Mechanical modeling with FreeFEM

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Signorini's contact problem [Signorini, 1933]

- The deformation of a body $\Omega \subset \mathbb{R}^3$ is described by the application $\boldsymbol{\phi} : \Omega \rightarrow \mathbb{R}^3$.
- The displacement field: $\boldsymbol{u} = \boldsymbol{\phi}(\boldsymbol{X}) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}$

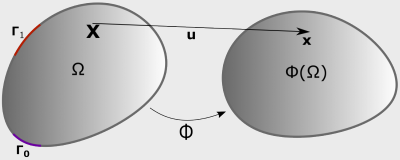

Figure: Initial and actual configurations

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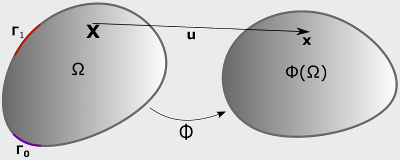


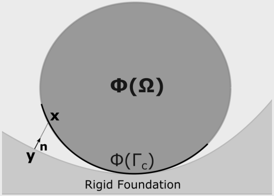
Figure: Initial and actual configurations

- Balance equations:

$$\begin{cases} \nabla.\boldsymbol{\sigma} + \mathbf{f} = 0 & \text{in } \Omega \\ \boldsymbol{\sigma} = C\epsilon & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_0 \\ \boldsymbol{\sigma}.\mathbf{n} = \mathbf{t} & \text{on } \Gamma_1 \end{cases} \quad (1)$$

- Contact conditions:

$$\begin{cases} g := (\mathbf{x} - \mathbf{y}).\mathbf{n} \geq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}.\mathbf{n}).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n.g = 0 & \text{on } \Gamma_C \end{cases} \quad (2)$$



Signorini's problem formulation

Linear elasticity

- The admissible set:

$$\mathbf{V} = \left\{ \mathbf{v} \in \left(H^1(\Omega) \right)^3 \;;\; \mathbf{v} = 0 \text{ on } \Gamma_0 \right\}$$

- Constitutive law of linear elasticity:

$$\boldsymbol{\sigma} = C\boldsymbol{\epsilon} \quad \text{where} \quad \boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla^T \boldsymbol{u} + \nabla \boldsymbol{u} \right) \tag{3}$$

- The total potential energy is defined by: $\mathcal{E}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v})$ where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega C\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\, dV \quad \text{and} \quad f(\mathbf{v}) = \int_\Omega \mathbf{f}.\mathbf{v}\, dV + \int_{\Gamma_1} \mathbf{t}.\mathbf{v}\, dA \tag{4}$$

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- The non-penetration set:

$$\mathbf{K} = \{ \mathbf{v} \in \mathbf{V} \; ; \; (\mathbf{x} - \mathbf{y}).\mathbf{n} \geq 0 \quad \forall \, \mathbf{x} \in \Phi(\Gamma_c) \} \tag{5}$$

- The displacement field $\boldsymbol{u}$ is a solution of the constrained minimization problem:

$$\mathbf{u} = \underset{v \in \mathbf{K}}{\operatorname{argmin}} \, \mathcal{E}(\mathbf{v}) \tag{6}$$

- Variational inequality (linear elasticity):

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall \, \mathbf{v} \in \mathbf{K} \tag{7}$$

Discretization

- Consider the following space

$$\mathbf{V}_h = \left\{ \mathbf{v} = (v_1, v_2) \in C^0(\Omega_h) \times C^0(\Omega_h) \mid \mathbf{v}_{|T_i} \in P_r \times P_r,\ \forall i = 1, \ldots, n_T \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0 \right\} \quad (8)$$

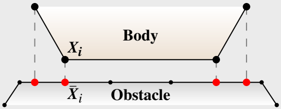
- Let $\mathbf{u}_h \in \mathbf{V}_h$, the displacement vector field on the mesh Ω_h, given by

$$\mathbf{u}_h = \sum_i \begin{pmatrix} U_i^x \\ U_i^y \end{pmatrix} \hat{w}_i \quad (9)$$

- The contact conditions:

$$(\mathbf{x}_i - \bar{\mathbf{x}}_i).\mathbf{n}_i \geq 0 \quad \forall\, i = 1, \ldots, nC$$

with $\mathbf{x}_i = \mathbf{X}_i + \mathrm{U}_i$ and $\bar{\mathbf{x}}_i = \bar{\mathbf{X}}_i$



- Another contact formulations describing the non-penetration, can be found in [Houssein et al., 2022, Houssein, 2022].

The Discretized problem

- The contact problem becomes

Small deformations

$$\hat{\mathbf{U}} = \begin{cases} argmin(\mathcal{E}_p(\mathbf{U})) \text{ s.t} \\ (\mathbf{x}_i - \bar{\mathbf{X}}_i).\mathbf{n}_i \geq 0 \quad \forall\, i = 1, \ldots, nC \end{cases}$$

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Large deformations case

Algorithm 2 Fixed point algorithm

while *error* $\geq \epsilon_{tol}$ **do**

1. Taking $\mathbf{x}^n = \mathbf{X}^n + \hat{\mathbf{U}}^n$

$$\hat{\mathbf{U}}^{n+1} = \begin{cases} argmin(\mathcal{E}_p(\mathbf{U})) \text{ s.t} \\ (\mathbf{x}_i - \bar{\mathbf{x}}_i^n).\mathbf{n}_i^n \geq 0 \quad \forall\, i = 1, \dots, nC \end{cases}$$

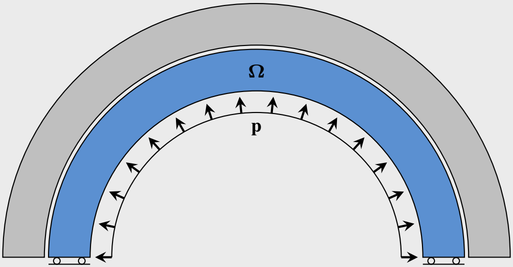
2. error$=\dfrac{\|\hat{\mathbf{U}}^{n+1} - \hat{\mathbf{U}}^n\|_\infty}{\|\hat{\mathbf{U}}_n\|_\infty}$

end while

- The minimization is done using the interior point method [Houssein et al., 2022, Houssein, 2022].

Contact algorithm example using IPOPT [Wächter and Biegler, 2006] in FreeFEM [Hecht, 2012]

- **Contact between an elastic arch and a rigid one**

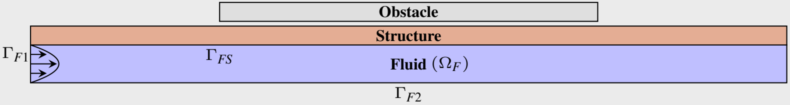


The scripts

- The **geometry**
- The linear elasticity **law**
- The contact **solver**

Fluid-structure interaction and contact

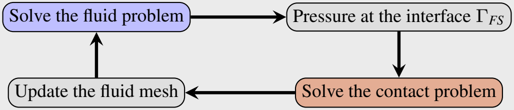
- The Geometry



- The Stokes system for the fluid (**u** is the velocity, p the pressure, $\mu = 1$)

$$\begin{cases} \mu\Delta\mathbf{u} - \nabla.p + \rho_F\mathbf{g} = 0 & \text{in } \Omega_F \\ \nabla.\mathbf{u} = 0 & \text{in } \Omega_F \\ \mathbf{u} = \mathbf{u}_\Gamma & \text{on } \Gamma = \Gamma_{F1} \cup \Gamma_{F2} \cup \Gamma_{FS} \end{cases} \tag{10}$$

- Strongly partitioned coupling scheme



Fluid-structure interaction and contact

- Force $f_S = \rho_S.g = 900.g$ and $\rho_F = 1000Kg/m^3$
- Young's modulus on $E_S = 4 \times 10^8$ Pa
- Velocity on Γ

$$\mathbf{u}_\Gamma = \begin{cases} \mathbf{0} & \text{on } \Gamma_{F2} \cup \Gamma_{FS} \\ 0.1(1 - \frac{(y-0.001)^2}{0.001^2})\mathbf{i} & \text{on } \Gamma_{F1} \end{cases} \tag{11}$$

The deformations

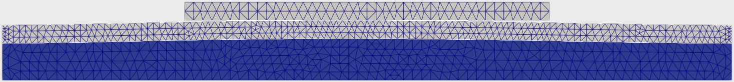


Figure: Actual configurations

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THANK YOU FOR YOUR ATTENTION