DISMA Dipartimento di Scienze Matematiche G. L. Lagrange

ECCELLENZA 2018 · 2022

“Scalloping” with friends

joint work with M. Morandotti, H. B. Gadehla

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POLITECNICO DI TORINO

Micro-scale swimming

AIM: Understand locomotion strategies of real microrganisms.

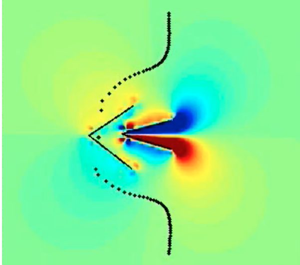
- Sperm cells exhibit collective or aggregate motion when swimming in groups
- Flagellar waveforms are modulated via hydrodynamic coupling with other flagella

Study of a simplified model made of two 2-link swimmers swimming together

Scallop Theorem

A swimmer that moves like a scallop, opening and closing reciprocally its valves, cannot achieve any net motion in a viscous fluid, because of the time reversibility of the equations.

Inertialess hydrodynamics is notorious for its time-reversibility constraints which leads to the celebrated Scallop Theorem. One way to overcome this constraint is increasing the number of swimmers: two scallops swimming together can in fact displace by a non-zero distance



Controllability

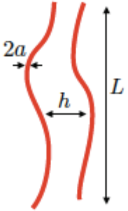
Is it possible to advance within the fluid, making cyclical shape changes?
More precisely is it possible to move between two fixed configurations opening and closing the to scallops?

$$\Downarrow$$

CONTROL THEORY

Forces Approximation

Immagine to have two almost vertical filaments close to each other each one representing a slender swimmer.



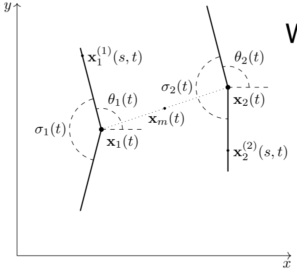
$$a << h << L$$

Denoting the force densities $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$, using Resistive-Force Theory (RFT) and taking into account the hydrodynamics interaction, we have

$$\mathbf{f}^{(1)} = -\Big(C_\perp \mathbf{I} + (C_\parallel - C_\perp)(\mathbf{t}^{(1)} \otimes \mathbf{t}^{(1)})\Big) \cdot \left(\frac{\partial \mathbf{r}^{(1)}}{\partial t} - \mathbf{v}^{(2)\to(1)}\right),$$

$$\mathbf{f}^{(2)} = -\Big(C_\perp \mathbf{I} + (C_\parallel - C_\perp)(\mathbf{t}^{(2)} \otimes \mathbf{t}^{(2)})\Big) \cdot \left(\frac{\partial \mathbf{r}^{(2)}}{\partial t} - \mathbf{v}^{(1)\to(2)}\right),$$

Man, Koens and Lauga.
Hydrodynamic interactions between nearby slender filaments.
EPL, 116 (2016), 24002.



We discretize each swimmer into 2-links.

- $\mathbf{x} = (x_i, y_i)$, for $i = 1, 2$, which is the position of the hinge of the i-th swimmer,
- θ_i, for $i = 1, 2$, angle that the upper link of the i-th swimmer forms with the positive x axis,
- σ_i opening angle of the i-th swimmer

Applying RFT and computing the interaction term $\mathbf{v}^{(i)\to(j)}$ we have

$$\mathbf{f}^{(i)}(s,t) = -\frac{1}{\Lambda(s,t)}\mathbf{J}^{(i)} \cdot \frac{\partial \mathbf{r}^{(i)}}{\partial t} - \lambda(s,t)\mathbf{J}^{(j)} \cdot \frac{\partial \mathbf{r}^{(j)}}{\partial t}$$

with $\lambda(s,t) = \frac{\ln\left(\frac{h}{L}\right)}{\ln\left(\frac{a}{L}\right)}$, $\Lambda(s,t) = 1 - \lambda^2$, and $\mathbf{J}^{(i)}$ is the i-th swimmer RFT operator.

Equations of motion

Writing the balance of total force and total torque acting on each swimmer we get

$$
-\Lambda \begin{pmatrix} \mathbf{F}_1(t) \\ T_1(t) \\ \mathbf{F}_2(t) \\ T_2(t) \end{pmatrix} = \left(\begin{array}{cc|cc} \mathbf{A}_1(t) & \mathbf{b}_1(t) & -\lambda \mathbf{A}_2(t) & -\lambda \mathbf{b}_2(t) \\ \mathbf{b}_1^\top(t) & \omega & -\lambda \mathbf{d}_1^\top(t) & -\lambda \varpi(t) \\ \hline -\lambda \mathbf{A}_1(t) & -\lambda \mathbf{b}_1(t) & \mathbf{A}_2(t) & \mathbf{b}_2(t) \\ -\lambda \mathbf{d}_2^\top(t) & -\lambda \varpi(t) & \mathbf{b}_2^\top(t) & \omega \end{array}\right) \begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot\theta_1(t) \\ \dot{\mathbf{x}}_2(t) \\ \dot\theta_2(t) \end{pmatrix}
$$

$$
+ \begin{pmatrix} \boldsymbol{\alpha}_1(t) \\ \omega/2 \\ -\lambda \boldsymbol{\alpha}_1(t) \\ -\lambda \beta(t) \end{pmatrix} \dot\sigma_1(t) + \begin{pmatrix} -\lambda \boldsymbol{\alpha}_2(t) \\ -\lambda \beta(t) \\ \boldsymbol{\alpha}_2(t) \\ \omega/2 \end{pmatrix} \dot\sigma_2(t) \qquad (1)
$$

$$
=: \mathcal{R}(t,\lambda) \begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot\theta_1(t) \\ \dot{\mathbf{x}}_2(t) \\ \dot\theta_2(t) \end{pmatrix} + \phi_1(t,\lambda)\dot\sigma_1(t) + \phi_2(t,\lambda)\dot\sigma_2(t) = 0
$$

Theorem (Invertibility)

There exists $\lambda_0 \in (0,1)$ *such that the matrix*

$$\mathcal{R}(t,\lambda) = \begin{pmatrix} \mathcal{R}_{11}(t) & -\lambda\mathcal{R}_{12}(t) \\ -\lambda\mathcal{R}_{21}(t) & \mathcal{R}_{22}(t) \end{pmatrix},$$

is invertible for every $\lambda \in [0,\lambda_0)$ *and for every* $t \in [0,+\infty)$.

Therefore the equations of motion read

$$\begin{pmatrix} \dot{\mathbf{x}}_1(t) \\ \dot\theta_1(t) \\ \dot{\mathbf{x}}_2(t) \\ \dot\theta_2(t) \\ \hline \dot\sigma_1(t) \\ \dot\sigma_2(t) \end{pmatrix} = \begin{pmatrix} -\mathcal{R}(t,\lambda)^{-1}\phi_1(t,\lambda) \\ \hline 1 \\ 0 \end{pmatrix} u_1(t) + \begin{pmatrix} -\mathcal{R}(t,\lambda)^{-1}\phi_2(t,\lambda) \\ \hline 0 \\ 1 \end{pmatrix} u_2(t) \tag{2}$$

$$\begin{aligned} &=: \mathbf{v}_1(\theta_1(t),\theta_2(t),\sigma_1(t),\sigma_2(t),\lambda)u_1(t) \\ &\quad + \mathbf{v}_2(\theta_1(t),\theta_2(t),\sigma_1(t),\sigma_2(t),\lambda)u_2(t), \end{aligned}$$

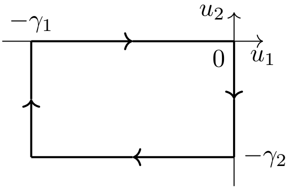
Controllability

We consider an initial configuration which is a perturbation of the aligned one:

$$\theta_i^\circ = \theta^\circ, \quad \sigma_i^\circ = \pi + \epsilon \cos((i-1)\phi),$$

and prescribe the following stroke in the time interval $[0, 4\tau]$, for a small $\tau > 0$ and for $\gamma_1, \gamma_2 > 0$,

$$t \mapsto \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} := \begin{cases} (0, -\gamma_2)^\top & \text{for } t \in [0, \tau), \\ (-\gamma_1, 0)^\top & \text{for } t \in [\tau, 2\tau), \\ (0, \gamma_2)^\top & \text{for } t \in [2\tau, 3\tau), \\ (\gamma_1, 0)^\top & \text{for } t \in [3\tau, 4\tau), \end{cases} \tag{3}$$



Classical tools in geometric control theory yield that the solution of the equations of motion with our initial conditions is given by

$$\begin{pmatrix} \mathbf{x}_1(4\tau) \\ \theta_1(4\tau) \\ \mathbf{x}_2(4\tau) \\ \theta_2(4\tau) \\ \hline \sigma_1(4\tau) \\ \sigma_2(4\tau) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1^\circ \\ \theta^\circ \\ \mathbf{x}_2^\circ \\ \theta^\circ \\ \hline \sigma_1^\circ \\ \sigma_2^\circ \end{pmatrix} - \gamma_1\gamma_2\tau^2[\mathbf{v}_1(\cdot,\lambda),\mathbf{v}_2(\cdot,\lambda)]\,|_{(\theta^\circ,\theta^\circ,\pi+\epsilon,\pi+\epsilon\cos\phi)} + o(\tau^2) \qquad (4)$$

with

$$[\mathbf{v}_1(\cdot,\lambda),\mathbf{v}_2(\cdot,\lambda)]\,|_{(\theta^\circ,\theta^\circ,\pi+\epsilon,\pi)} = \begin{pmatrix} \xi_1(\phi,\theta^\circ)\epsilon^2 + o(\epsilon^2) \\ \eta_1(\phi,\theta^\circ)\epsilon^2 + o(\epsilon^2) \\ \vartheta_1(\phi)\epsilon + o(\epsilon^2) \\ \xi_2(\phi,\theta^\circ)\epsilon^2 + o(\epsilon^2) \\ \eta_2(\phi,\theta^\circ)\epsilon^2 + o(\epsilon^2) \\ \vartheta_2(\phi)\epsilon + o(\epsilon^2) \\ \hline 0 \\ 0 \end{pmatrix}$$

in powers of ϵ

The two swimmers rotate counter-clockwise by the same amount, which is of order ϵ, and there is a net motion of order ϵ^2 along both axes, which vanishes (up to order $o(\epsilon^2)$) according to the value of θ^0.
We can estimate the global net displacement of the system by tracking the midpoint $t \mapsto \mathbf{x}_m(t) = (x_m(t), y_m(t))$ of the line connecting the two hinges

$$\Delta \mathbf{x}_m = \mathbf{x}_m(4\tau) - \mathbf{x}_m^\circ = \frac{-\gamma_1\gamma_2\tau^2}{2}\begin{pmatrix} C\epsilon^2 \cos\theta^\circ \sin^2\phi + o(\epsilon^2) \\ C\epsilon^2 \sin\theta^\circ \sin^2\phi + o(\epsilon^2) \end{pmatrix},$$

where $C = C(L, \lambda, C_\parallel, C_\perp)$ is given by

$$C = \frac{L\lambda\left(C_\perp^2(1+\lambda) - 3C_\parallel C_\perp + 3C_\parallel^2\right)}{128 C_\parallel C_\perp (1-\lambda^2)}. \tag{5}$$

Theorem

Let $\epsilon, \tau > 0$ be given small parameters, let $\phi \in \mathbb{R}$, and let $\gamma_1, \gamma_2 > 0$. Consider the initial configuration near the aligned one and the previous control stroke. Then the maximal displacement for the pair of scallops is obtained for strokes that have a phase difference of $\phi = \pi/2$.

Proof.

The net displacement of the midpoint $\mathbf{x}_m$ is given by

$$\delta_m(\phi) := |\Delta \mathbf{x}_m| = \frac{C\gamma_1\gamma_2\tau^2\epsilon^2 \sin^2\phi}{2}(1 + o(\epsilon^2)),$$

which is clearly maximum for the phase difference $\phi = \frac{\pi}{2}$. □

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Numerics

To integrate numerically the equations we choose the following shape deformation

$$t \mapsto \sigma_i(t) = \pi + \epsilon \cos\left(\omega t + (i-1)\phi\right) = \pi + \epsilon \cos\left(\frac{\pi t}{2\tau} + (i-1)\phi\right),$$

The choice of the frequency $\omega = \pi/2\tau$ is made so that in the time interval $[0, 4\tau]$ the angles σ_i have returned to their initial value after spanning only one period.

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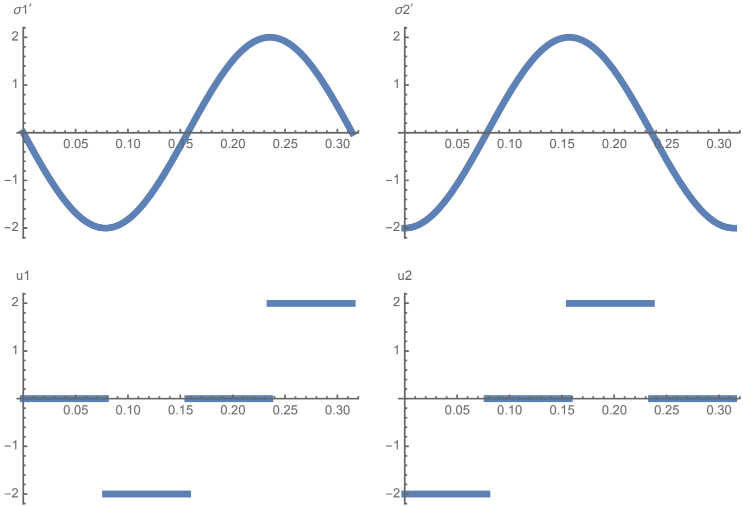

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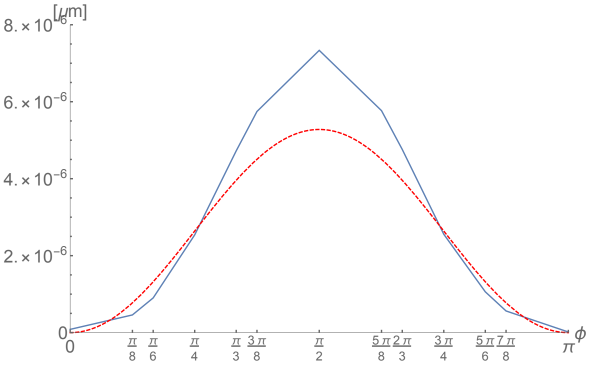
ϵ	0.1
ω	20
L	10μm
h	1μm
a	0.25μm
$C_\perp$	2Ns/μm^2
$C_\parallel$	1Ns/μm^2



We obtain results in good agreement with theoretical predictions

Zoppello, Morandotti, Gadhela.
Controlling non controllable scallops.
Submitted (2022)

Conclusions and perspectives

- Two almost aligned scallops are able to achieve a net motion making periodic shape changes,
- The maximal displacement is obtained for the swimmers beating in anti-phase.

Perspectives

- Gain a general controllability result for the coupled swimmers,
- Generalize the results to more than 2 swimmers.

Thank you for your attention