “An Arbitrary-Order and Compact-Stencil Discretization of Diffusion on General Meshes Based on Local Reconstruction Operators" *Comput. Methods Appl. Math.* 2014

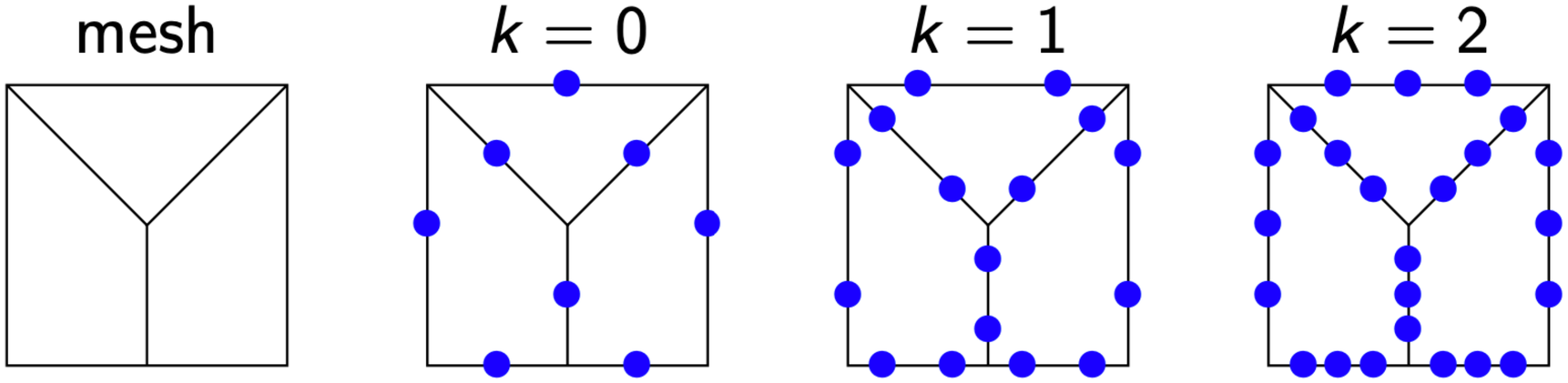
[2] B. Cockburn, D. A. Di Pietro, and A. Ern. “Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods”. ES*AIM Math. Model. Numer. Anal.* 2016

[3] G. Giorgiani, S. Fernández-Méndez, and A. Huerta. "Hybridizable discontinuous Galerkin with degree adaptivity for the incompressible Navier–Stokes equations." *Computers & Fluids.* 2014

HHO method



The dof's in the cells can be eliminated by a local Schur complement technique (static condensation)



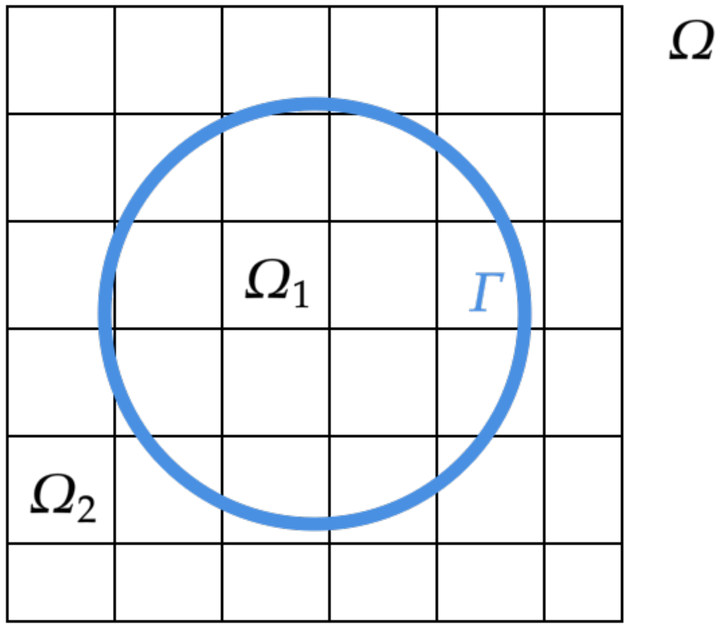
Unfitted methods

The mesh is independent from the interface

The interface is represented by means of a level-set function φ such that

$$\Gamma = \{\mathbf{x} : \varphi(\mathbf{x}) = 0\}$$

$$\varphi(x, y) = x^2 + y^2 - R^2$$

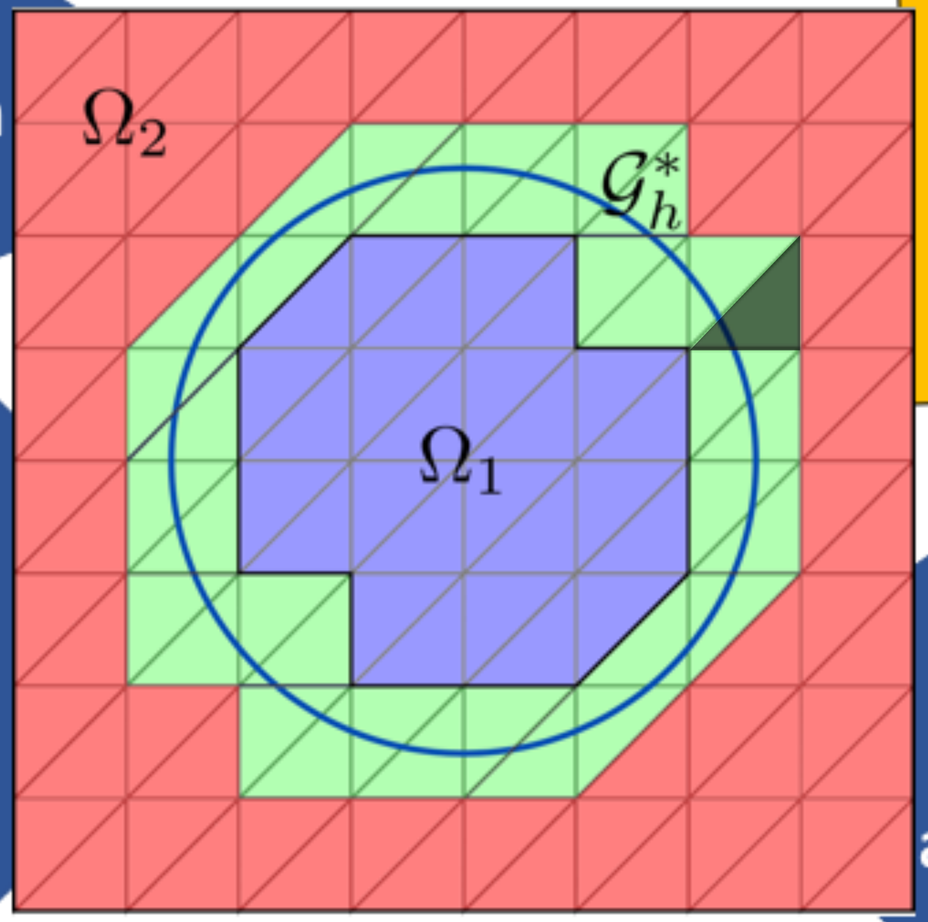


Unfitted HHO method

Doubling of unknowns on cut cells

Consistent penalty method requiring integration on cut cells

Nitsche's penalty



Simple mesh for complex geometries

Singular cuts handled by cell agglomeration (HHO-HDG)

[4] E. Burman, M. Cicuttin, G. Delay, and A. Ern. "An unfitted hybrid high-order method with cell agglomeration for elliptic interface problems." *SIAM J. Sci. Comput.* 2021

Advantages from HHO

- **Local conservation** at the cell level
- **High-order** and non-uniform degree discretisation
- **Computational efficiency** owing to static condensation

Advantages from unfitted

- **General meshes** with polyhedral cells
- **Interface** precisely tracked
- Suited for **large deformation** problem

Outline

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- ☑ **Interface Stokes problem**
  - Equilibrium droplet in shear flow
- ☐ **Unknown equilibrium interface**
  - Moving interface: **level-set scheme**

Interface Stokes problem

Let $\boldsymbol{\sigma}_i := 2\mu_i \nabla^s \boldsymbol{u}_i - p_i \mathbb{I}$, $i \in \{1,2\}$, be the **Cauchy stress tensor**

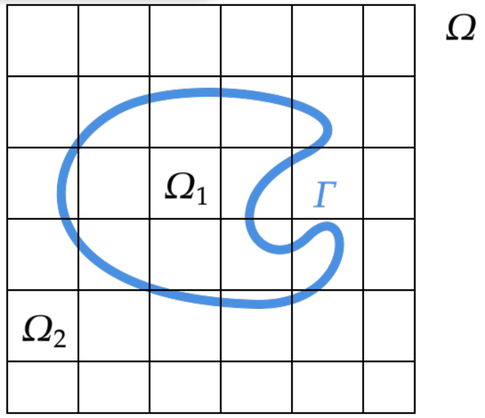
Interface Stokes problem

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}_i &= \boldsymbol{f}_i && \text{in } \Omega_i, && i \in \{1,2\},\\
\nabla \cdot \boldsymbol{u}_i &= 0 && \text{in } \Omega_i, && i \in \{1,2\},\\
\boldsymbol{u}_2 &= \boldsymbol{g} && \text{on } \partial\Omega,\\
[\![\boldsymbol{u}]\!] = \boldsymbol{0}, \quad [\![\boldsymbol{\sigma}]\!] \cdot \boldsymbol{n}_\Gamma &= \gamma H \boldsymbol{n}_\Gamma && \text{on } \Gamma,
\end{aligned}
$$

Interface condition

Data:

- γ the **surface tension**, H and $\boldsymbol{n}_\Gamma$ the **interface curvature** and **normal**
- μ_i the **viscosity**
- $[\![\cdot]\!] := \cdot_{|\Omega_1} - \cdot_{|\Omega_2}$ the **jump** across Γ
- $\nabla^s \boldsymbol{u}_i := \frac{1}{2}(\nabla \boldsymbol{u}_i + \nabla \boldsymbol{u}_i^T)$ the symmetric gradient



[5] E. Burman, G. Delay, and A. Ern. "An unfitted hybrid high-order method for the Stokes interface problem." *IMA J. Numer. Anal.* 2021

Interface equilibrium Stokes

Interface Stokes problem

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\end{aligned}
$$

Equilibrium

$+$ $\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma = 0$ on Γ

Circular equilibrium: $\mathbf{g} = \mathbf{0}$

Elliptic equilibrium: $\mathbf{g} = \varepsilon(x, -y)^{\mathrm{T}}$

Interface equilibrium Stokes

Interface Stokes problem

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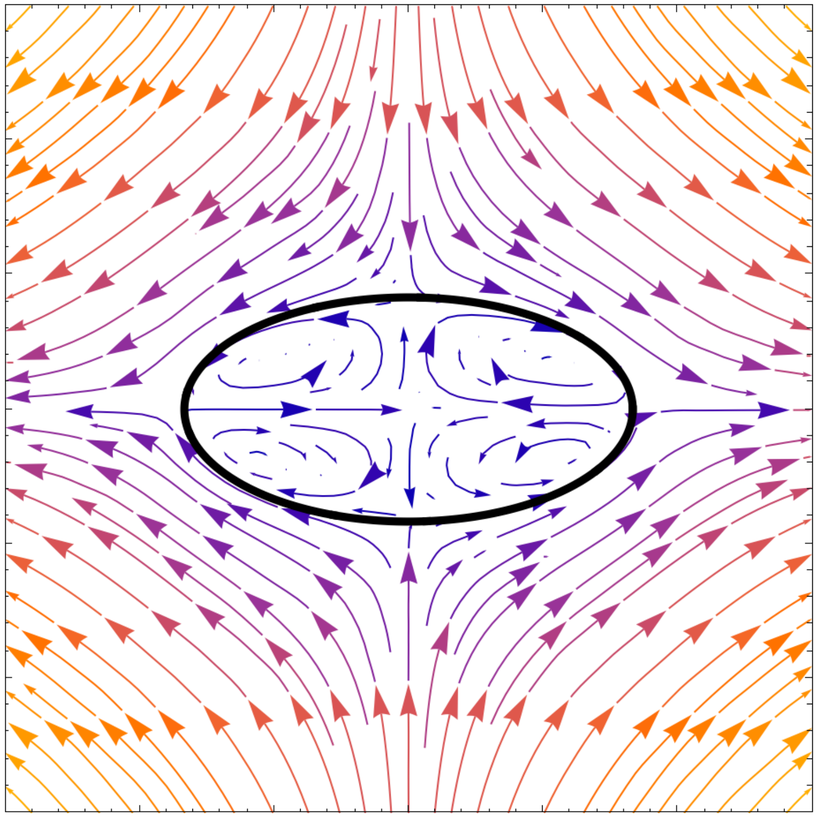

- **Deformation parameter** $D = \dfrac{R_2 - R_1}{R_1 + R_2}$, $R_1 < R_2$ ellipse radii
- **Viscosity ratio** $\lambda = \dfrac{\mu_1}{\mu_2}$
- **Capillary number** $Ca = \mu_2 \dfrac{\varepsilon L_{\mathrm{ref}}}{\gamma}$

Interface equilibrium Stokes

Interface Stokes problem

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Equilibrium

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Circular equilibrium: $\mathbf{g} = \mathbf{0}$

Elliptic equilibrium: $\mathbf{g} = \varepsilon(x, -$

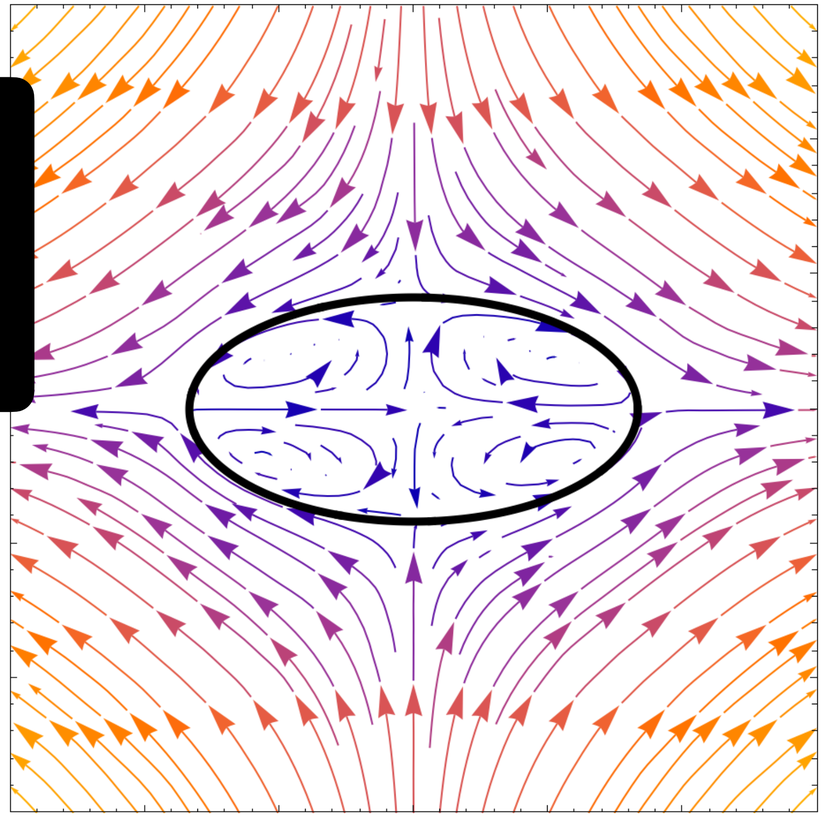
OBJECTIVE

$$D = D(Ca, \lambda)$$

- **Deformation parameter** $D = \dfrac{R_2 -}{R_1 + R_2}$

- **Viscosity ratio** $\lambda = \dfrac{\mu_1}{\mu_2}$

- **Capillary number** $Ca = \mu_2 \dfrac{\varepsilon L_{\text{ref}}}{\gamma}$



Elliptic equilibrium

Interface Stokes problem

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\end{aligned}
$$

Equilibrium

$+$ $\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma = 0$ on Γ

With $\mathbf{g} = \varepsilon(x, -y)^{\mathrm{T}}$

The velocity field can be decomposed: $\mathbf{u}(\varepsilon, \gamma) = \mathbf{u}(\varepsilon,0) + \mathbf{u}(0,\gamma)$

Fixed:

$$\lambda = \frac{\mu_1}{\mu_2}$$

$$D = \frac{R_2 - R_1}{R_1 + R_2}$$

Equilibrium

$\mathbf{u}(\varepsilon,0) \cdot \mathbf{n}_\Gamma = -\mathbf{u}(0,\gamma) \cdot \mathbf{n}_\Gamma$ on Γ

Elliptic equilibrium

Interface Stokes problem

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\begin{aligned}
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Fixed:

$$\lambda = \frac{\mu_1}{\mu_2}$$

$$D = \frac{R_2 - R_1}{R_1 + R_2}$$

Equilibrium

$$\mathbf{u}(\varepsilon,0) \cdot \mathbf{n}_\Gamma = -\,\mathbf{u}(0,\gamma) \cdot \mathbf{n}_\Gamma \text{ on } \Gamma$$

Equilibrium

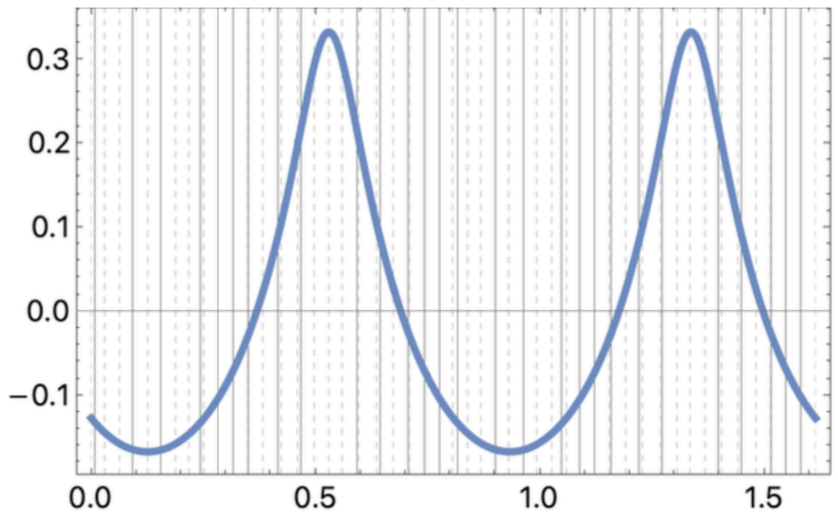
$$\mathbf{u}(0,\gamma = 1) \cdot \mathbf{n}_\Gamma = m\mathbf{u}(\varepsilon = 1,0) \cdot \mathbf{n}_\Gamma \text{ on } \Gamma$$

To be found

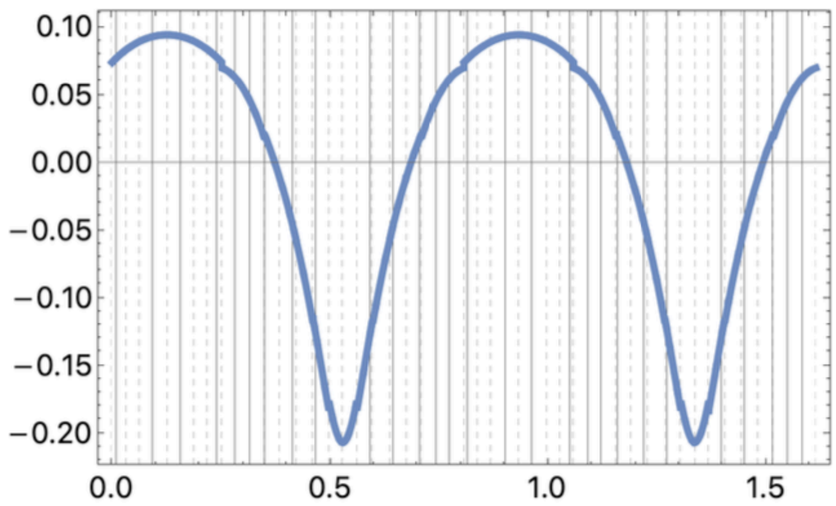
Elliptic equilibrium

$$\lambda = \frac{\mu_1}{\mu_2} = 1$$

$$D = \frac{R_2 - R_1}{R_1 + R_2} = 1/3$$



(a) $\boldsymbol{u}(\varepsilon = 1, 0) \cdot \boldsymbol{n}_\Gamma$.

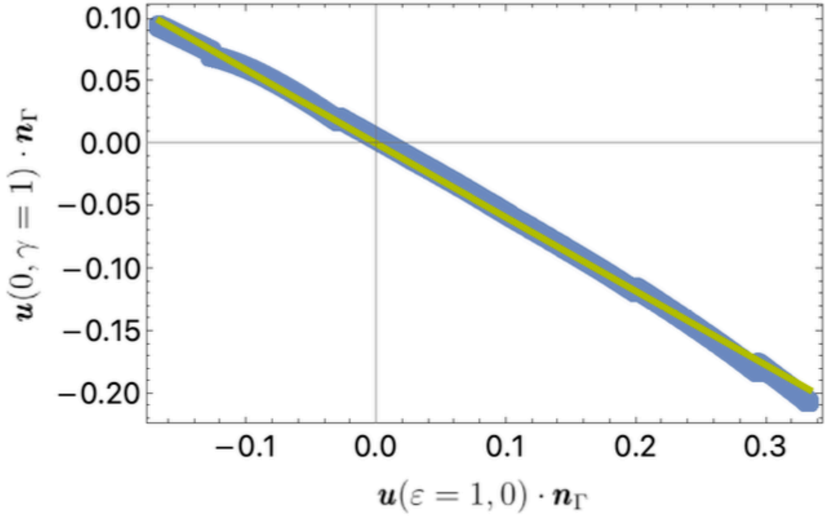


(b) $\boldsymbol{u}(0, \gamma = 1) \cdot \boldsymbol{n}_\Gamma$.

$$m \approx -0.59$$

$$\varepsilon_{\text{eq}} = -m, \gamma = 1$$

$$Ca = \mu_2 \frac{\varepsilon_{\text{eq}} L_{\text{ref}}}{\gamma} \approx 0.28$$



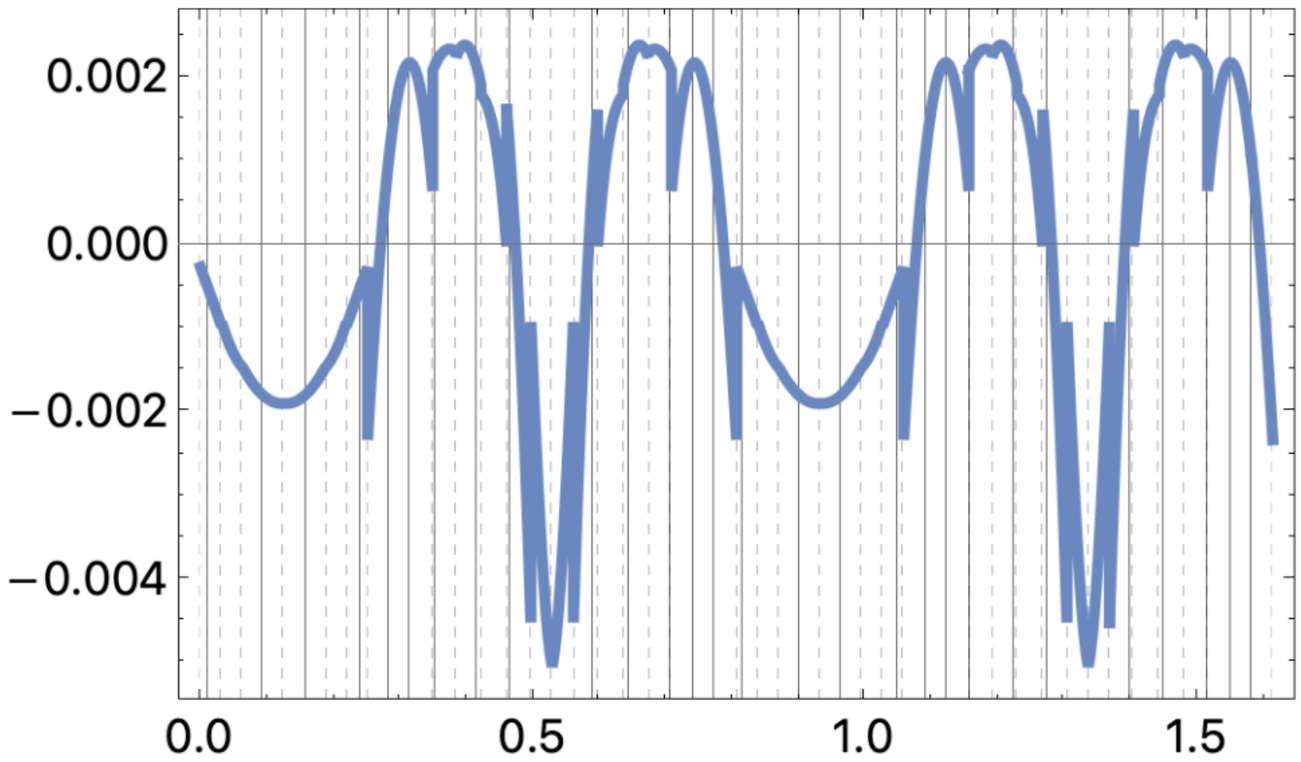
(c) $\boldsymbol{u}(0, \gamma = 1) \cdot \boldsymbol{n}_\Gamma$ w.r.t. $\boldsymbol{u}(\varepsilon = 1, 0) \cdot \boldsymbol{n}_\Gamma$

Elliptic equilibrium

$$\lambda = \frac{\mu_1}{\mu_2} = 1$$

$$D = \frac{R_2 - R_1}{R_1 + R_2} = 1/3$$

$Ca = 0.28$



Normal velocity **reduced by a factor 100** in the domain $\Omega = (-2,2)^2$

Notice:

- **Finer meshes** better approximate maximum curvature zones
- The **size of the domain** (if too small) can affect the shear-surface equilibrium

Elliptic equilibrium

Varying

$$\lambda = \frac{\mu_1}{\mu_2}$$

$$D = \frac{R_2 - R_1}{R_1 + R_2}$$

$\rightarrow$ D/Ca

λ	0.01	0.1	1	10	100
our work	1.06	1.08	1.18	1.24	1.25
ref. [6], see (1)	1.00	1.02	1.09	1.17	1.19

In good agreement with Taylor theoretical estimate $D = Ca\frac{19\lambda + 16}{16\lambda + 16}$ (1).

[6] G. I. Taylor. “The formation of emulsions in definable fields of flow”. *Proc. Roy. Soc. London Ser. A*. 1934.

Outline

- ☐ **Unfitted HHO method**
- ☐ **Interface Stokes problem**
  - Equilibrium droplet in shear flow
- ☑ **Unknown equilibrium interface**
  - Moving interface: **level-set scheme**

Research equilibrium interfaces

Fixed-point scheme

At any m

Given the interface Γ^m,
interface Stokes problem $(\mathbf{u}^m, p^m)$

Unfitted HHO method

Iteratively

Given the flow field $\mathbf{u}^m$,
interface evolution $\Gamma^m \to \Gamma^{m+1}$

Level-set transport problem

Convergence condition

$$\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma = 0 \text{ on } \Gamma$$

Research equilibrium interfaces

Fixed-point scheme

At any m

Given the interface Γ^m,
interface Stokes problem $(\mathbf{u}^m, p^m)$

Iteratively

Given the flow field $\mathbf{u}^m$,
interface evolution $\Gamma^m \to \Gamma^{m+1}$

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma}_i^m &= \boldsymbol{f}_i && \text{in } \Omega_i^m, \quad i \in \{1,2\},\\
\nabla \cdot \boldsymbol{u}_i^m &= 0 && \text{in } \Omega_i^m, \quad i \in \{1,2\},\\
\boldsymbol{u}_2^m &= \boldsymbol{g} && \text{on } \partial\Omega,\\
[\![\boldsymbol{u}^m]\!] = \mathbf{0}, \quad [\![\boldsymbol{\sigma}^m]\!] \cdot \boldsymbol{n}_\Gamma^m &= \gamma H^m \boldsymbol{n}_\Gamma^m && \text{on } \Gamma_{\mathcal{T}}^m.
\end{aligned}
$$

To be smoothed

Smoothed cell component HHO velocity

$$\boxed{\widetilde{\boldsymbol{f}}^m(\boldsymbol{x},t) := \boldsymbol{u}_{\mathcal{T}}^{m,c}(\boldsymbol{x})\widetilde{\Phi}_{\mathcal{T}}^m(\boldsymbol{x},t)}$$

$$
\begin{aligned}
\partial_t \widetilde{\Phi}_{\mathcal{T}}^m + \nabla \cdot \widetilde{\boldsymbol{f}}^m &= 0 && \text{in } \Omega \times (0, \Delta t^m),\\
\widetilde{\Phi}_{\mathcal{T}}^m(\boldsymbol{x}) &= \Phi_{\mathcal{T}}^0(\boldsymbol{x}) && \text{on } \partial\Omega_{\text{in}} \times (0, \Delta t^m),\\
\widetilde{\Phi}_{\mathcal{T}}^m(\boldsymbol{x}) &= \Phi_{\mathcal{T}}^0(\boldsymbol{x}) && \text{in } \Omega,
\end{aligned}
$$

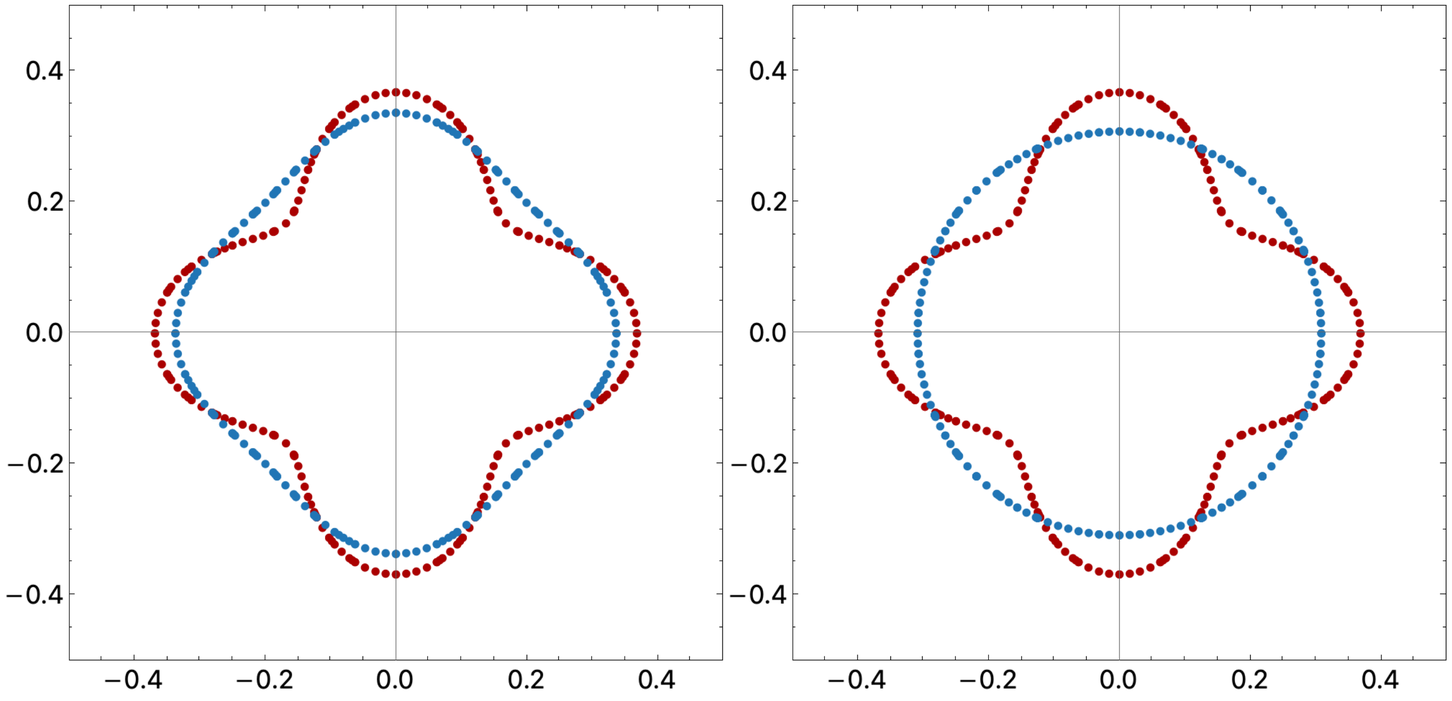
Solved using continuous FEM with graph viscosity [7]

[7] J. Guermond, M. De Luna, and T. Thompson. "A conservative anti-diffusion technique for the level set method." *IMA J. Numer. Anal.* 2017

Research equilibrium interfaces

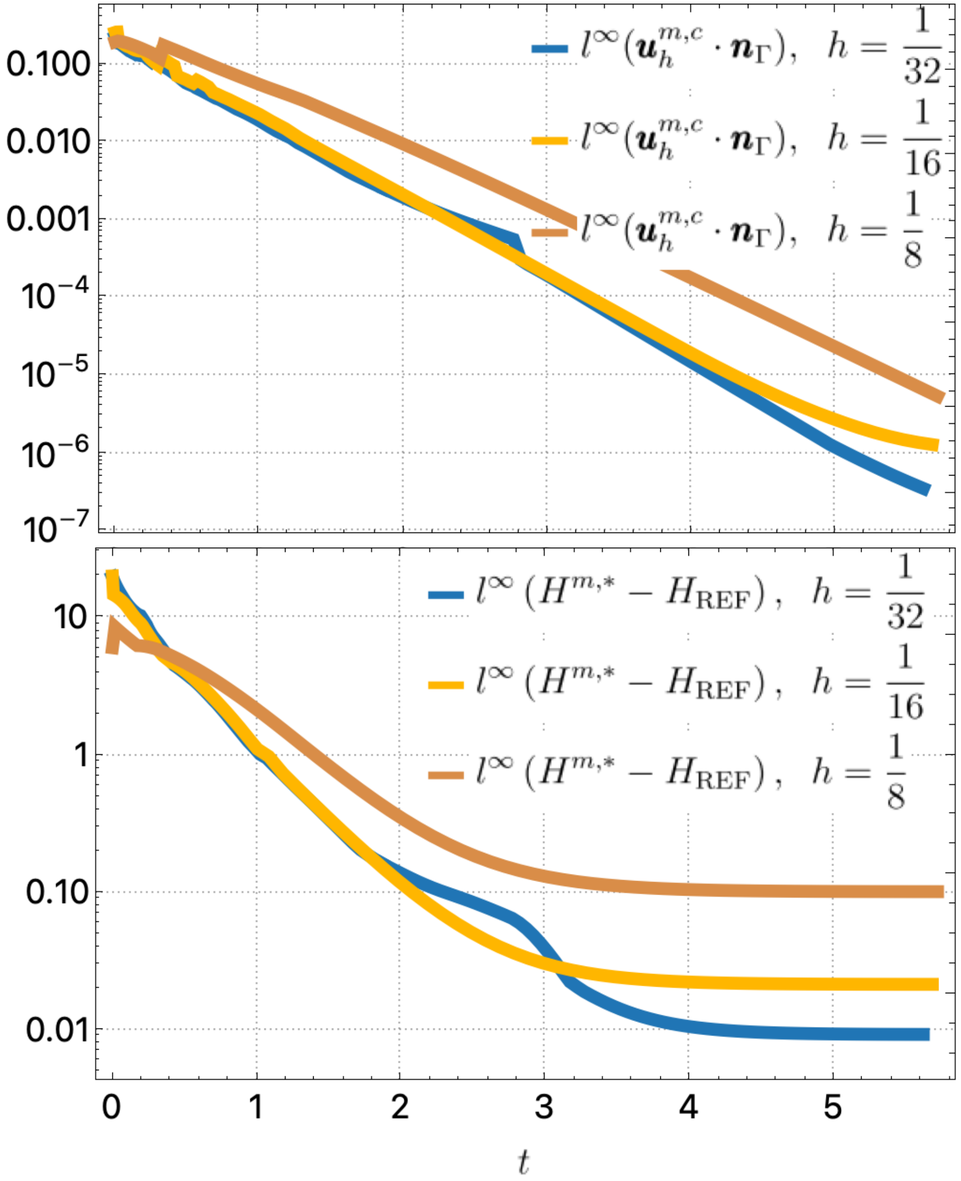
Circular equilibrium: $\mathbf{g} = \mathbf{0}$

Initial interface (red) - evolved interface (blue)



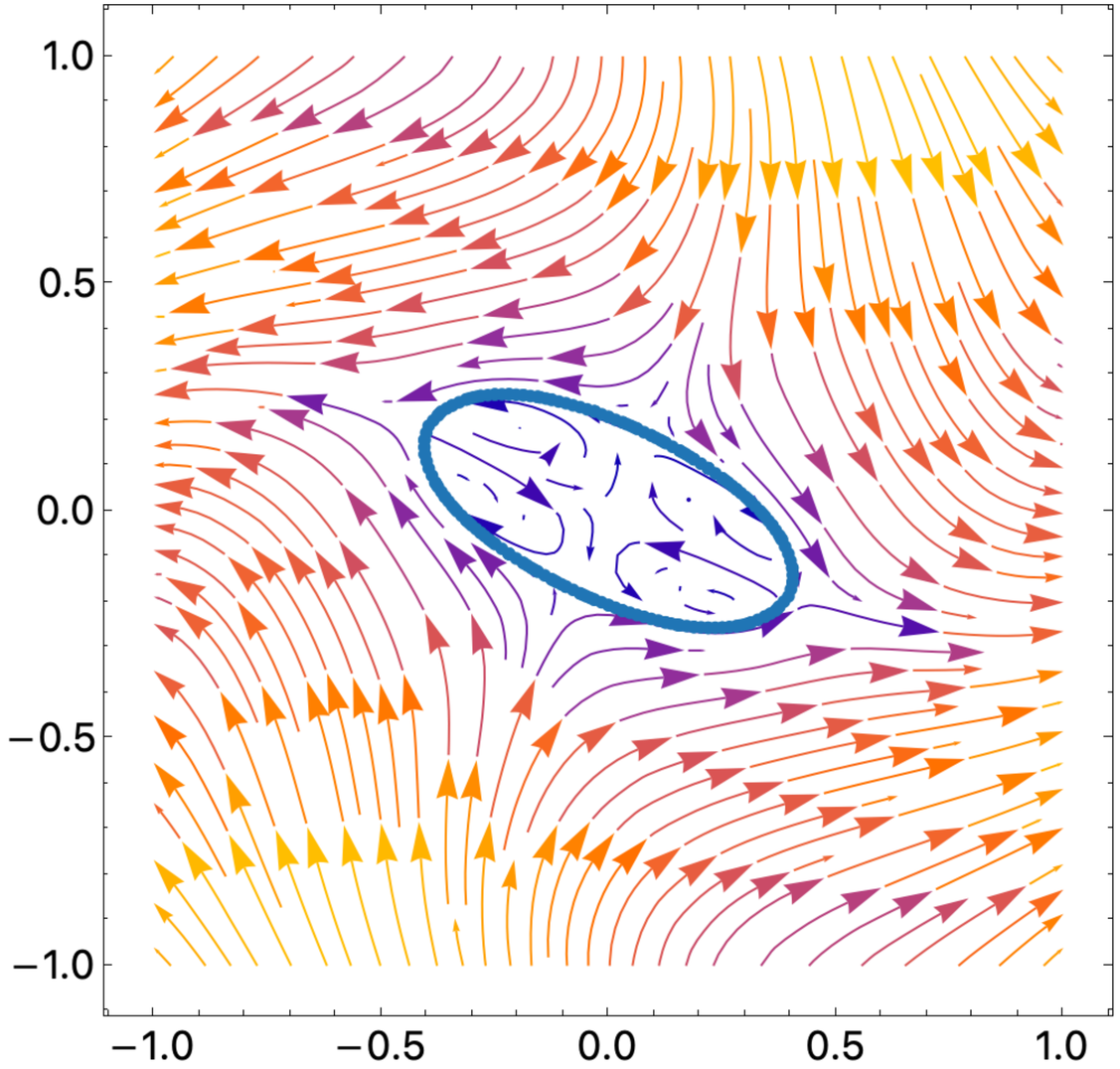
Iteration 50 Iteration 210

$$\Phi_{\mathcal{T}}^{m} \in \mathcal{P}_2^{\mathrm{c}}(\mathcal{T}), \quad \mathrm{HHO}(\mathrm{k}=1)$$

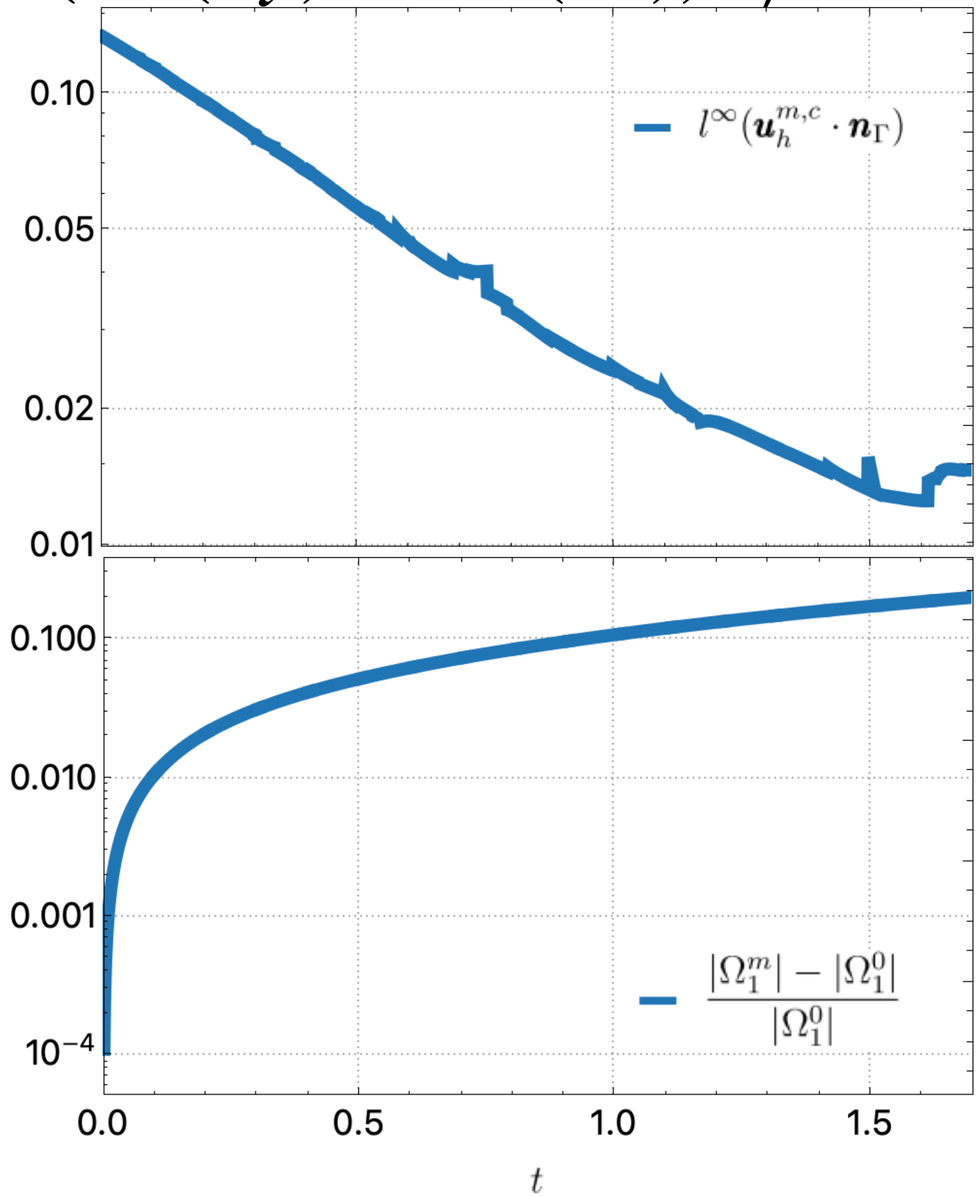


Research equilibrium interfaces

Unknown equilibrium: $\mathbf{g} = \varepsilon\left((x, -y)^{\mathrm{T}} + 0.5(sin(\pi y) + sin(\pi x))^{\mathrm{T}}\right)$



$\Phi_{\mathcal{T}}^{m} \in \mathcal{P}_2^{\mathrm{c}}(\mathcal{T}), \quad \mathrm{HHO(k=1)}$



Conclusions

Principal outcomes obtained

- Extension of the **unfitted HHO method** towards **evolving multi-phase problems**
- **Interface equilibrium Stokes**: fast algorithm to achieve elliptic equilibrium droplets
- Research of **unknown equilibrium** droplets in Stokes flows

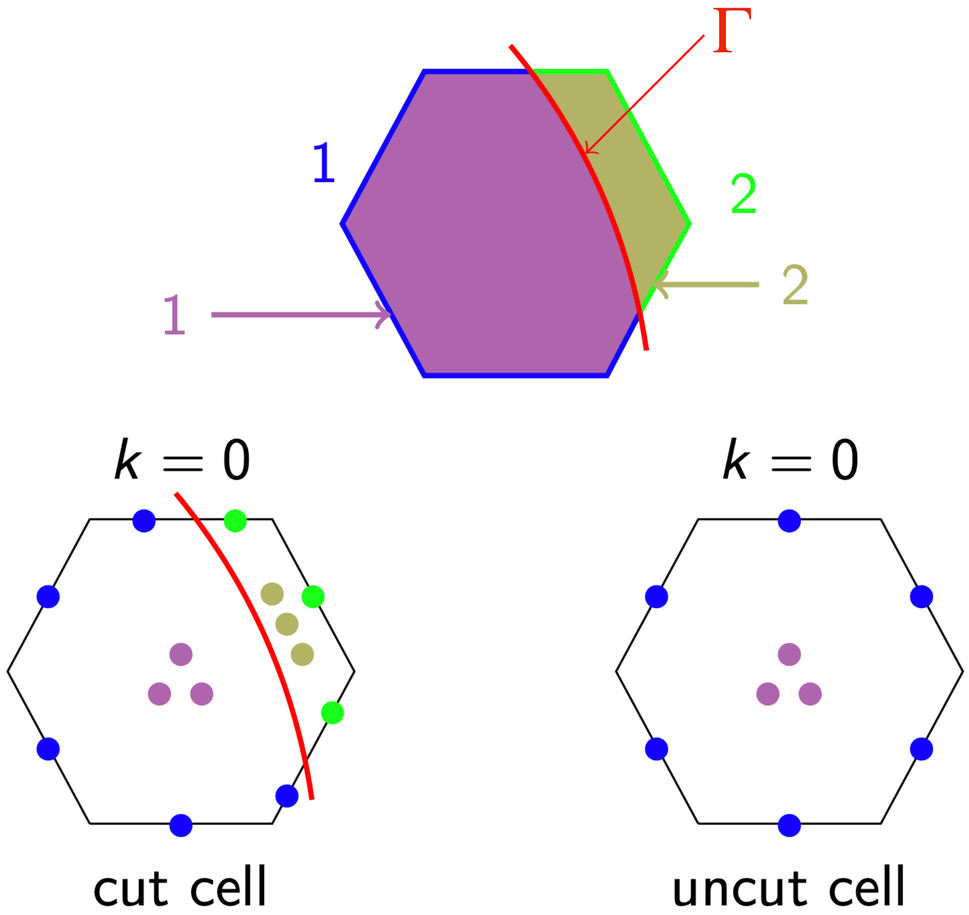
Future possible developments

- Improvement of the robustness with respect to the **mass loss** → level-set scheme employing divergence-free velocity fields
- Extension to 3D
- Extension to Navier-Stokes multi-phase problems

Thank you for your attention!!!

Unfitted HHO method

We consider polynomials of degree k on the faces and $(k+1)$ on the cells



Notice: doubling of the unknowns on cut cells/faces

Unfitted HHO - Interface Stokes

Velocity and pressure approximation

$$\widehat{\boldsymbol{v}}_T := (\boldsymbol{v}_{T^1}, \boldsymbol{v}_{T^2}, \boldsymbol{v}_{(\partial T)^1}, \boldsymbol{v}_{(\partial T)^2}) \in \widehat{\boldsymbol{U}}_T^k,$$
$$p_T := (p_{T^1}, p_{T^2}) \in P_T^k := \mathbb{P}_d^k(T^1) \times \mathbb{P}_d^k(T^2),$$

Gradient reconstruction $(\mathbb{E}_{T^i}^k(\widehat{\boldsymbol{v}}_T), \mathbf{q})_{T^i} := (\nabla^{\mathrm{s}} \boldsymbol{v}_{T^i}, \mathbf{q})_{T^i} + (\boldsymbol{v}_{(\partial T)^i} - \boldsymbol{v}_{T^i}, \mathbf{q}\boldsymbol{n}_T)_{(\partial T)^i} - \alpha_i([\![\boldsymbol{v}_T]\!], \mathbf{q}\boldsymbol{n}_T)_{T^\Gamma},$

Divergence reconstruction $D_{T^i}^k(\widehat{\boldsymbol{v}}_T) := \mathrm{trace}(\mathbb{E}_{T^i}^k(\widehat{\boldsymbol{v}}_T))$

Stabilizations $s_T^{1,2}(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) := \sum_{i \in \{1,2\}} \mu_i h_T^{-1} (\Pi_{(\partial T)^i}^k(\boldsymbol{v}_{T^i}) - \boldsymbol{v}_{(\partial T)^i}, \Pi_{(\partial T)^i}^k(\boldsymbol{w}_{T^i}) - \boldsymbol{w}_{(\partial T)^i})_{(\partial T)^i}$

$$s_T^\Gamma(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) := \mu_\# h_T^{-1} ([\![\boldsymbol{v}_T]\!], [\![\boldsymbol{w}_T]\!])_{T^\Gamma}, \qquad \mu_\# := \min(\mu_1, \mu_2),$$

$$a_T(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) := \sum_{i \in \{1,2\}} 2\mu_i (\mathbb{E}_{T^i}^k(\widehat{\boldsymbol{v}}_T), \mathbb{E}_{T^i}^k(\widehat{\boldsymbol{w}}_T))_{T^i} + s_T^\Gamma(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) + s_T^{1,2}(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) \qquad b_T(\widehat{\boldsymbol{w}}_T, r_T) := \sum_{i \in \{1,2\}} (r_{T^i}, D_{T^i}^k(\widehat{\boldsymbol{w}}_T))_{T^i}$$

$$A_T((\widehat{\boldsymbol{v}}_T, r_T), (\widehat{\boldsymbol{w}}_T, q_T)) := a_T(\widehat{\boldsymbol{v}}_T, \widehat{\boldsymbol{w}}_T) - b_T(\widehat{\boldsymbol{w}}_T, r_T) + b_T(\widehat{\boldsymbol{v}}_T, q_T), \qquad l_T(\widehat{\boldsymbol{w}}_T) := \sum_{i \in \{1,2\}} \left\{ (\boldsymbol{f}, \widehat{\boldsymbol{w}}_{T^i})_{T^i} + \alpha_i (\boldsymbol{g}_N, \widehat{\boldsymbol{w}}_{T^i})_{T^\Gamma} \right\},$$