Homogenization of stiff inclusions through network approximation

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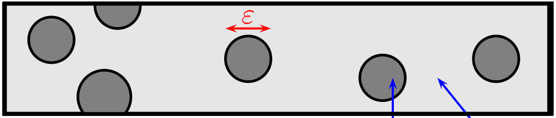
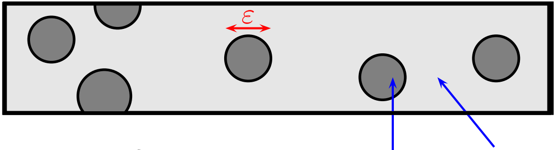
Introduction



- framework :study of electric potential in an *homogeneous medium* with random inclusions F_ε of *infinite conductivity* and small size ε
- a criterion ensuring homogenization was already given by Zhikov



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- a criterion ensuring homogenization was already given by Zhikov

Goal : relax this criterion using idea from network approximation.

$\Longrightarrow$ inspired by the work on network approximation of Leonid Berlyand [Berlyand and Kolpakov, 2001].

Microscopic problem

- $F(\omega) = \cup_{I \in \mathrm{CC}(F)} I(\omega) \subset \mathbb{R}^3$ is the **random set of inclusions**, $\mathrm{CC}=$ *Connected Components*
- domain $U \subset \mathbb{R}^3$
- $I_\varepsilon = \varepsilon I$ and $F_\varepsilon = \cup_{I \in \mathrm{CC}(F)} I_\varepsilon(\omega)$ such that "$\varepsilon I \subset U$"

Consider **electric potentials** $(u_\varepsilon)_{\varepsilon>0} \in H_0^1(U)$ such that

$$\begin{cases} -\Delta u_\varepsilon = f \quad \text{in} \quad U \setminus F_\varepsilon, \ \textit{conductivity} = \textit{1} \\ \nabla u_\varepsilon = 0 \quad \text{in} \quad F_\varepsilon \\ \displaystyle\int_{\partial I_\varepsilon} \partial_\nu u_\varepsilon = 0, \quad \forall I_\varepsilon \in \mathrm{CC}(F_\varepsilon) \\ u_\varepsilon = 0 \quad \text{on} \quad \partial U \end{cases} \tag{E_ε}$$

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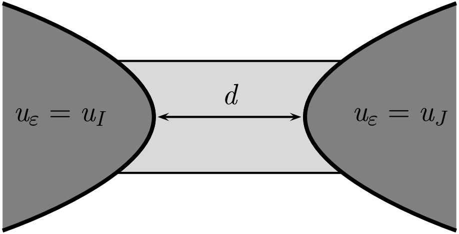
- u_ε equals a constant C_I in each inclusion, determined by **the zero-flux condition**

$\Longrightarrow$ The main goal is to pass to the limit when $\varepsilon \to 0$ (*homogenization result*)

Heuristic difficulty

Naive heuristic difficulty - *two inclusions set-up*



the conductivity u_ε must pass quickly from u_I to u_J
$\implies$ explosion of $\int |\nabla u_\varepsilon|^2$ in the contact zone $\approx |\ln d||u_I - u_J|^2$.

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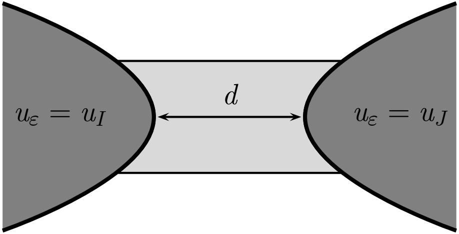

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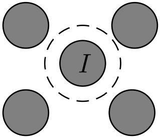
clusters/dense settings may prevent homogenization
(the energy might blows up, cf work of Berlyand. L).

Existing results

Well-separated inclusions $\Longrightarrow$ homogenization holds



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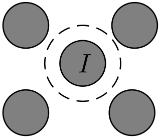

Theorem (Zhikov, [Jikov et al., 1994])

Assume that all the inclusions are balls of unit radius, and that, almost surely

$$\limsup_{N\to+\infty} \frac{1}{N^3} \sum_{I\subset Q_N} \mu_I < +\infty, \quad \textit{where} \quad \mu_I := |\ln d(I, F\setminus I)|, Q_N = [-N/2, N/2]^3 \qquad \text{(Z)}$$

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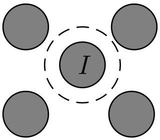

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Then $(u_\varepsilon)_\varepsilon$ converges weakly in $H^1_0(U)$ to u_0 solution of the ***effective problem***

$$\begin{cases} -\mathit{div}\, A_0 \nabla u_0 = (1-\lambda) f & \textit{in} \quad U \\ u_0 = 0 & \textit{on} \quad \partial U \end{cases} \qquad (E_0)$$

where A_0 is the ***effective conductivity matrix*** *and λ is the* ***density of the inclusions*** ***($\lambda = \mathbb{E}\left[1_F\right]$)***.

Effective conductivity

The effective conductivity matrix is constant and defined for any $\xi \in \mathbb{R}^3$:

$$A_0\xi \cdot \xi := \mathbb{E} \fint_{Q_1} |\xi + \nabla\phi_\xi|^2$$

where $\phi_\xi(\omega, x)$ is a random *corrector* which satisfies for almost all ω

$$\begin{cases} -\Delta\phi_\xi = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus F, \\ \nabla\phi_\xi = -\xi \quad \text{in} \quad F, \\ \displaystyle\int_{\partial I} \partial_\nu \phi_\xi = 0, \quad \forall I \in \mathrm{CC}(F). \end{cases} \tag{E_C}$$

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$\Longrightarrow$ <u>classic difficulty of stochastic homogenization</u> :

it is unclear that this problem is well-posed !

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Geometric Assumptions

We restrict to a specific class of inclusions

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The inclusions are regular and the gaps are well-separated

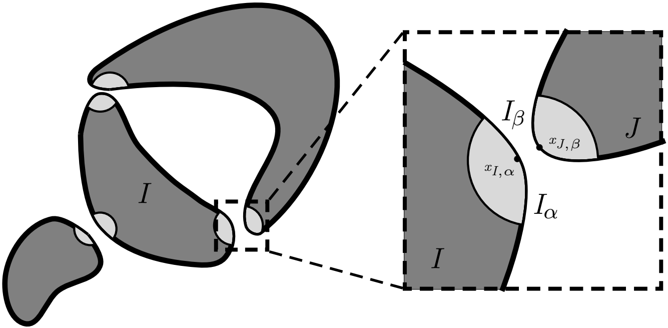


Figure – Geometry of the inclusions with a close-up on a gap.

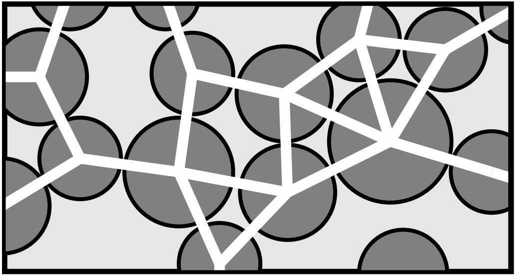
- no constraint so far on the inclusion's diameter !

Multigraph of the inclusions

Definition

The δ-multigraph of inclusions associated to F is the unoriented multigraph $Gr(F)$ with

- the vertices of $Gr(F)$ are the inclusions of F
- there is an edge between I and J for each δ-close gaps
- the corresponding edge e has a weight $\mu_e = |\ln|x_{I,\alpha} - x_{J,\beta}||$



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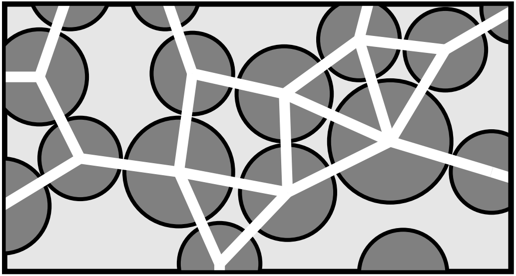


- **cluster** : the union of all the inclusions that are the nodes of a same connected component of $Gr(F)$

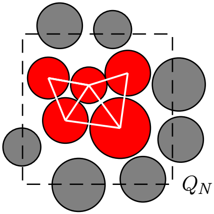
Discrete energy on the network

Goal : using the multigraph of inclusions, relax the Zhikov assumption.

Discrete energy : Let $F_N := \cup_{I \in \mathrm{CC}(F),\ I \subset Q_N} I$ and

- a family $\{u_I\}$ indexed by the vertices I
- a family $\{b_{IJe}\}$ indexed by vertices I, J and an edge e linking I to J

$$\boxed{\mathcal{E}\Big(F_N, \{u_I\}, \{b_{IJe}\}\Big) := \sum_{I,J,e} \mu_e |b_{IJe} - b_{JIe} + u_I - u_J|^2 + \sum_I |I||u_I|^2}$$



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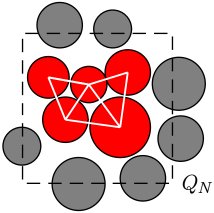


If $b_{IJe} = \xi \cdot x_I$, this energy is an upper bound for the energy $||u||_{H^1(Q_N)}$ where u that is **harmonic outside** F_N**, and satisfies** $u = u_I + \xi \cdot x_I$ **on each** $I \in \mathrm{CC}(F_N)$
$\Longrightarrow$ similar to ϕ_ξ !

Effective conductivity - corrector result

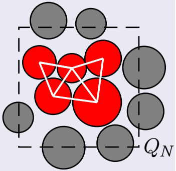
Let $x_I = \fint_I x$ and I_0 the inclusion that contains 0 (with $I_0 = \emptyset$ if $0 \notin F$).

Theorem ([Gérard-Varet and Girodroux-Lavigne, 2021])

If for any $\xi \in \mathbb{R}^3$, a.s,

$$\limsup_{N \to +\infty} \inf_{\{u_I\}} \frac{1}{|Q_N|} \mathcal{E}\left(F_N, \{u_I\}, \{b_{IJe} = \xi \cdot x_I\} \right) < +\infty, \qquad \text{(H1)}$$

*and $\mathbb{E}\,(\text{diam}\, I_0)^2 < +\infty$, then the corrector problem has a solution ϕ_ξ with good properties and **the conductivity matrix A_0 is well-defined.***



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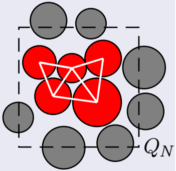

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- (H1) appears as a natural condition to get a uniform bound on the mean energy of the corrector over Q_N
- (Z) condition implies that $\limsup_{N\to+\infty} \frac{1}{|Q_N|}\mathcal{E}\Big(F_N, \{0\}, \{\xi \cdot x_I\}\Big) < +\infty$
- minimizing $\inf_{\{u_I\}} \mathcal{E}\Big(F_N, \{u_I\}, \{\xi \cdot x_I\}\Big) \implies$ solving the **weighted laplacian problem** on the graph $Gr(F_N)$: $(\Delta_{F_N} + I_d)\mathbb{U} = \mathbb{B}_\xi$

Corollary - bounded diameter condition

Let $\mathcal{C}_0$ the cluster that contains 0 ($\mathcal{C}_0 = \emptyset$ if $0 \notin F$). We can derive a convenient assumption on the inclusions

Corollary

If $\mathbb{E}\,(\textit{diam}\,\mathcal{C}_0)^2 < +\infty$ *then* (H1) *is verified.*

Proof. Take $u_I := -\xi \cdot (x_I - x_{\mathcal{C}})$. With this choice

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$$\mathcal{E}(F_N, \{u_I\}, \{\xi \cdot x_I\}) = \sum_{I \in \mathrm{CC}(F_N)} |I||u_I|^2 \lesssim \sum_{\substack{\mathcal{C} \\ \text{cluster of } F_N}} \sum_{I \in \mathcal{C}} |I||x_I - x_{\mathcal{C}}|^2$$

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Eventually, any F such that the $\mathbb{P}(0 \in \text{cluster of diameter } N) \leq 1/N^{3+\alpha}$ is admissible.

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Eventually, any F such that the $\mathbb{P}(0 \in \text{cluster of diameter } N) \leq 1/N^{3+\alpha}$ is admissible. Similar results were established recently in [Duerinckx and Gloria, 2021] for the Stokes problem.

Homogenization

Theorem ([Gérard-Varet and Girodroux-Lavigne, 2021])

We assume that there exists $s \in (2, +\infty)$, such that F satisfies almost surely

$$\limsup_{N \to +\infty} \frac{1}{|Q_N|} \sup_{||b_{IJe}||_s = 1} \inf_{\{u_I\}} \mathcal{E}\Big(F_N, \{u_I\}, \{b_{IJe}\}\Big) < +\infty, \tag{H2}$$

where $||b_{IJe}||_s^{s/2} = \frac{1}{|Q_N|} \sum_{I,J,e} |b_{IJe}|^s$, then there exists $p(s)$ such that if $\mathbb{E}\,(\mathit{diam}\, I_0)^p < +\infty$, then homogenization holds.

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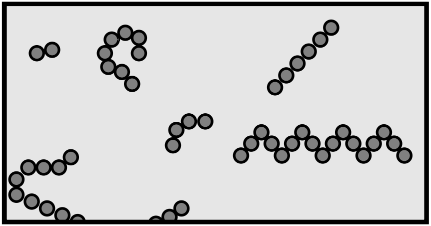
- (H2) $\implies$ (H1)
- (H2) is implied by a slightly improved Zhikov condition
- *Proof : modified oscillatory test function method*

Corollary - cycle-free inclusions

Corollary

Assume that $Gr(F)$ is ***cycle-free****, $\sup_I I < +\infty$ and that $\mathbb{E}\,\sharp\mathcal{C}_0^p < +\infty$ with $p > 2$, then, F satisfies (H2) with exponent $s = \frac{2p}{p-2} \in (2, +\infty)$.*



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Thank you for listening !

Precised corrector result

Theorem ([Gérard-Varet and Girodroux-Lavigne, 2021])

If almost surely,

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where $F_N := \cup_{I\in \mathrm{CC}(F),\ I\subset Q_N} I$*, and if* $\mathbb{E}\,\mathit{diam}(I_0)^2 < +\infty$*, then there exists a scalar field* $\phi_\xi(\omega, x) \in L^2(\Omega, H^1_{loc}(\mathbb{R}^3))$ *with stationary gradient s.t*

i) $\nabla\phi_\xi(\omega,\cdot)$ *satisfies* (E_C) *a.s*

ii) ϕ *is sub-linear, in the sense that* $\varepsilon\phi(\cdot/\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$ *in* $L^s_{loc}(\mathbb{R}^3)$ *for any* $s < 6$

iii) $\mathbb{E}\int_{Q_1} \nabla\phi_\xi = 0$, $\mathbb{E}\int_{Q_1} |\nabla\phi_\xi|^2 < +\infty$, $\int_{Q_1} \phi_\xi = 0$

iv) *up to a constant,* ϕ_ξ *is the unique minimizer of the variational problem*

$$\inf\left\{\mathbb{E}\int_{Q_1} |\nabla\phi + \xi|^2,\ \phi \in L^2(\Omega, H^1_{loc}(\mathbb{R}^3)),\ \nabla\phi\ \textit{stationary},\ \nabla\phi + \xi|_F = 0,\ \mathbb{E}\nabla\phi = 0\right\}$$

Sketch of the homogenization proof

- Step 1 : thanks to (H2), we prove the following lemma of extension outside the inclusion , using local surgery on the gaps

Lemma

Almost surely, there exists $C > 0$ *independent of* ε *such that for all* $\varphi^\varepsilon \in W^{1,s}(F_\varepsilon)$*, one can find a field* $\phi^\varepsilon \in H_0^1(U)$ *such that*

$$\nabla \phi^\varepsilon = \nabla \varphi^\varepsilon \quad \textit{in} \;\; F_\varepsilon, \quad \|\nabla \phi^\varepsilon\|_{L^2(U)} \leq C \|\nabla \varphi^\varepsilon\|_{L^s(F_\varepsilon)}, \;\; 2 \leq s < +\infty$$

- step 2 : by a duality argument, we obtain an extension theorem for divergence-free vector outside the inclusions , that we use to extend ∇u_ε.
- step 3 : we introduce for $\varphi \in C_0^\infty(U)$ the *oscillatory test function* $\varphi^\varepsilon(x) := \varphi(x) + \varepsilon \sum_i \phi_{e_i}(x/\varepsilon)\partial_i \varphi(x)$ and we want to use it as a test function in (E_ε)
- step 4 : since $\nabla \varphi^\varepsilon = \varepsilon \sum_i \nabla \partial_i \varphi(x) \phi_{e_i}(x/\varepsilon) \neq 0$ in F_ε, we correct it using the lemma
- step 5 : passing to the limit using ergodic's theorem, div/curl lemma and sublinearity of the corrector.