Low-amplitude transient elastic waves in a 1D periodic array of non-linear interfaces: homogenization and time-domain simulations

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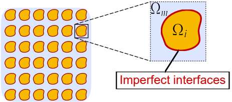
C. Bellis[2], B. Lombard[2], M. Touboul[1,2] and R. Assier[1]

[1] Department of Mathematics, The University of Manchester, Oxford Road, Manchester, M13 9PL, UK
[2] Aix Marseille Univ, CNRS, Centrale Marseille, LMA, Marseille, France

Imperfect interfaces: background and objective

- Wave propagation in microstructured media ($\boldsymbol{U}$ displacement, $\boldsymbol{\Sigma}$ stress)
- Jump conditions to model imperfect contacts between the matrix and inclusions



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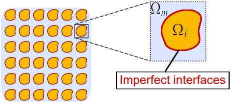

➢ Homogenization of thin layers:

$$\begin{cases} [\![U]\!]_a = B_1 \left\langle\!\left\langle \boldsymbol{\Sigma} \cdot \boldsymbol{n} \right\rangle\!\right\rangle_a + B_2 \left\langle\!\left\langle \dfrac{\partial U}{\partial X_2} \right\rangle\!\right\rangle_a \\ [\![\boldsymbol{\Sigma} \cdot \boldsymbol{n}]\!]_a = S \left\langle\!\left\langle \dfrac{\partial^2 U}{\partial t^2} \right\rangle\!\right\rangle_a + \boldsymbol{C} \cdot \left\langle\!\left\langle \dfrac{\partial \boldsymbol{\Sigma}}{\partial X_2} \right\rangle\!\right\rangle_a \end{cases}$$

$[\![f]\!]$ jump across the interface
$\langle\!\langle f \rangle\!\rangle$ mean value across the interface

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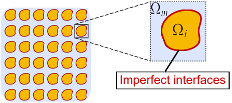

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➢ Limit behaviour $B_2 = \boldsymbol{C} = 0 \rightarrow$ **Spring-mass** jump conditions, cf phenomenological models [Jones, Whittier, Tattersall, Sevostianov, Licht, Lebon, Rizzoni]

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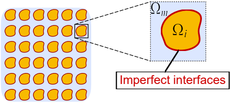

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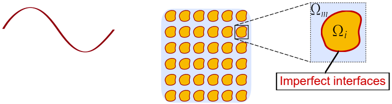

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Objective: Effective behaviour in the long-wavelength regime

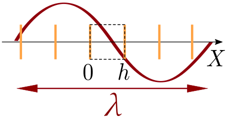
Homogenization with spring-mass interfaces (possibly non-linear) in a 1D setting

Outline

Microstructured problem

- 1D array of periodicity h
- Conservation equations

$$\begin{cases} \rho_h \dfrac{\partial^2 U_h}{\partial t^2} = \dfrac{\partial \Sigma_h}{\partial X} + F \\ \Sigma_h = \mu_h \dfrac{\partial U_h}{\partial X} \end{cases}$$



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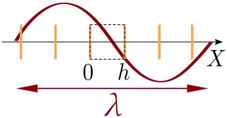

- **Spring-mass jump conditions** at $X_n = nh,\ n \in \mathbb{Z}$:

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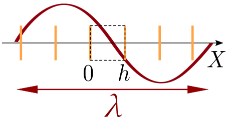

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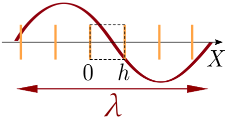

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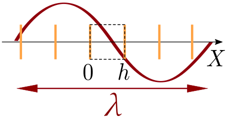

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$\mathcal{R} : (-d, +\infty) \longrightarrow \mathbb{R}$
($d \in R + \cup \{+\infty\}$ maximum compressibility length)
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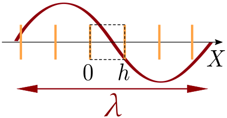

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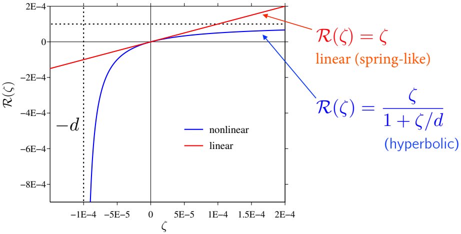

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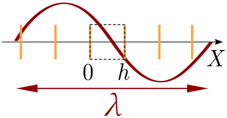

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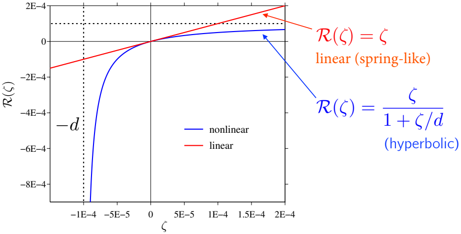

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- Assumptions on the source F

➢ low-frequency: $h \ll \lambda \Longrightarrow$ small parameter $\eta \ll 1$

➢ low-amplitude when non linear (no additional scaling in η at short times)

Microstructure behaviour

- Energy analysis

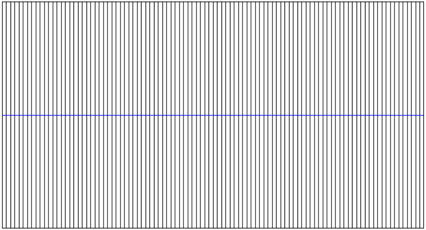
Energy conservation in the 1D array

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{E}_h^m + \mathscr{E}_h^i) = 0$$

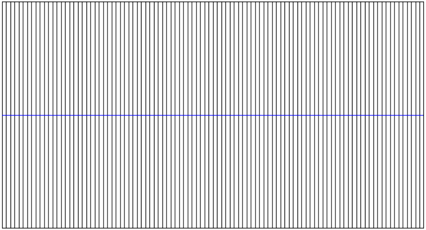
→ Medium mechanical energy: $\mathscr{E}_h^m(t) = \frac{1}{2}\int_I \left\{\rho_h(X)V_h(X,t)^2 + \frac{1}{\mu_h(X)}\Sigma_h(X,t)^2\right\}\mathrm{d}X$

→ Interface energy: $\mathscr{E}_h^i(t) = \sum_{X_n^I}\left\{\frac{1}{2}M\,\langle\!\langle V_h(\cdot,t)\rangle\!\rangle^2_{X_n^I} + K\int_0^{\mathcal{R}^{-1}\left(\langle\!\langle \Sigma_h(\cdot,t)\rangle\!\rangle_{X_n^I}/K\right)}\mathcal{R}(\zeta)\,\mathrm{d}\zeta\right\}$

- Numerical simulations for the hyperbolic law $\mathcal{R}(\zeta) = \frac{\zeta}{1+\zeta/d}$ and varying amplitude A



$A = 0.1$



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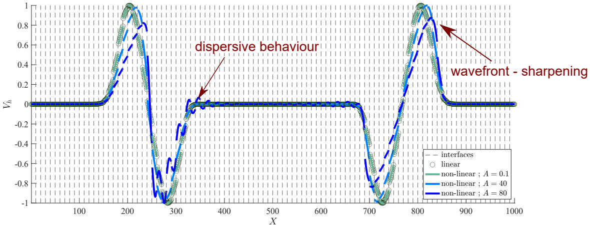

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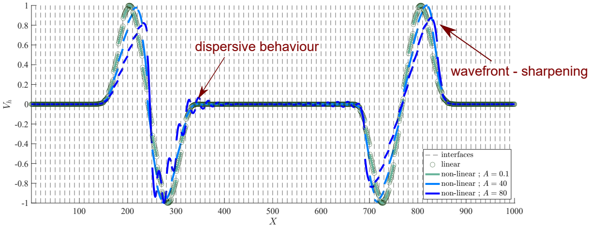

⇒ 1st order approximation for small amplitudes and small frequencies (at short finite times)

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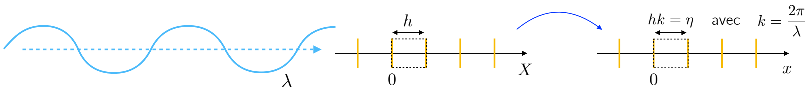

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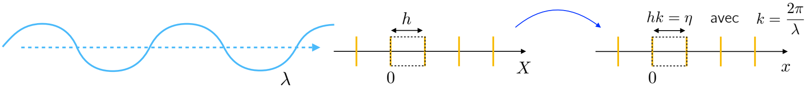

- wave equation: $\alpha\left(\frac{x}{\eta}\right)\frac{\partial^2 u_\eta}{\partial \tau^2}(x,\tau) = \frac{\partial}{\partial x}\left(\beta\left(\frac{x}{\eta}\right)\frac{\partial u_\eta}{\partial x}(x,\tau)\right) + f(x,\tau)$
- interface conditions (contributions in η):

$$\begin{cases} \eta \left\langle\!\left\langle \dfrac{\partial^2 u_\eta}{\partial \tau^2}(\cdot,\tau) \right\rangle\!\right\rangle_{x_n} = \left[\!\left[\beta \dfrac{\partial u_\eta}{\partial x}(\cdot,\tau) \right]\!\right]_{x_n} \\ \left\langle\!\left\langle \beta \dfrac{\partial u_\eta}{\partial x}(\cdot,\tau) \right\rangle\!\right\rangle_{x_n} = \dfrac{\kappa}{h}\,\mathcal{R}\left(\dfrac{h}{\eta}\,[\![u_\eta(\cdot,\tau)]\!]_{x_n}\right) \end{cases}$$

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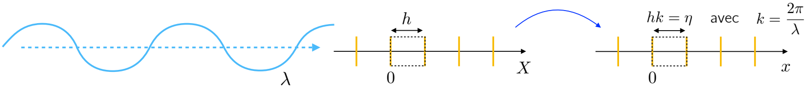

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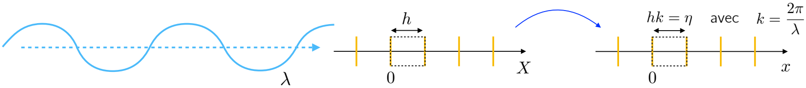

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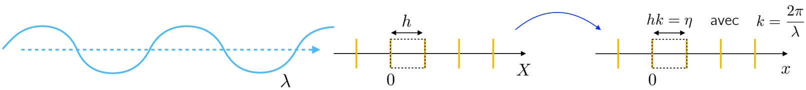
➢ **Formal asymptotic expansion**: $u_\eta(x,\tau) = u_0(x,\tau) + \sum_{j\geqslant 1} \eta^j u_j(x,y,\tau)$

Two-scale analysis

➢ **Non-dimensionalization**: $(X,t) \rightsquigarrow (x,\tau)$



- wave equation: $\alpha\left(\frac{x}{\eta}\right)\frac{\partial^2 u_\eta}{\partial \tau^2}(x,\tau) = \frac{\partial}{\partial x}\left(\beta\left(\frac{x}{\eta}\right)\frac{\partial u_\eta}{\partial x}(x,\tau)\right) + f(x,\tau)$
- interface conditions (contributions in η):

$$\begin{cases} \eta \left\langle\!\left\langle \frac{\partial^2 u_\eta}{\partial \tau^2}(\cdot,\tau) \right\rangle\!\right\rangle_{x_n} = \left[\!\left[\beta \frac{\partial u_\eta}{\partial x}(\cdot,\tau) \right]\!\right]_{x_n} \\ \left\langle\!\left\langle \beta \frac{\partial u_\eta}{\partial x}(\cdot,\tau) \right\rangle\!\right\rangle_{x_n} = \frac{\kappa}{h}\mathcal{R}\left(\frac{h}{\eta}[\![u_\eta(\cdot,\tau)]\!]_{x_n}\right) \end{cases}$$

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⚠ Specificities of the setting under study:

- $\mathcal{R}$ smooth $\rightarrow \mathcal{R}\left(\frac{h}{\eta}[\![u_\eta]\!]_{y_n}\right) = \sum_{\ell\geqslant 0}\frac{(h\eta)^\ell}{\ell!}\left(\sum_{j\geqslant 2}\eta^{j-2}[\![u_j(x,\cdot,\tau)]\!]_{y_n}\right)^\ell \mathcal{R}^{(\ell)}\left(h[\![u_1(x,\cdot,\tau)]\!]_{y_n}\right)$
- 1D setting $\rightarrow$ direct integrations and $\left\langle \frac{\mathrm{d}g}{\mathrm{d}y}\right\rangle = \int_0^1 \frac{\mathrm{d}g}{\mathrm{d}y}(y)\mathrm{d}y = -[\![g]\!] = -(g(0^+) - g(1^-))$

Field approximation

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  - continuous in X

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- **Limit cases**
  - → Linear interfaces, i.e. $\mathcal{R}(\zeta) = \zeta$:

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  → Perfect interfaces, i.e. $M = 0$ and $K = +\infty$:

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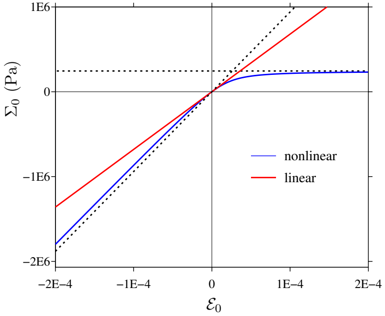
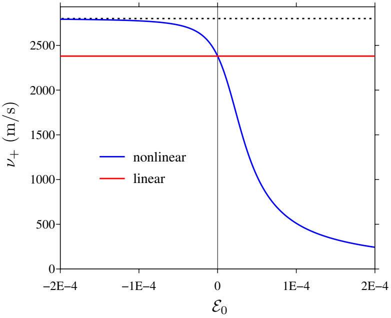
→ except when linear interfaces, the system is genuinely non-linear





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$$\frac{\mathrm{d}}{\mathrm{d}t}(\underbrace{\mathscr{E}_0^m + \mathscr{E}_0^i}_{\mathscr{E}_0}) = 0 \text{ *for sufficiently smooth fields* and without source}$$

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$\Longrightarrow$ $t^\star \approx \frac{1}{\mathcal{E}_{\text{max}}\gamma\,\omega_c} + \frac{1}{2f_c}$ for a forcing at $\omega_c = 2\pi f_c$ and an amplitude $\mathcal{E}_{\text{max}}$

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$$U_1(X,t) = \underbrace{\overline{U}_1(X,t)}_{\text{mean field}} + \underbrace{\mathcal{P}\big(y, \mathcal{E}_0(X,t)\big)}_{\text{cell function}} \, \mathcal{E}_0(X,t)$$

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- Source term $\mathcal{S}(U_0)$ and stiffness $\mathcal{G}'_{\text{eff}}(\mathcal{E}_0)$ (may) depend non-linearly on U_0
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→ Cell function for $X \in (nh, (n+1)h)$:

$$\mathcal{P}\big(y, \mathcal{E}_0(X,t)\big) = \left(\tfrac{1}{2} - y\right) + \big(b(y) - \mathcal{B}\big) \left\langle \tfrac{1}{E} \right\rangle \frac{\mathcal{G}_{\text{eff}}\big(\mathcal{E}_0(X,t)\big)}{\mathcal{E}_0(X,t)} \text{ with } y = (X - nh)/h$$

- (Possibly) non-linear function of U_0

Outline

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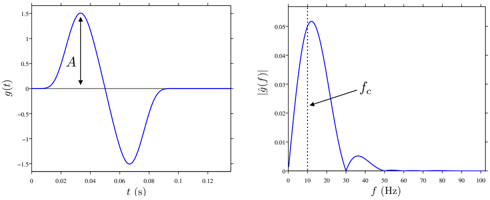
- time-domain signal $g(t)$ combination of sinusoids
- wide-band signal with central frequency f_c



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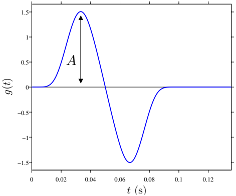
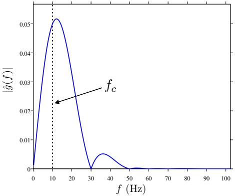
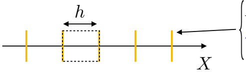




→ **Medium considered**:



$$\begin{cases} M = 0 \\ \mathcal{R}(\zeta) = \dfrac{\zeta}{1 + \zeta/d} \end{cases}$$

Numerical illustration: **linear** case $\langle\langle \Sigma_h \rangle\rangle = K \; [\![U_h]\!]$

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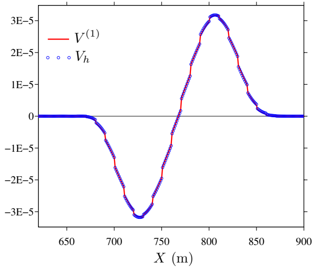

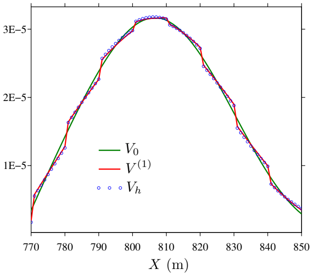

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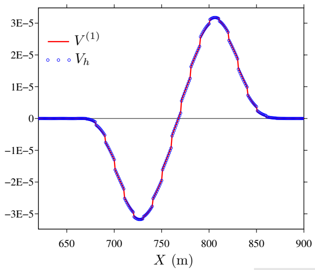
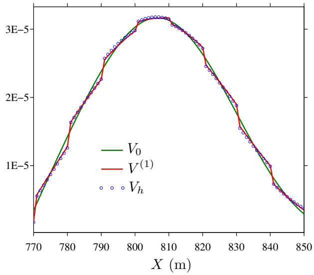
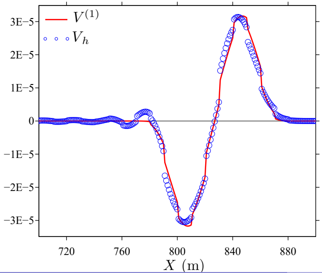


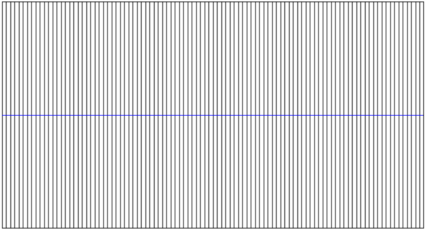


- Dispersive effects increase with η ($\eta = 0.52$):

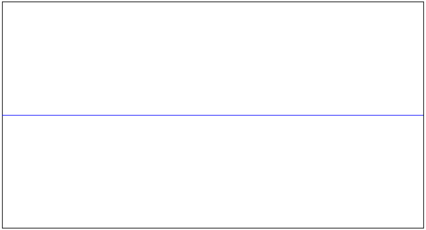


Numerical illustration: **non-linear** case $\langle\langle \Sigma_h \rangle\rangle = \frac{K[\![U_h]\!]}{1+[\![U_h]\!]/d}$

✓ Moderate frequency $\eta = 0.26$ and moderate amplitude $A = 40$:

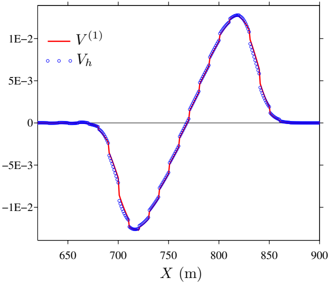


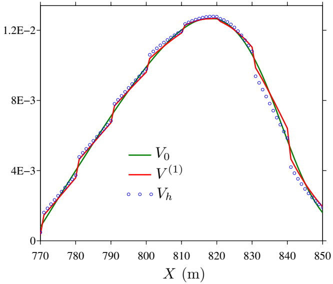
Microstructured medium



Homogenized medium

$\Longrightarrow$ Snapshots for exact velocity V_h and homogenized velocity $V^{(1)} = V_0 + hV_1$:





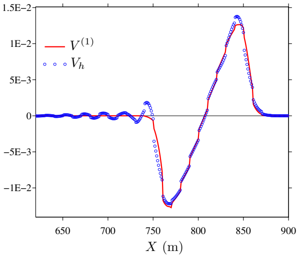
Numerical illustration: **non-linear** case $\langle\langle \Sigma_h \rangle\rangle = \frac{K[\![U_h]\!]}{1+[\![U_h]\!]/d}$

✗ Beyond the model validity (larger values of η or A)

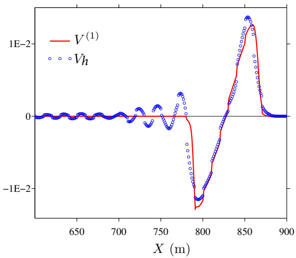
Numerical illustration: **non-linear** case $\langle\langle \Sigma_h \rangle\rangle = \frac{K[\![U_h]\!]}{1+[\![U_h]\!]/d}$

✗ Beyond the model validity (larger values of η or A)

- Fixed amplitude
  $A = 40$



$\eta = 0.39$



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Numerical illustration: **non-linear** case $\langle\langle \Sigma_h \rangle\rangle = \frac{K[\![U_h]\!]}{1+[\![U_h]\!]/d}$

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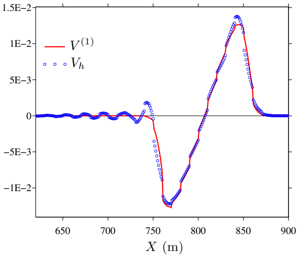

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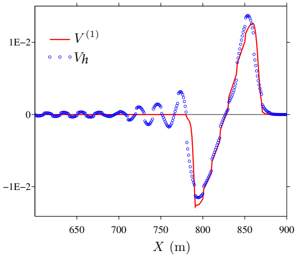
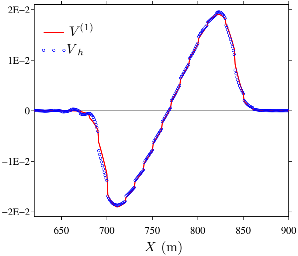
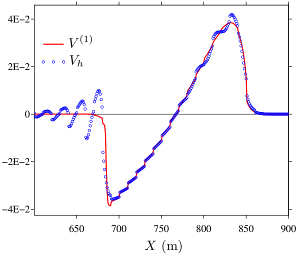


$\eta = 0.52$

- Fixed frequency $\eta = 0.26$



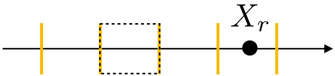
$A = 60$

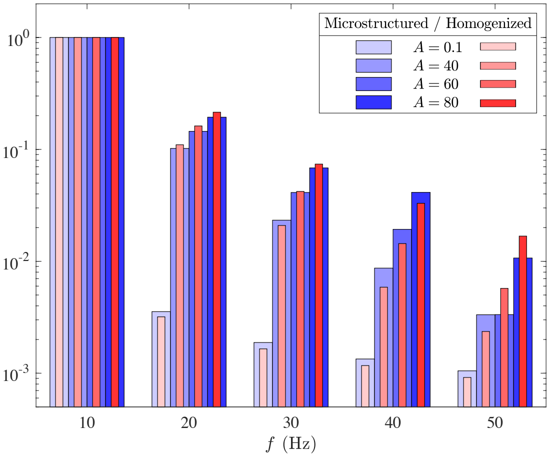


$A = 120$

Accuracy of the harmonics?

Monochromatic source ($f_c = 10$Hz) and spectrum of recorded signal at X_r





- ✓ amplitudes of multiple harmonics increase when A increases
- ✓ stronger non-linear effects when A increases
- ✓ overall good agreement between harmonics for the microstructure and the zeroth-order model

Energy and formation of shocks

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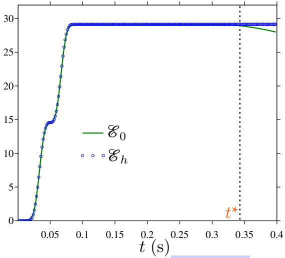
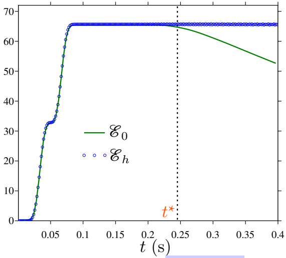


$\eta = 0.26$ and $A = 40$



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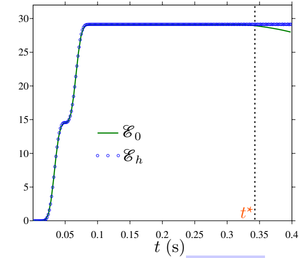

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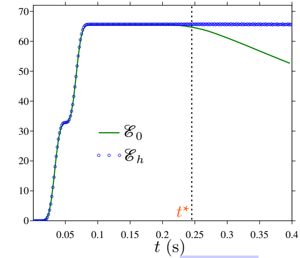

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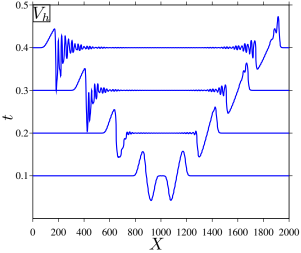
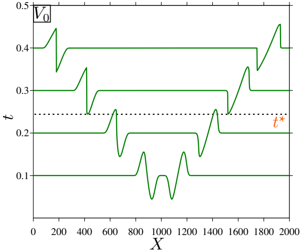
- ✓ Estimation of $t^\star \approx$ upper bound of the time of validity of the effective model





Outline

Outline and perspectives

Conclusions

- ✓ 1st order homogenized model accurate at low frequency, low amplitude, and short times
- ✓ Upper bound of the time of validity
- ✓ Non-linear zeroth-order model: shocks occur
- ✗ Dispersive effects increase with source amplitude and time in the microstructure and are not captured by the model

C. Bellis, B. Lombard, M. Touboul, R. Assier, *Journal of the Mechanics and Physics of Solids* (2021)

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Perspectives

- Derivation of a model accounting for dispersive effects (2nd order model?)
- Homogenization for larger source amplitudes
- Derivation of an effective model in 2D and 3D?

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Thank you for your attention!