# A unified shallow water/Dupuit-Forchheimer approach to solve large scale Aquifer flows.

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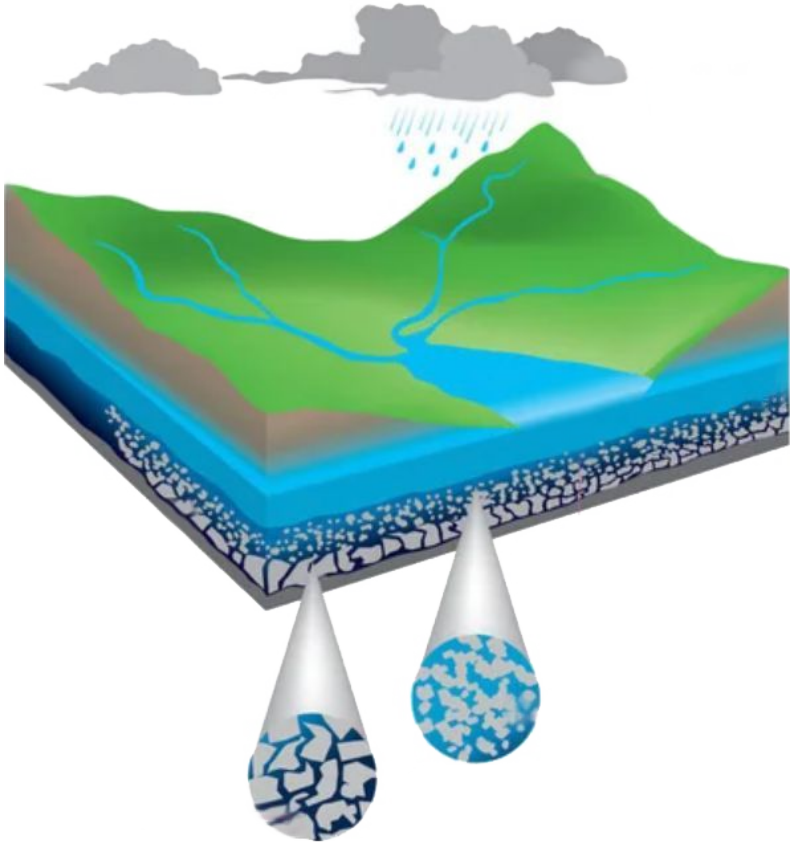
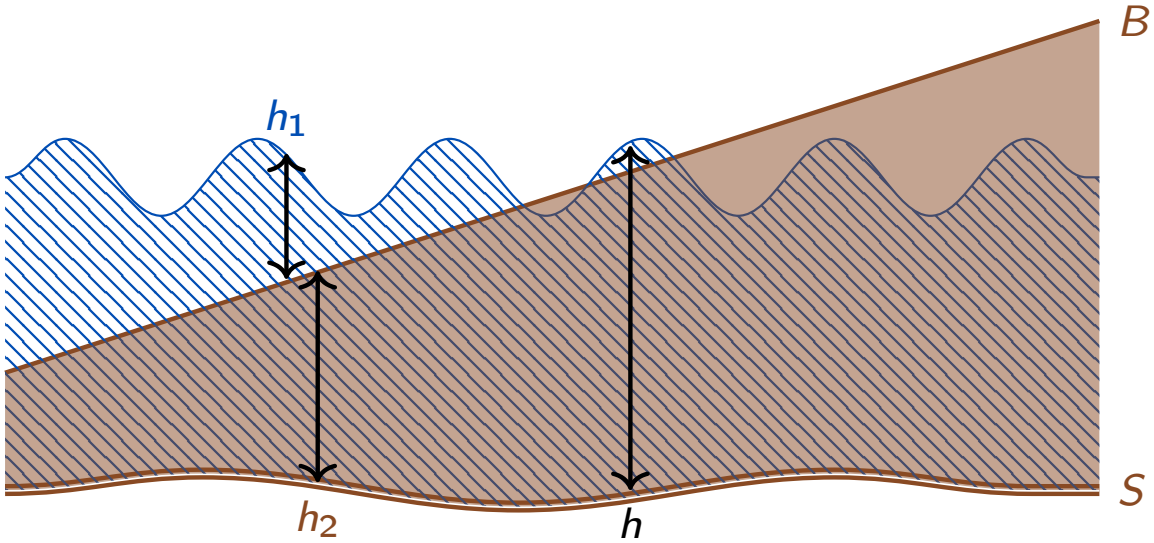
- free flow surface on a porous media ;
- large aquifer ;
- long time simulations ;
  - flooding,
  - sand beach,
  - water infiltration,
- more complicated topology ;
  - multi-layers free surface/porous media,
  - congestion, roof and floating body,
  - confined aquifer.



$$\begin{cases} \partial_t V + \nabla \cdot (V_1 \bar{\mathrm{u}} - \bar{\kappa} V_2 \nabla \phi) = 0 \\ \partial_t (V_1 \bar{\mathrm{u}}) + \nabla \cdot (V_1 \bar{\mathrm{u}} \otimes \bar{\mathrm{u}}) = -V_1 \nabla \phi - \dfrac{\bar{\mathrm{u}}}{2} (G)_+ \\ G = -(\partial_t V_1 + \nabla \cdot (V_1 \bar{\mathrm{u}})) \end{cases}$$




The unknowns :

- $\phi$, the potential ;
- $\bar{\mathrm{u}}$, horizontal velocity.

Secondary variables :

- $h = {}^{\phi}/_{g}$-S ;
- $h_2 = \min(h, B - S)$ ;
- $h_1 = h - h_2$.
- $\kappa$ hydraulic conductivity
- $V_1 = \bar{s}_1 h_1$ ;
- $V_2 = \bar{s}_2 h_2$ ;
- $V = V_1 + V_2$.
- $\overline{V} = \bar{s}_2 (B - S)$

Initial data : $h(0, x) = h_0(x)$ and $\bar{\mathrm{u}}(0, x) = u_0(x)$.

$$(E) \xrightarrow{\epsilon << 1} (SW_2) \xrightarrow{\overline{\kappa} << 1} (SW/DF)$$

- Can also be integrate from the Navier-Stokes model ;
- **mono-valuated** free-surface ;
  - $\eta$ the water table and $\overline{s}$ the space that the water can occupied
- $V_1(t,x) = \int_{B(x)}^{\infty} s(t,x,z)\,\mathrm{d}z$ and $V_2(t,x) = \int_{S(x)}^{B(x)} s(t,x,z)\,\mathrm{d}z$,
- $\overline{\mathrm{u}}_1(t,x) = \frac{1}{V_1}\int_{B}^{\infty} s\mathrm{u}\,\mathrm{d}z$ and $\overline{\mathrm{u}}_2(t,x) = \frac{1}{V_2}\int_{S}^{B} s\mathrm{u}\,\mathrm{d}z$,
- uniform horizontal velocity ;

$$\begin{aligned}
\partial_t s + \nabla\cdot(s\mathrm{u}) + \partial_z(sw) &= 0\\
s\left(\partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} + w\partial_z\mathrm{u}\right) &= s\left(-\frac{\nabla p}{\rho} - \frac{\mathrm{u}}{\kappa}\right)\\
s\left(\partial_t w + \mathrm{u}\cdot\nabla w + w\partial_z w\right) &= s\left(-\frac{\partial_z p}{\rho} - \frac{w}{\kappa} - g\right).
\end{aligned}$$

$$(E) \xrightarrow{\epsilon << 1} (SW_2) \xrightarrow{\overline{\kappa} << 1} (SW/DF)$$

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- uniform horizontal velocity ;

$$\partial_t s + \nabla\cdot(s\mathrm{u}) + \partial_z(sw) = 0$$

$$\downarrow$$

$$\begin{aligned} \partial_t V_1 + \nabla\cdot(V_1\overline{\mathrm{u}}_1) &= -G \\ \partial_t V_2 + \nabla\cdot(V_2\overline{\mathrm{u}}_2) &= G \end{aligned}$$

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- uniform horizontal velocity ;
- **hydrostatic** pressure ;
  - $p = \rho g(\eta - z) + O(\varepsilon^2)$,
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$$\downarrow$$

$$\begin{aligned} \partial_t(V_1\overline{\mathrm{u}}_1) + \nabla\cdot(V_1\overline{\mathrm{u}}_1\otimes\overline{\mathrm{u}}_1) &= -gV_1\nabla(h+S) - \frac{V_1\overline{\mathrm{u}}_1}{\overline{\kappa}_1} - \mathrm{u}_B G \\ \partial_t(V_2\overline{\mathrm{u}}_2) + \nabla\cdot(V_2\overline{\mathrm{u}}_2\otimes\overline{\mathrm{u}}_2) &= -gV_2\nabla(h+S) - \frac{V_2\overline{\mathrm{u}}_2}{\overline{\kappa}_2} + \mathrm{u}_B G \end{aligned}$$

$$(E) \xrightarrow{\epsilon << 1} (SW_2)^1 \xrightarrow{\overline{\kappa} << 1} (SW/DF)$$

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1. Audusse, Emmanuel et al. "A multilayer Saint-Venant system with mass exchanges for shallow water flows.

$$(E) \xrightarrow{\epsilon << 1} (SW_2) \xrightarrow{\overline{\kappa} << 1} (SW/DF)$$

- $\overline{s}(x,z) = \begin{cases} \overline{s}_1(x) & \text{if } B \leq z, \\ \overline{s}_2(x) & \text{if } S \leq z < B. \end{cases}$
- $\overline{\kappa}_1(x) = \infty \quad \text{and} \quad \overline{\kappa}_2(x) = O(K_2)$
- $V_2 = \sum_{i=0}^{\infty} K_2^i V_2^{(i)}, \quad \overline{\mathrm{u}}_2 = \sum_{i=0}^{\infty} K_2^i \overline{\mathrm{u}}_2^{(i)} \quad \text{and} \quad G = \sum_{i=0}^{\infty} K_2^i G^{(i)}$

Main Order

$$\kappa_2 \left( \partial_t \left( V_2^{(0)} \overline{\mathrm{u}}_2^{(0)} \right) + \nabla \cdot \left( V_2^{(0)} \overline{\mathrm{u}}_2^{(0)} \otimes \overline{\mathrm{u}}_2^{(0)} \right) \right) = -g\kappa_2 V_2^{(0)} \nabla \left( h^{(0)} + S \right) - V_2^{(0)} \overline{\mathrm{u}}_2^{(0)} + \kappa_2 \mathrm{u}_B^{(0)} G^{(0)}$$

$$\boxed{\overline{\mathrm{u}}_2^{(0)} = 0}$$

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Second Order

$$0 = -gV_2^{(0)}\nabla\left(h^{(0)} + S\right) - \frac{V_2^{(0)}\overline{\mathrm{u}}_2^{(1)}}{\kappa_2^{(1)}} + \mathrm{u}_B^{(0)} G^{(0)}$$

$$\partial_t V_2^{(0)} = G^{(0)}$$

$$\overline{\kappa}_2^{(1)} = \frac{K_2}{\overline{\kappa}_2}$$

$$V_2\overline{\mathrm{u}}_2 = -g\overline{\kappa}_2 V_2 \nabla(h+S) + O(K_2)$$

$$G^{(0)} = 0$$

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Energy dissipation law

$$\partial_t(\mathcal{E} + V_1 \frac{|\overline{u}_1|^2}{2}) + \nabla(\phi(V_1\overline{u}_1 - \overline{\kappa}_2 V_2 \nabla\phi) + V_1\overline{u}_1 \frac{|\overline{u}_1|^2}{2}) \leq -\kappa_2 V_2 |\nabla\phi|^2$$

- stability of the model

CPR scheme [2]

- IMEX-scheme
- stable & consistent
- asymptotic preserving

2. Edwige Godlewski et al. "Congested shallow water model: on floating body". In : SMAI Journal of Computational Mathematics (2021). doi : 10.5802/smai-jcm.67. url : https://hal.inria.fr/hal-01871708.

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CFL

$$\frac{\delta_t}{\delta_x} \sum_{f \in \mathbb{F}_k} \left( F_{1,f} \cdot N_k^{k_f} \right)^- < \frac{V_{1,k}^{SW}}{2}.$$

- $F_{1,f}$ the shallow water flux at the interfaces
- $N_k^{k_f}$ normal from the face $k$ to $k_f$
- $\delta_t$ time step
- $\delta_x$ space interval
- $V_{1,k}^{SW}$ shallow water volume explicitly calculate with only shallow water flux

Energy dissipation law

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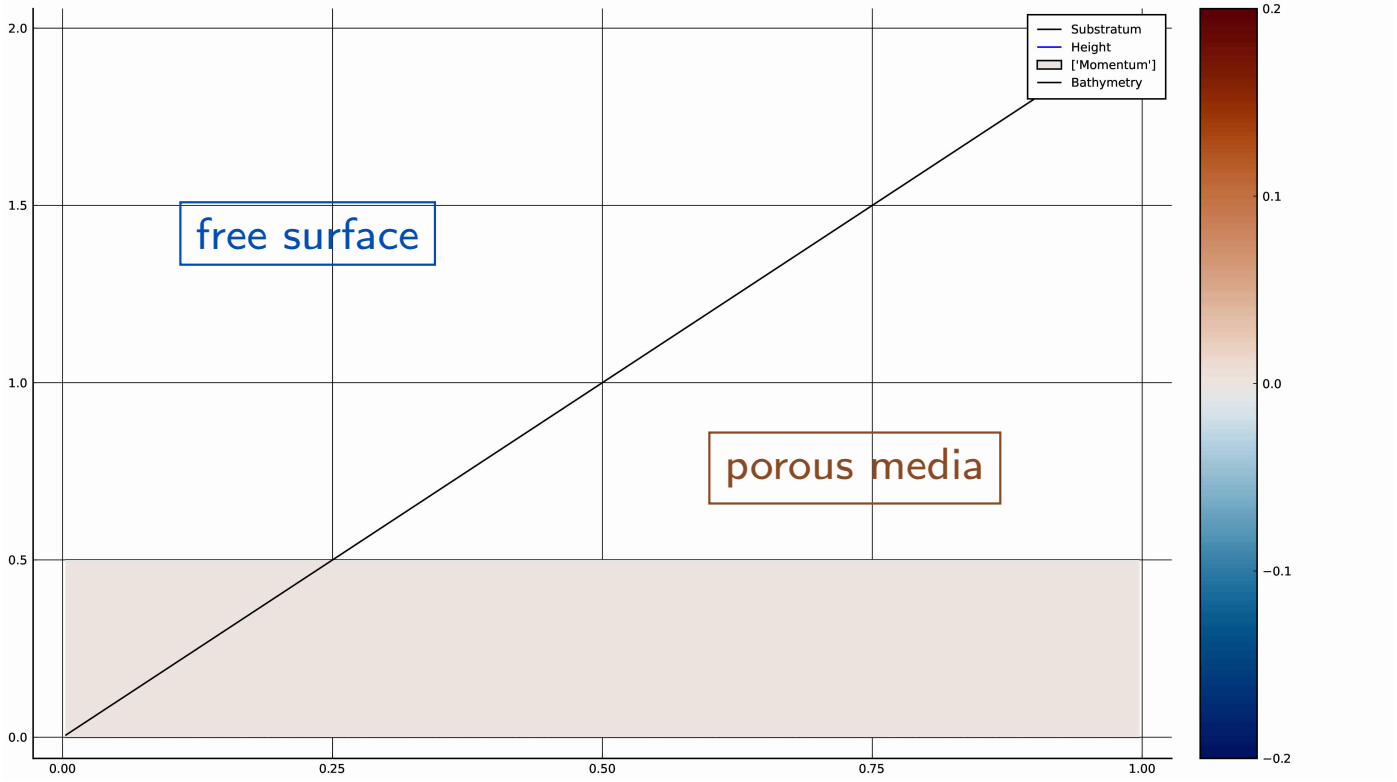
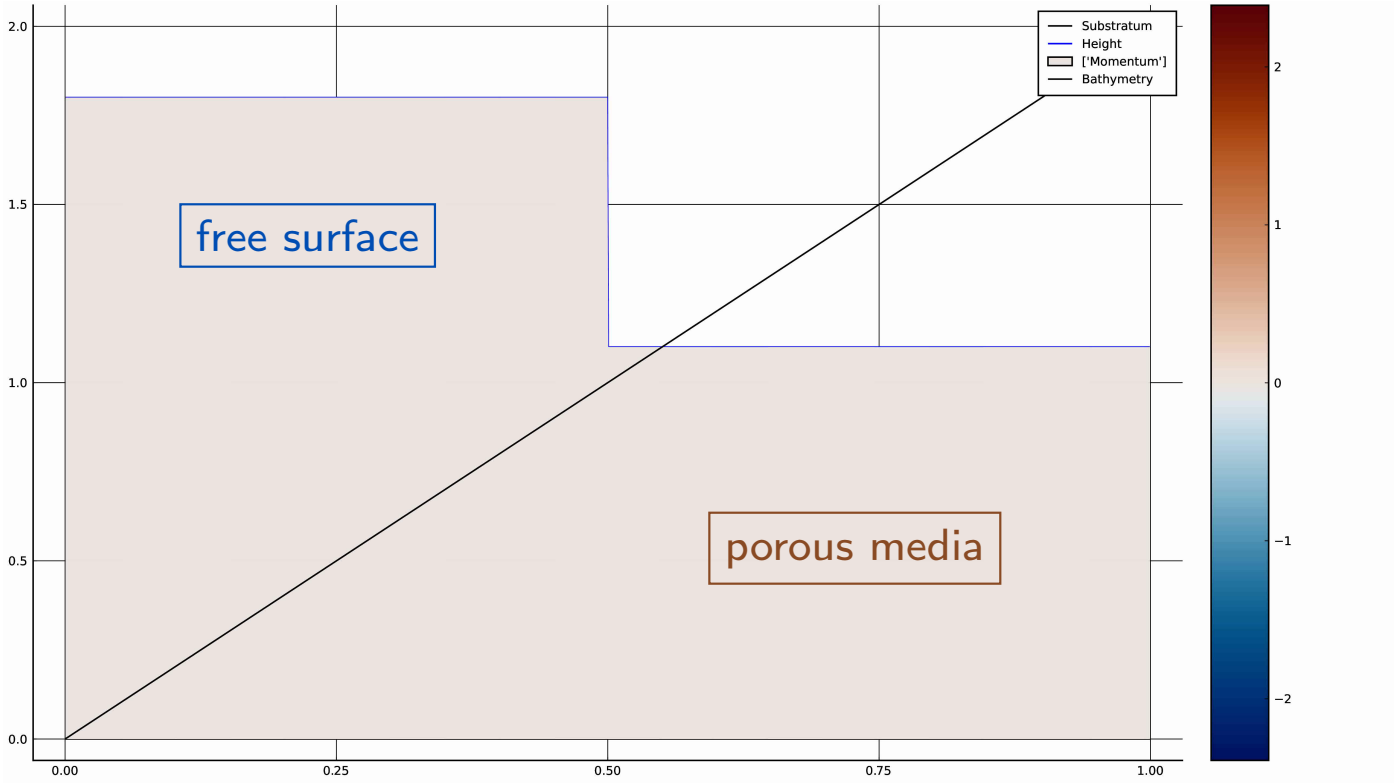
when $V_{1,k}^{SW} \longrightarrow 0$




- $B(x) = 2x$
- $S(x) = 0$
- $h_0(x) = 0.5$
- $T^{max} = 5000$
- $\delta_x = 5.10^{-3}$
- $s_2 = 0.5$
- $h(1,t) = 1.2 + 0.3\frac{\cos(\pi t)}{5000/2}$
- $h(0,t)$ : wall
- restriction on $\delta_t$




- $B(x) = 2x$
- $S(x) = 0$
- $h_0(x) = \begin{cases} 1.8 \text{ if } x < 0.5 \\ 1.1 \end{cases}$

- $T^{max} = 5000$
- $\delta_x = 5.10^{-3}$
- $s_2 = 0.8$
- $\gamma_{CPR} = 10$

- $h(0,t)$ : wall
- $h(1,t)$ : wall
- restriction on $\delta_t$

## Video Links

- take an experiment
- choose a phenomenon to highlight
- check the behavior of the scheme

Youtube source : https://www.youtube.com/watch?v=bgEWf_ps5xs

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End of the thesis objectives :

- roof
- non-permeable part in the porous media
- free surface part in the porous media
- thin free surface flow on the porous media

**Youtube source** : `https://www.youtube.com/watch?v=bgEWf_ps5xs`

Audusse, Emmanuel et al. "A multilayer Saint-Venant system with mass exchanges for shallow water flows. Derivation and numerical validation". In : ESAIM: M2AN 45.1 (2011), p. 169-200. doi : 10.1051/m2an/2010036. url : https://doi.org/10.1051/m2an/2010036.

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