

Optimisation de forme pour la résistance hydrodynamique en régime de Stokes

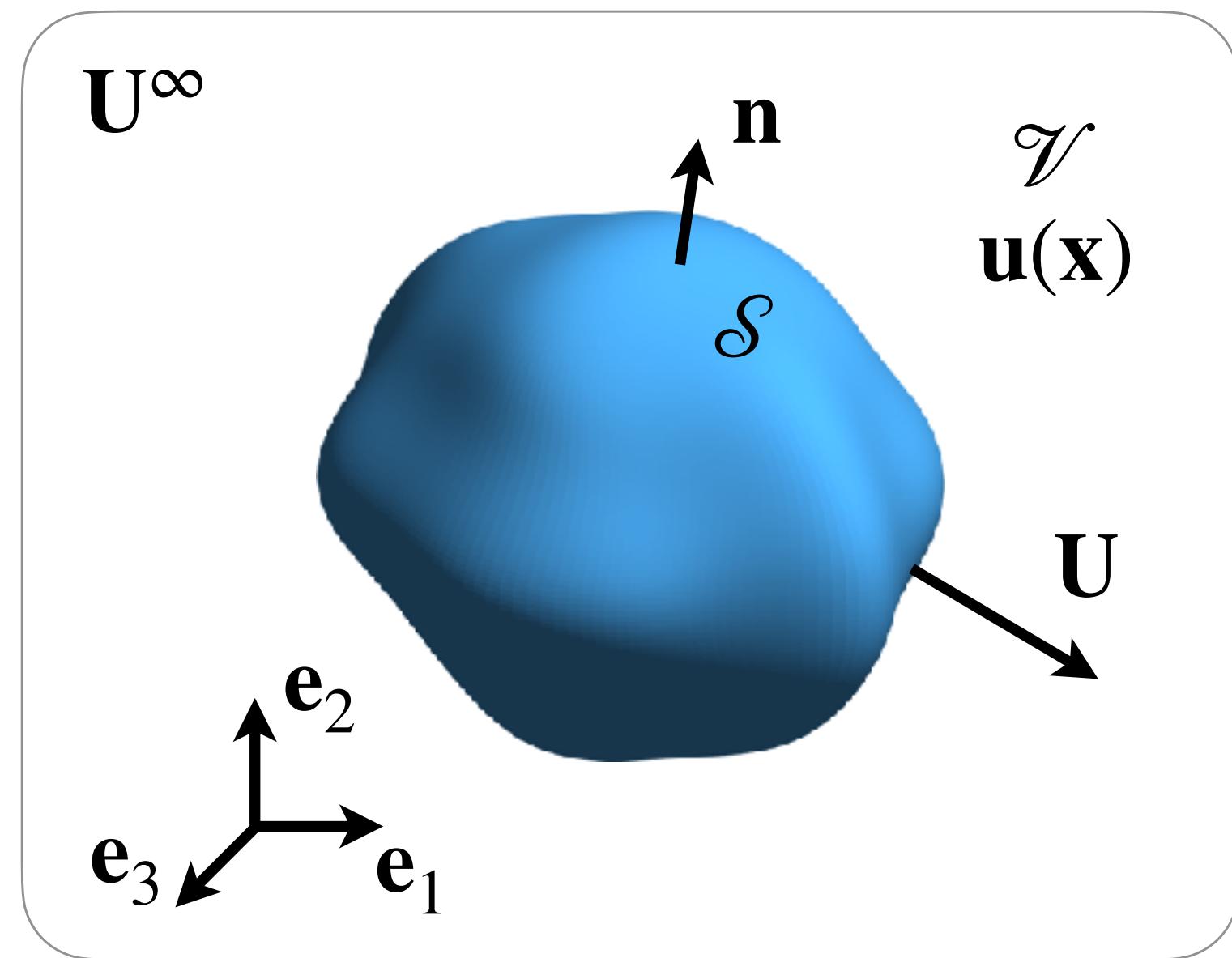
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Description du problème

Résistance hydrodynamique pour un objet rigide en mouvement dans un fluide de Stokes



$$\begin{cases} \mu\Delta\mathbf{u} - \nabla p = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathcal{V}, \\ \mathbf{u} = \mathbf{U} & \text{on } \mathcal{S}, \\ \mathbf{u} = \mathbf{U}^\infty & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

Stokes equation

$$\mathbf{F}^h + \mathbf{F} = 0, \quad \mathbf{T}^h + \mathbf{T} = 0.$$

Force and torque balance

Stress tensor:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \frac{1}{2}\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

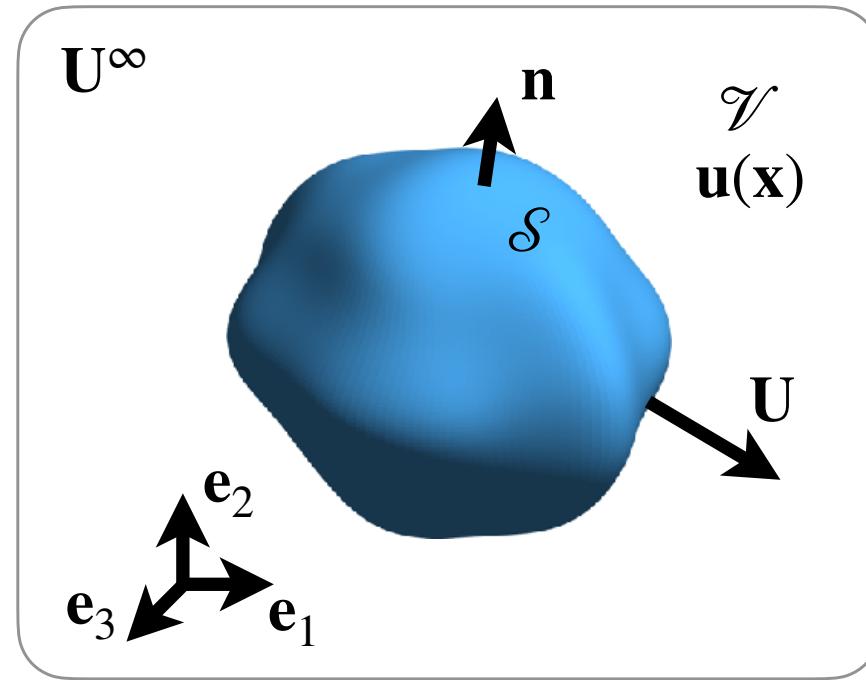
Hydrodynamic drag force and torque:

$$\mathbf{F}^h = \int_{\mathcal{S}} \boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{n} dS,$$

$$\mathbf{T}^h = \int_{\mathcal{S}} \mathbf{x} \times (\boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{n}) dS.$$

How to minimise/maximise the hydrodynamic drag?

Grand Resistance Tensor



Rigid body motion

$$\mathbf{U} = \mathbf{Z} + \boldsymbol{\Omega} \times \mathbf{x}$$

Linear background flow

$$\mathbf{U}^\infty = \mathbf{Z}^\infty + \boldsymbol{\Omega}^\infty \times \mathbf{x} + \mathbf{E}^\infty$$

$$\begin{cases} \mu \Delta \mathbf{u} - \nabla p = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathcal{V}, \\ \mathbf{u} = \mathbf{U} & \text{on } \mathcal{S}, \\ \mathbf{u} = \mathbf{U}^\infty & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

$$\mathbf{F}^h = \int_S \sigma(\mathbf{u}, p) \mathbf{n} dS,$$

$$\mathbf{T}^h = \int_S \mathbf{x} \times (\sigma(\mathbf{u}, p) \mathbf{n}) dS.$$

Grand Resistance Tensor

$$\begin{pmatrix} \mathbf{F}^h \\ \mathbf{T}^h \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{Z}^\infty - \mathbf{Z} \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\Omega} \\ \mathbf{E}^\infty \end{pmatrix} = \boxed{\begin{pmatrix} \mathbf{K} & \mathbf{C} & \boldsymbol{\Gamma} \\ \tilde{\mathbf{C}} & \mathbf{Q} & \boldsymbol{\Lambda} \end{pmatrix}} \begin{pmatrix} \mathbf{Z}^\infty - \mathbf{Z} \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\Omega} \\ \mathbf{E}^\infty \end{pmatrix}$$

\mathbf{K}_{ij} : **force** along \mathbf{e}_i generated in resistance to **translation** along \mathbf{e}_j

\mathbf{Q}_{ij} : **torque** along \mathbf{e}_i generated in resistance to **rotation** along \mathbf{e}_j

\mathbf{C}_{ij} : **force** along \mathbf{e}_i generated in resistance to **rotation** along \mathbf{e}_j

$\tilde{\mathbf{C}}_{ij}$: **torque** along \mathbf{e}_i generated in resistance to **translation** along \mathbf{e}_j ($\tilde{\mathbf{C}} = \mathbf{C}^T$)

$\boldsymbol{\Gamma}_{ijk}$: **force** along \mathbf{e}_i generated in resistance to **shear** along \mathbf{e}_j

$\boldsymbol{\Lambda}_{ijk}$: **torque** along \mathbf{e}_i generated in resistance to **shear** along \mathbf{e}_j

- The GRT depends **only on the shape of the object**
- **Optimise** a shape with respect to a given entry?

Grand Resistance Tensor

$$\begin{cases} \mu\Delta\mathbf{u} - \nabla p = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathcal{V}, \\ \mathbf{u} = \mathbf{U} & \text{on } \mathcal{S}, \\ \mathbf{u} = \mathbf{U}^\infty & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

One entry = one Stokes problem

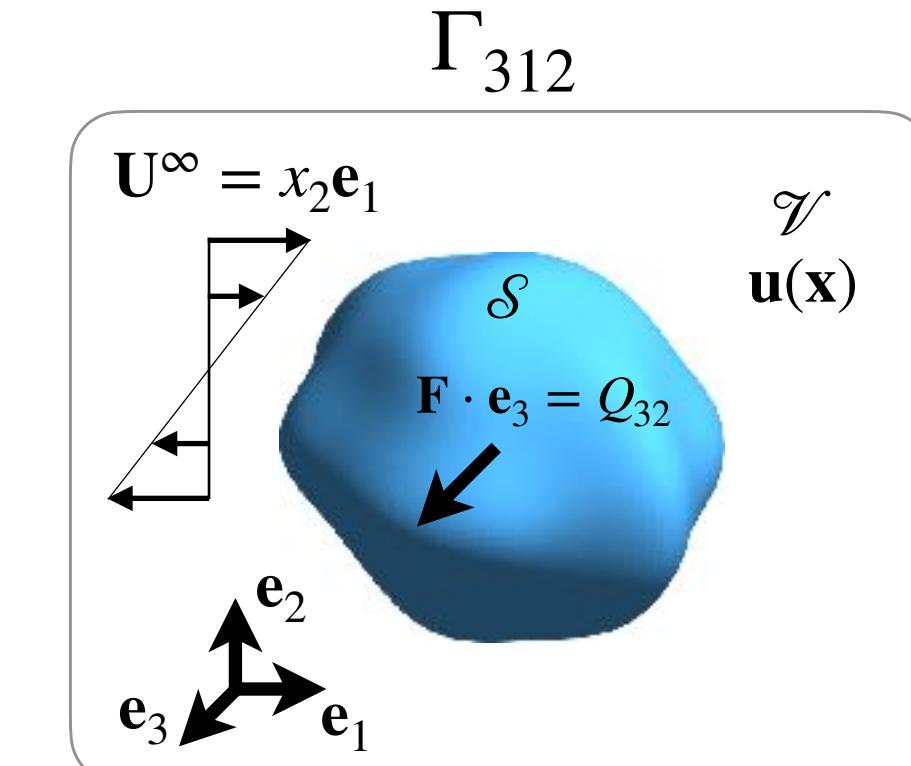
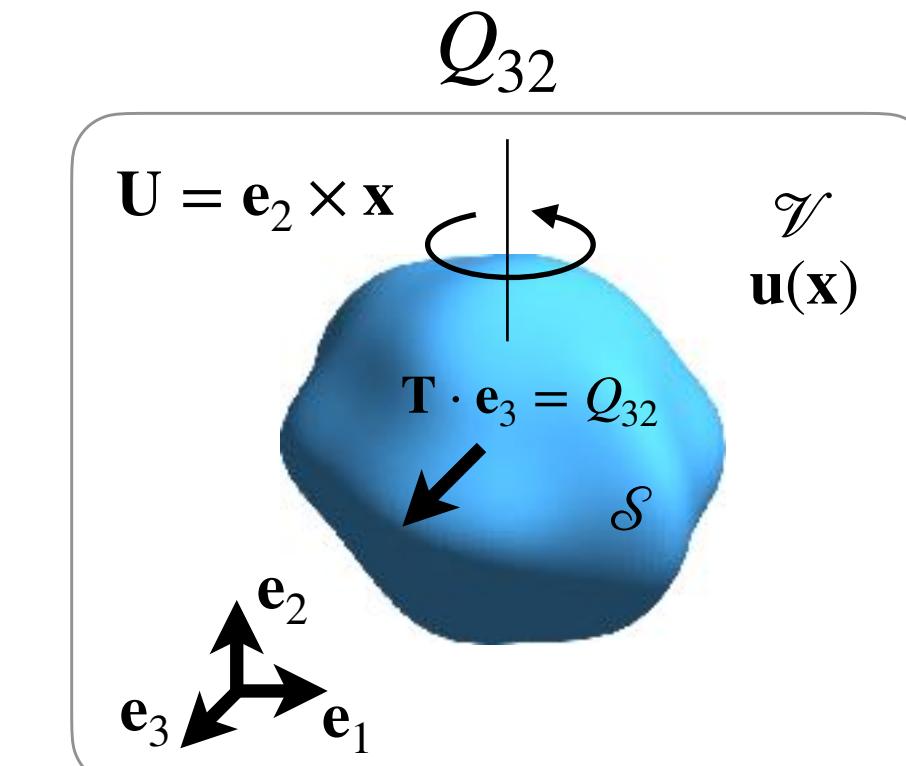
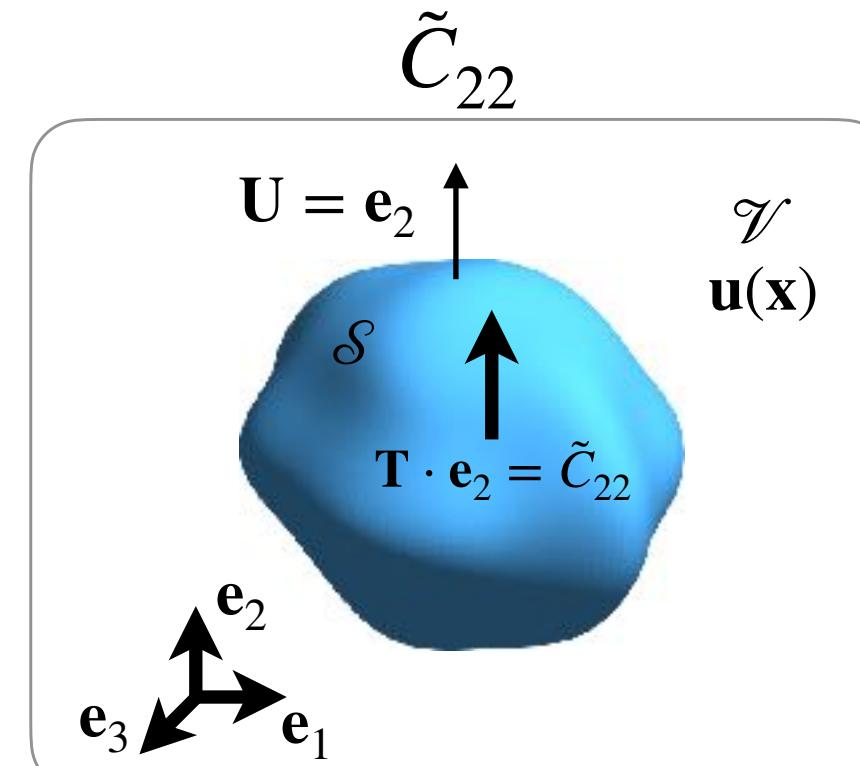
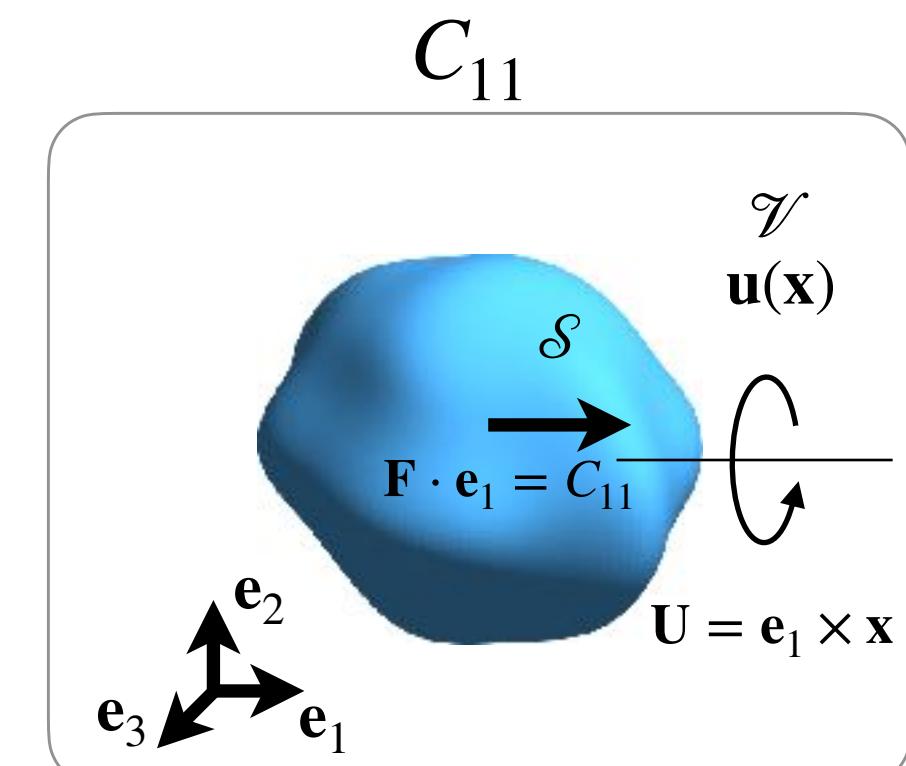
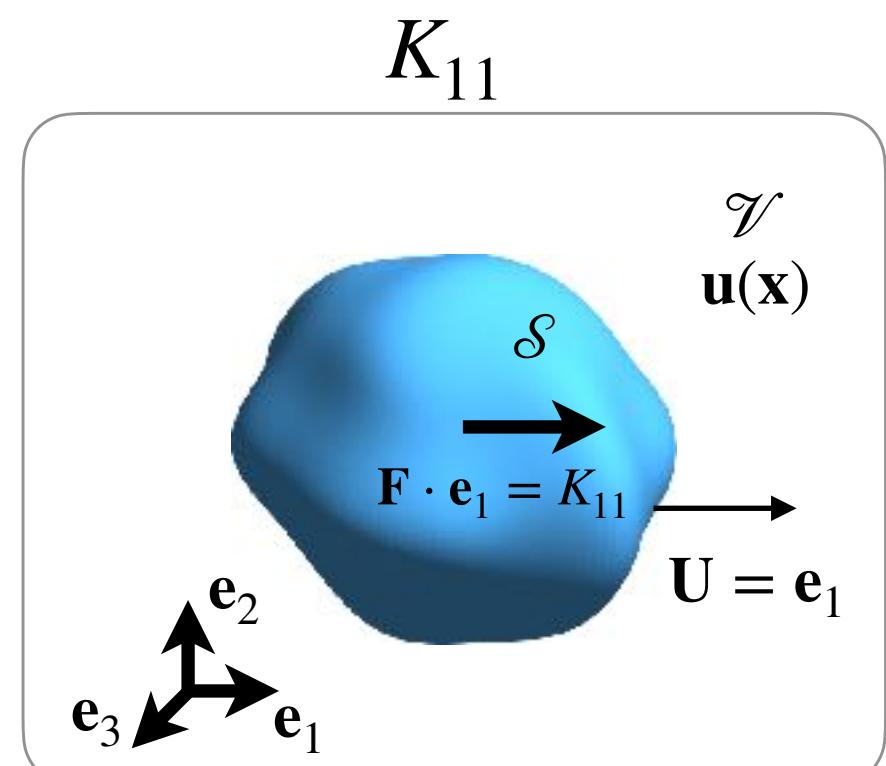
On définit

$$J_V(\mathcal{S}) = - \int_{\mathcal{S}} \sigma(\mathbf{u}, p) \mathbf{n} \cdot \nabla d\mathcal{S} \rightarrow$$

Examples

J_V	\mathbf{U}	\mathbf{V}	\mathbf{U}^∞
K_{ij}	\mathbf{e}_j	\mathbf{e}_i	$\mathbf{0}$
C_{ij}	$\mathbf{e}_j \times \mathbf{x}$	\mathbf{e}_i	$\mathbf{0}$
\tilde{C}_{ij}	\mathbf{e}_j	$\mathbf{e}_i \times \mathbf{x}$	$\mathbf{0}$
Q_{ij}	$\mathbf{e}_j \times \mathbf{x}$	$\mathbf{e}_i \times \mathbf{x}$	$\mathbf{0}$
Γ_{ijk}	$\mathbf{0}$	\mathbf{e}_i	$x_k \mathbf{e}_j$
Λ_{ijk}	$\mathbf{0}$	$\mathbf{e}_i \times \mathbf{x}$	$x_k \mathbf{e}_j$

Optimisation problem:
 $\min_{\mathcal{S}} J_V(\mathcal{S})?$



État de l'art

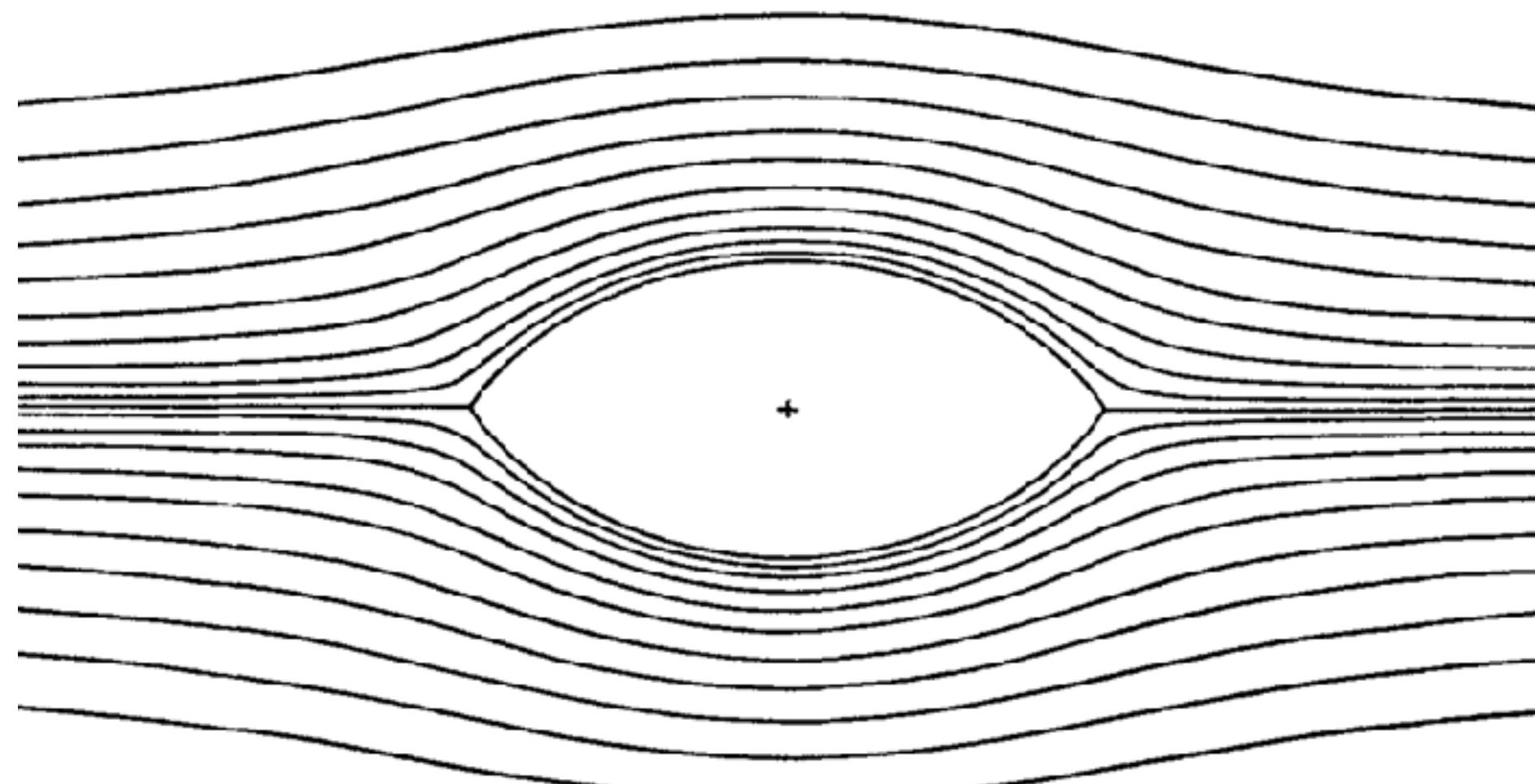
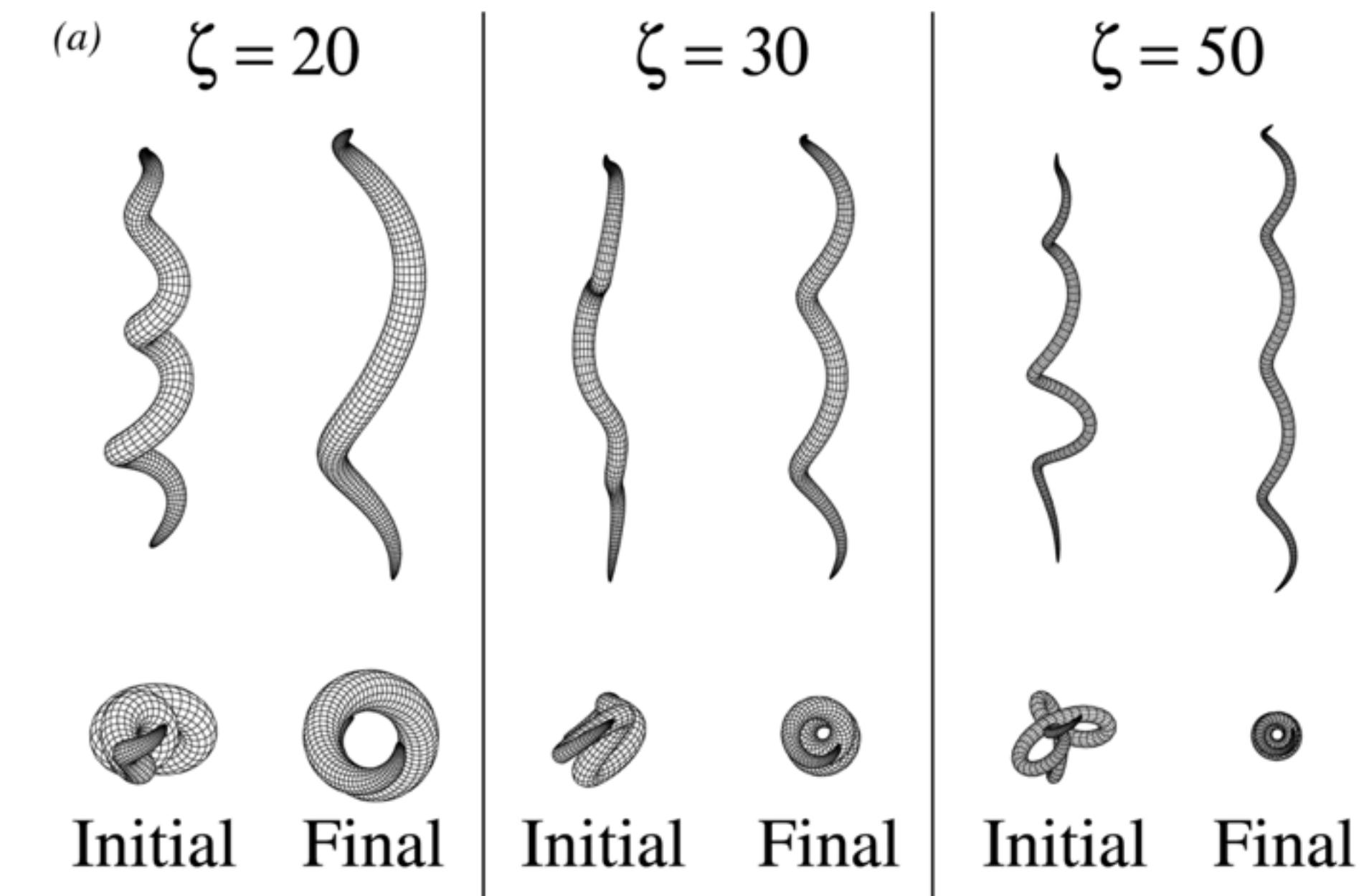


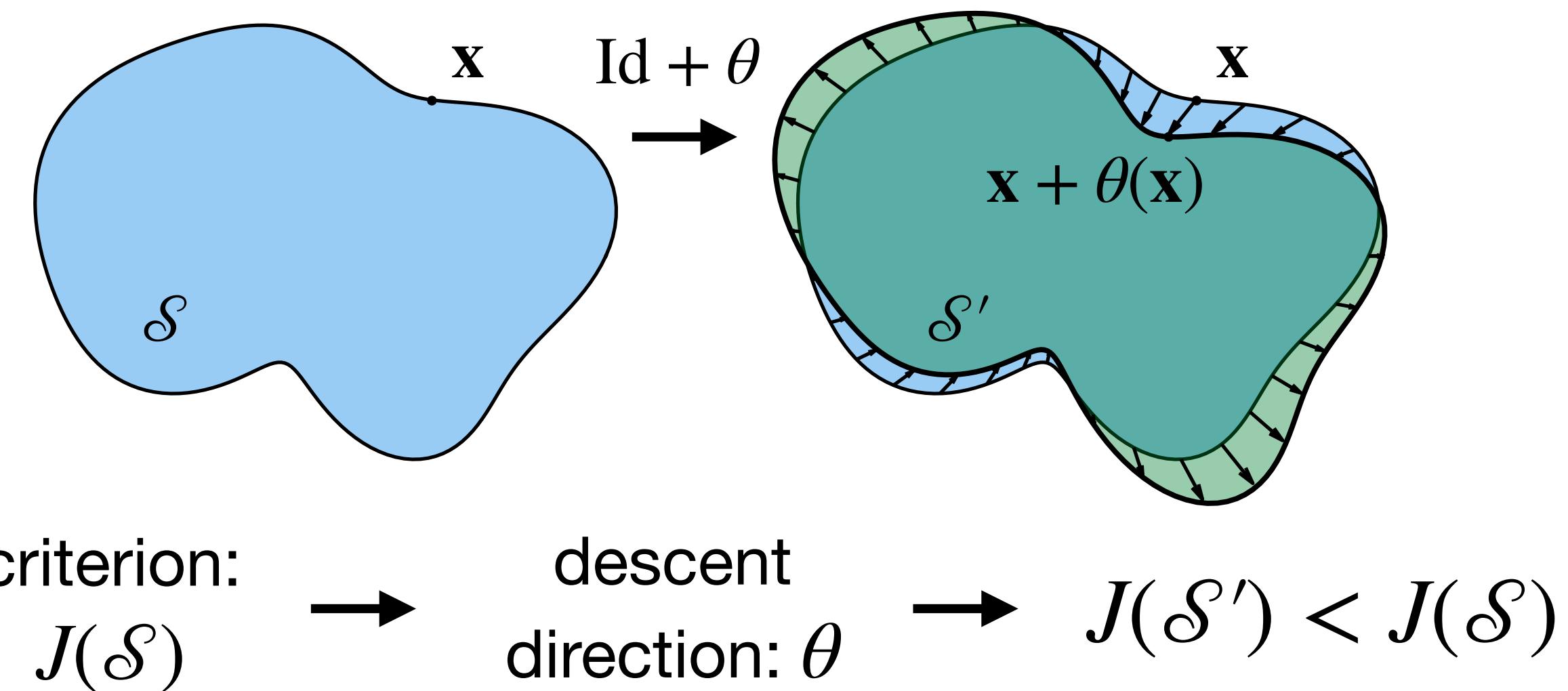
FIGURE 4

K_{11} : traînée optimale
Pironneau 1974, Bourot 1974
+ divers raffinements



C_{11} : Keaveny et al. 2013
classe de formes restreinte

Calcul de la dérivée de forme



- dérivée de forme :

$$DJ(\mathcal{S}) \cdot \theta = \lim_{\varepsilon \rightarrow 0} \frac{J(\mathcal{S} + \varepsilon \theta) - J(\mathcal{S})}{\varepsilon}$$

- idéalement on voudrait l'écrire comme

$$DJ(\mathcal{S}) \cdot \theta = \int_{\mathcal{S}} F(\mathbf{x}) \theta \cdot \mathbf{n} \quad \text{pour un certain } F \text{ (le \textcolor{blue}{gradient de forme})}$$

Calcul de la dérivée de forme

Proposition

La dérivée du critère

$$J_{\mathbf{V}}(\mathcal{S}) = - \int_{\mathcal{S}} \sigma(\mathbf{u}, p) \mathbf{n} \cdot \mathbf{V} d\mathcal{S}$$

dans la direction θ est donnée par

$$DJ_{\mathbf{V}}(\mathcal{S}) \cdot \theta = - 2\mu \int_{\mathcal{S}} (-\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \mathbf{e}(\mathbf{U}) : \mathbf{e}(\mathbf{v}) + \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{V})) (\theta \cdot \mathbf{n}) d\mathcal{S}.$$

En particulier, si \mathbf{U} et \mathbf{V} sont linéaires,

$$DJ_{\mathbf{V}}(\mathcal{S}) \cdot \theta = \int_{\mathcal{S}} (2\mu \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v})) (\theta \cdot \mathbf{n}) d\mathcal{S}$$

gradient de forme F

Démonstration

$$J_{\mathbf{V}}(\mathcal{S}) = - \int_{\mathcal{S}} \sigma(\mathbf{u}, p) \mathbf{n} \cdot \mathbf{V} d\mathcal{S}$$

1. on écrit le **problème adjoint**

$$\begin{cases} \mu \Delta \mathbf{v} - \nabla q = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \mathcal{V}, \\ \mathbf{v} = \mathbf{V} & \text{on } \mathcal{S}, \\ \mathbf{v} = 0 & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

2. on exprime le critère sous la forme d'une **intégrale de volume**

$$J_{\mathbf{V}}(\mathcal{S}) = - 2\mu \int_{\mathcal{V}} \mathbf{e}(u) : \mathbf{e}(v) d\mathcal{V}$$

3. on calcule la **dérivée de forme** en θ :

$$DJ_{\mathbf{V}}(\mathcal{S}) \cdot \theta = - 2\mu \left(\int_{\mathcal{S}} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v})(\theta \cdot \mathbf{n}) d\mathcal{S} + \int_{\mathcal{V}} \mathbf{e}(\mathbf{u}') : \mathbf{e}(\mathbf{v}) d\mathcal{V} + \int_{\mathcal{V}} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}') d\mathcal{V} \right)$$

Démonstration

$$DJ_V(\mathcal{S}) \cdot \theta = -2\mu \left(\int_{\mathcal{S}} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v})(\theta \cdot \mathbf{n}) d\mathcal{S} + \int_{\mathcal{V}} \mathbf{e}(\mathbf{u}') : \mathbf{e}(\mathbf{v}) d\mathcal{V} + \int_{\mathcal{V}} \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}') d\mathcal{V} \right)$$

The Eulerian derivatives \mathbf{u}' and \mathbf{v}' satisfy the auxiliary equations:

$$\begin{cases} \mu\Delta\mathbf{v}' - \nabla q' = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{v}' = 0 & \text{in } \mathcal{V}, \\ \mathbf{v}' = -\nabla(\mathbf{v} - \mathbf{V})\mathbf{n}(\theta \cdot \mathbf{n}) & \text{on } \mathcal{S}, \\ \mathbf{v}' = 0 & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

$$\begin{cases} \mu\Delta\mathbf{u}' - \nabla p' = 0 & \text{in } \mathcal{V}, \\ \nabla \cdot \mathbf{u}' = 0 & \text{in } \mathcal{V}, \\ \mathbf{u}' = -\nabla(\mathbf{u} - \mathbf{U})\mathbf{n}(\theta \cdot \mathbf{n}) & \text{on } \mathcal{S}, \\ \mathbf{u}' = 0 & \text{when } |\mathbf{x}| \rightarrow \infty. \end{cases}$$

Using these relations, a few more i.p.p. and some tweaking, we get:

$$DJ_V(\mathcal{S}) \cdot \theta = -2\mu \int_{\mathcal{S}} \boxed{(-\mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}) + \mathbf{e}(\mathbf{U}) : \mathbf{e}(\mathbf{v}) + \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{V}))(\theta \cdot \mathbf{n})} d\mathcal{S}$$

F(x)

□

Implémentation numérique

Direction de descente

$$DJ_{\mathbf{V}}(\mathcal{S}) \cdot \theta = \int_{\mathcal{S}} \boxed{(2\mu \mathbf{e}(\mathbf{u}) : \mathbf{e}(\mathbf{v}))} (\theta \cdot \mathbf{n}) d\mathcal{S}$$

shape gradient F

formulation variationnelle $\forall \psi, \int_{\mathcal{V}} \nabla \theta : \nabla \psi d\mathcal{V} = - DJ_{\mathbf{V}}(\mathcal{S}) \cdot \psi$

then for $\psi = \theta$: $DJ_{\mathbf{V}}(\mathcal{S}) \cdot \theta = - \int_{\mathcal{V}} |\nabla \theta|^2 d\mathcal{V} < 0$

Equation de Laplace

$$\begin{cases} -\Delta \theta = 0 & \text{in } \mathcal{V}, \\ \theta = 0 & \text{when } |\mathbf{x}| \rightarrow \infty, \\ (\nabla \theta) \mathbf{n} = -F \mathbf{n} & \text{on } \mathcal{S}. \end{cases}$$

Implémentation numérique

Contrainte de volume constant

Contrainte $V(\mathcal{S}) = V_0$.

Augmented Lagrangian

$$\mathcal{L}(\mathcal{S}, \ell, b) = J(\mathcal{S}) - \ell(V(\mathcal{S}) - V_0) + \frac{b}{2}(V(\mathcal{S}) - V_0)^2$$

↑
criterion ↑
 Lagrange
 multiplier ↑
 penalization
 parameter

augmented
shape gradient: $\phi = F - \ell + b(V(\mathcal{S}) - V_0)$

$$\begin{cases} -\Delta\theta = 0 & \text{in } \mathcal{V}, \\ \theta = 0 & \text{when } |\mathbf{x}| \rightarrow \infty, \\ (\nabla\theta)\mathbf{n} = -\boxed{\phi}\mathbf{n} & \text{on } \mathcal{S}. \end{cases}$$

+ updating at each
deformation step

$$\begin{aligned} \ell &\leftarrow \ell - b(V(\mathcal{S}) - V_0) \\ b &\leftarrow ab, a > 1 \quad \text{until } b = b_{\max} \end{aligned}$$

Implémentation numérique

Résolution des EDP par intégrale de frontière

Integral representation of the solution of Stokes equation

$$\mathbf{u}(\mathbf{x}) = \int_{\mathcal{S}} \mathbf{G}(\mathbf{x} - \mathbf{x}_0) \mathbf{f}(\mathbf{x}_0) d\mathbf{x}_0 \quad \text{with}$$

$$\mathbf{G}(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} \left(\mathbf{I} + \frac{\mathbf{x} \otimes \mathbf{x}}{\|\mathbf{x}\|^2} \right)$$

Green kernel

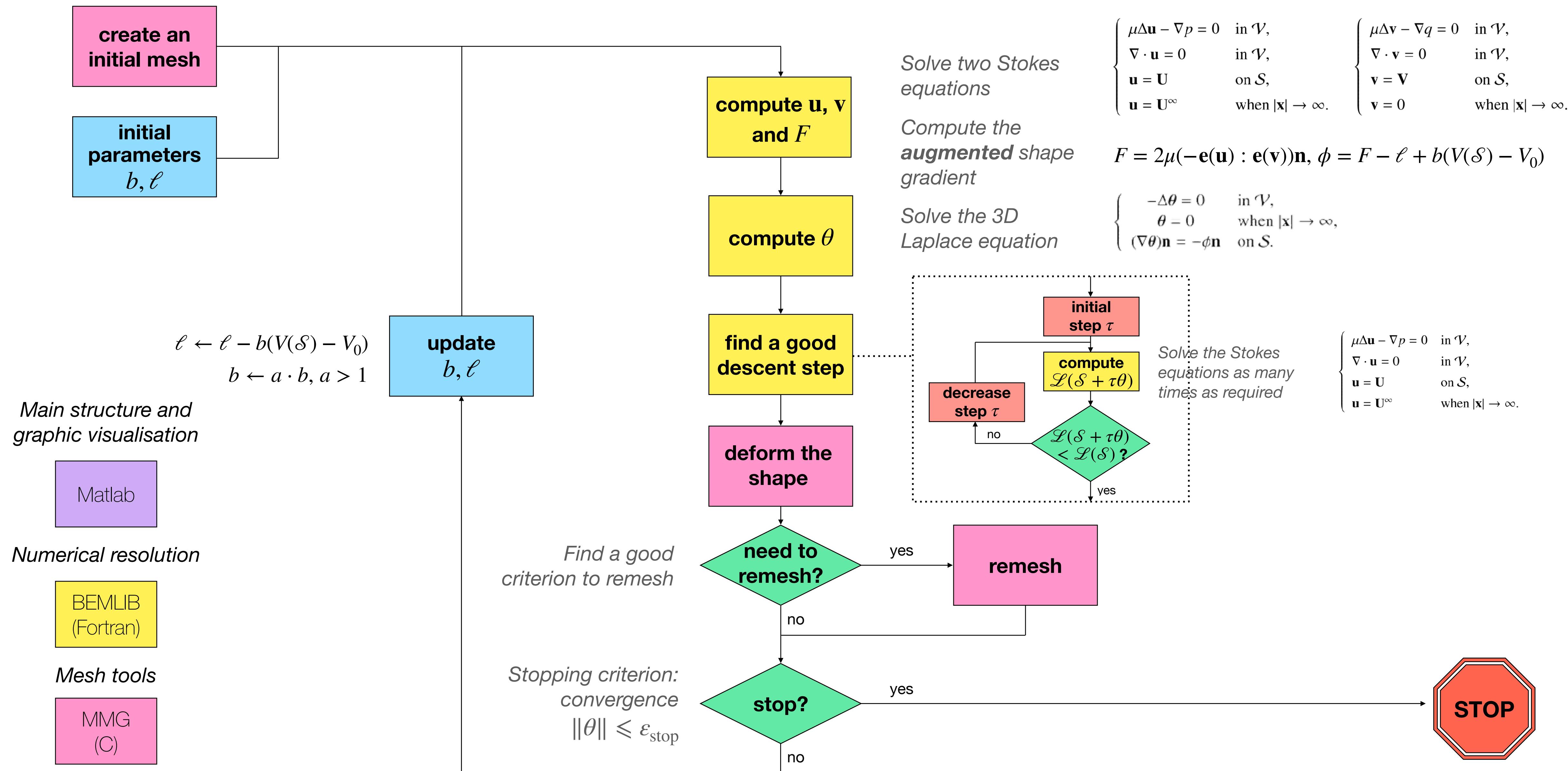
Boundary integral method

Mesh the surface – break down the integral above into a sum on the elements – express each integral as a discrete weighted sum – handle singularities – invert the linear system

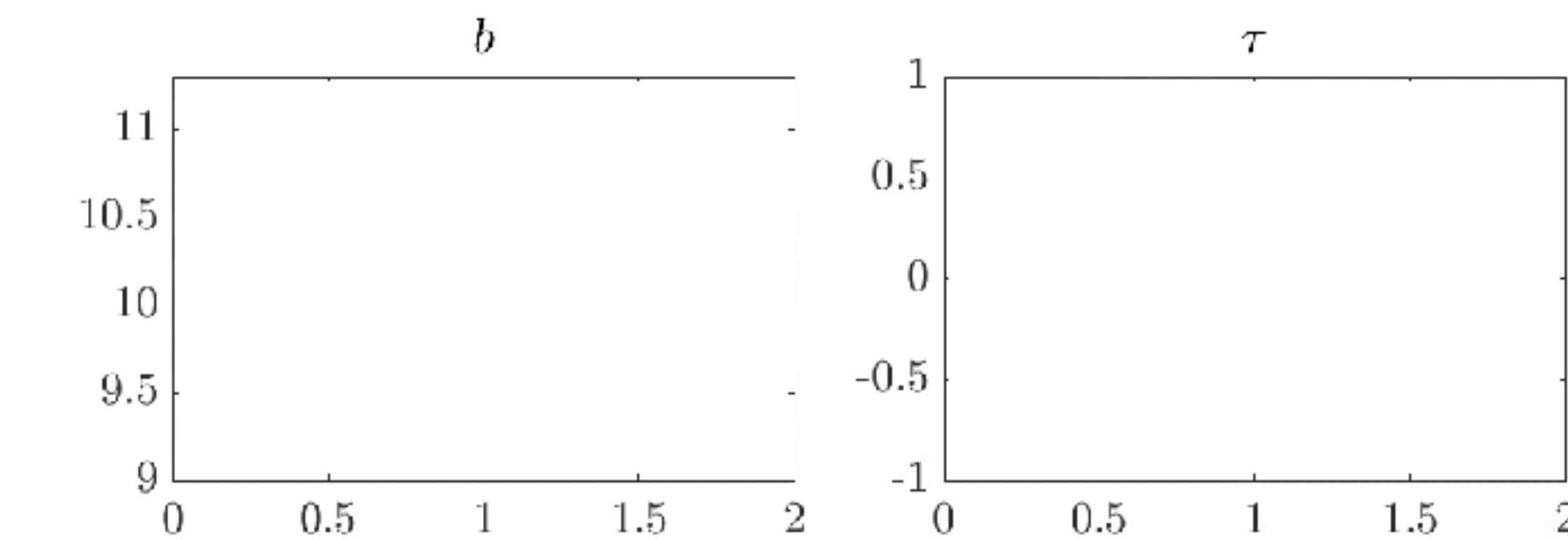
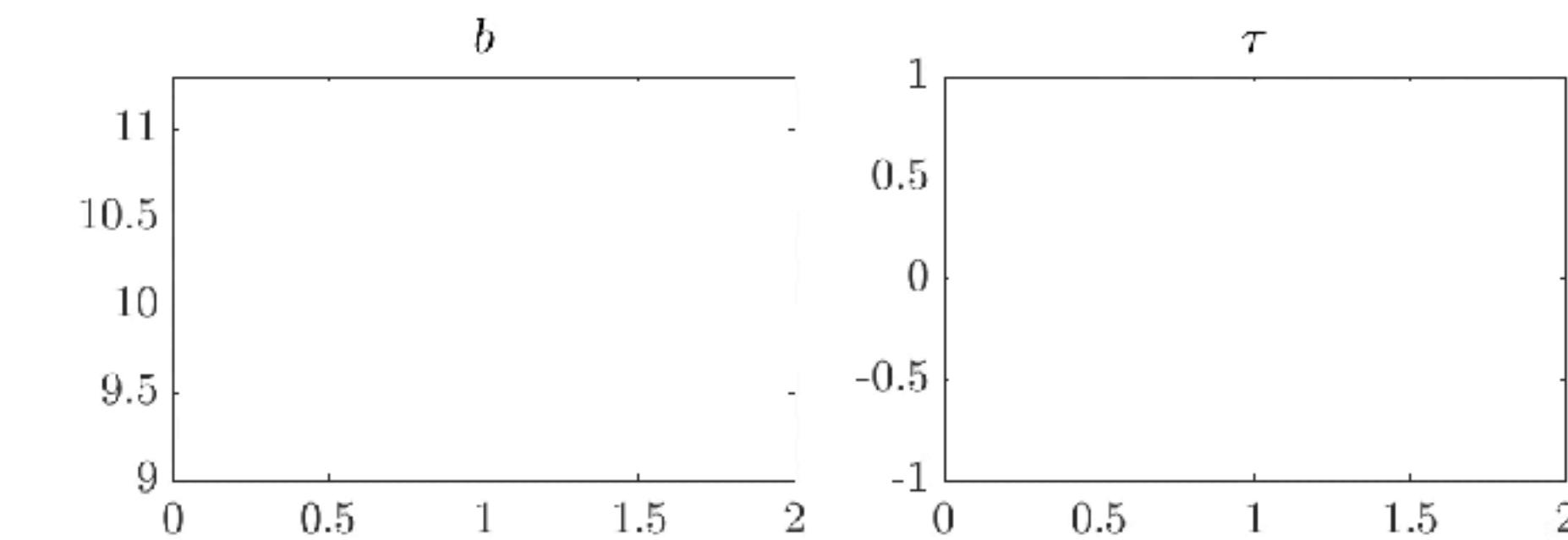
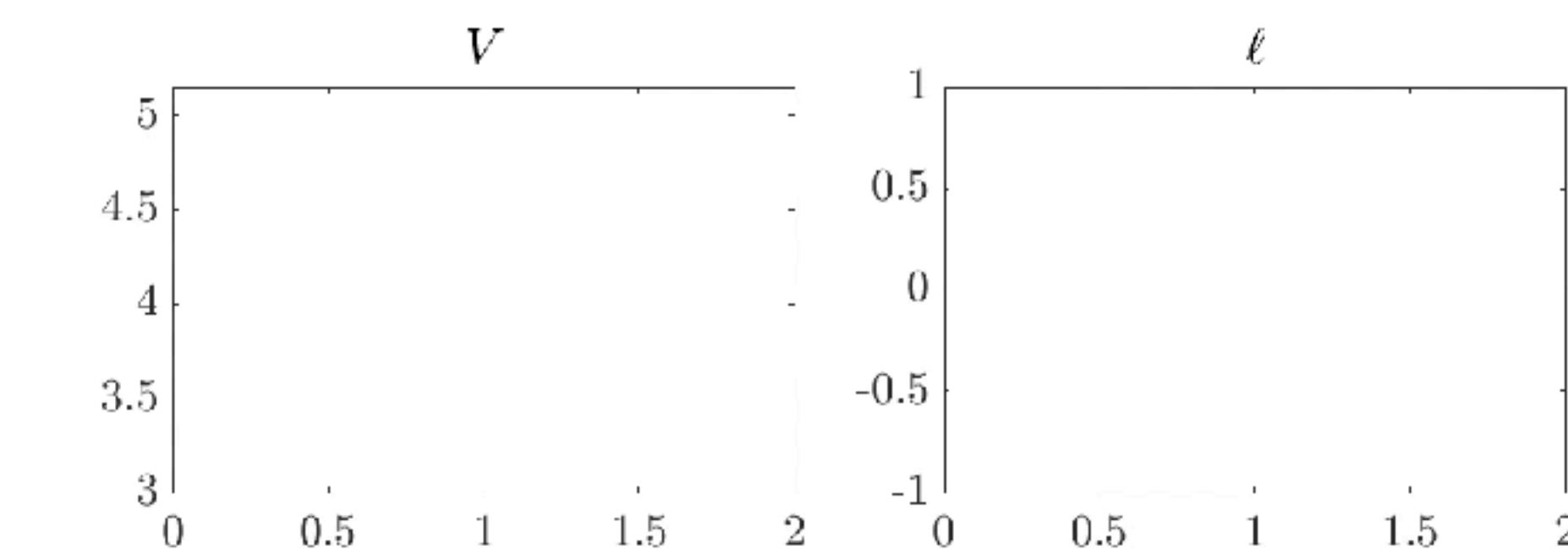
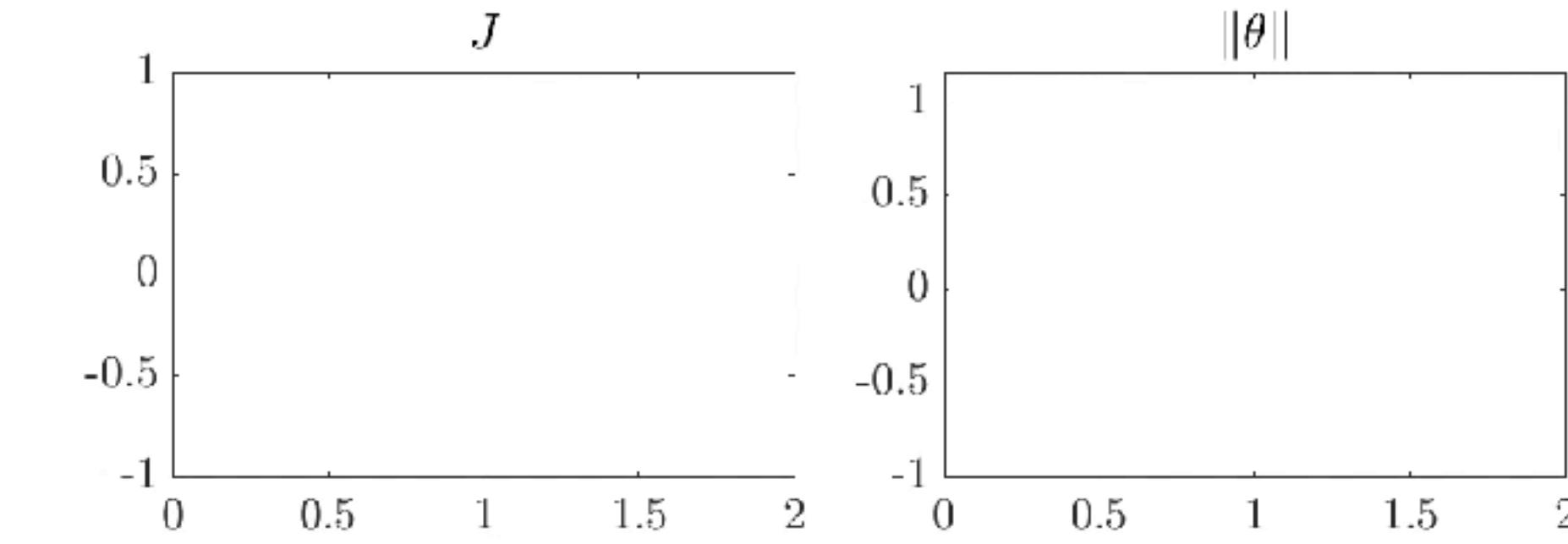
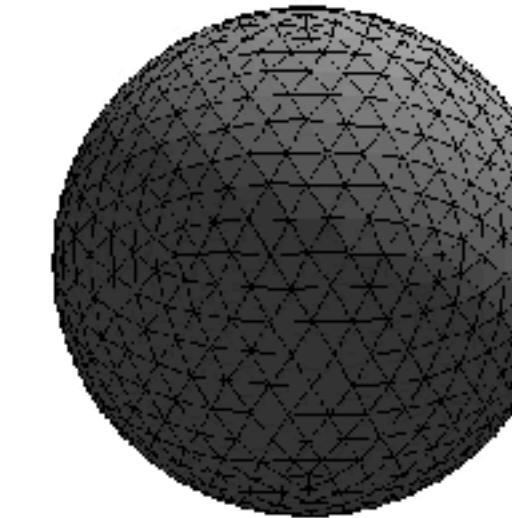
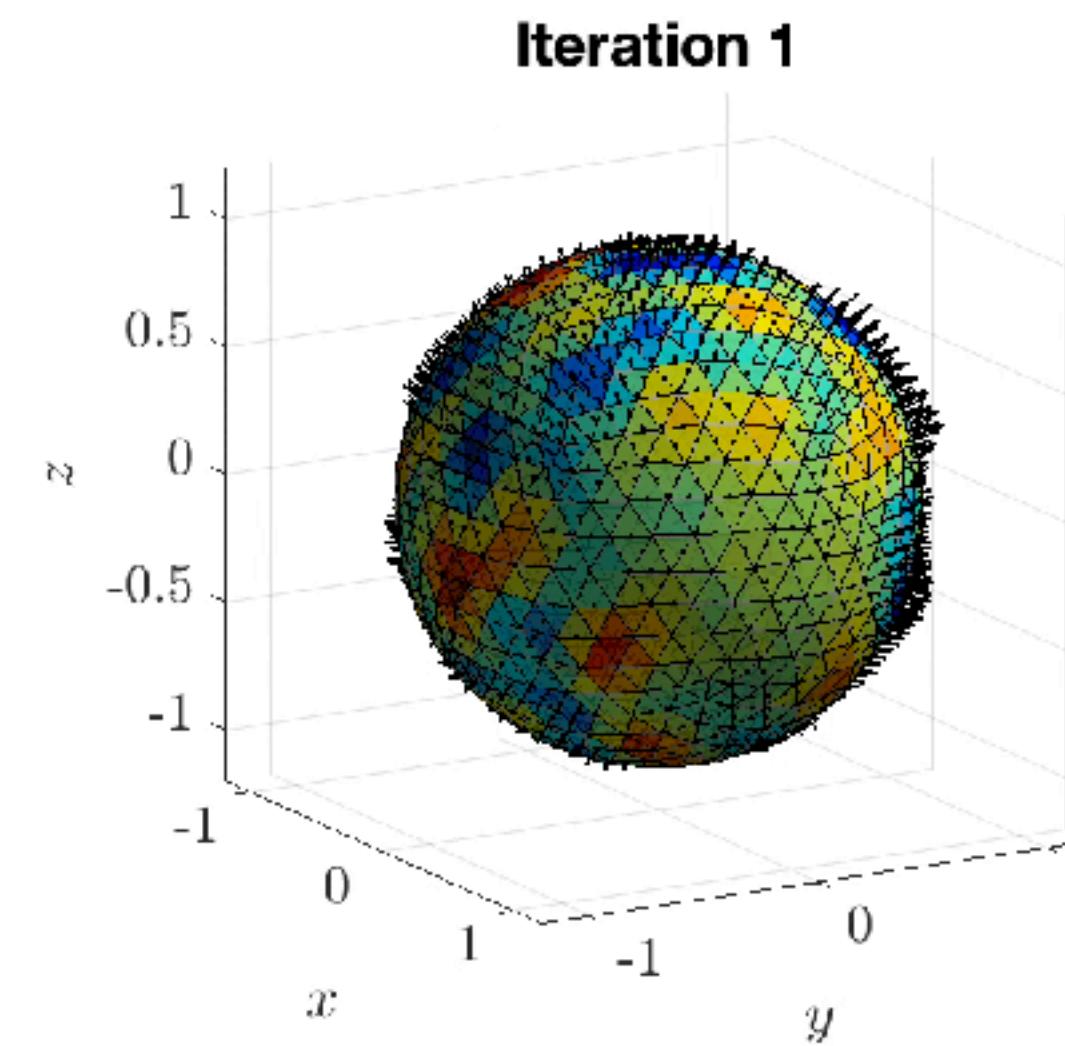
Integral representation of the rate-of-strain tensor

$$e_{ij}(\mathbf{x}) = \int_{\mathcal{S}} \left(\frac{1}{|r|^3} \delta_{ij} r_k - \frac{3}{|r|^5} r_i r_j r_k \right) f_k(\mathbf{x}_0) d\mathbf{x}_0$$
$$r = \mathbf{x} - \mathbf{x}_0$$

Shape optimisation algorithm with augmented Lagrangian

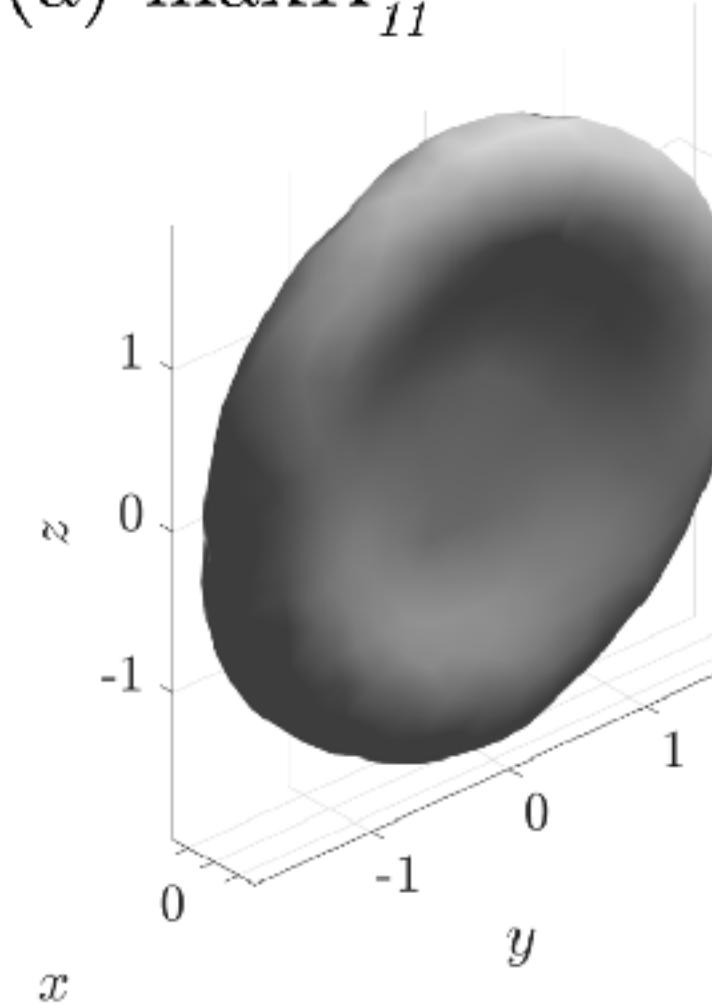


Numerical results: example on C_{11}

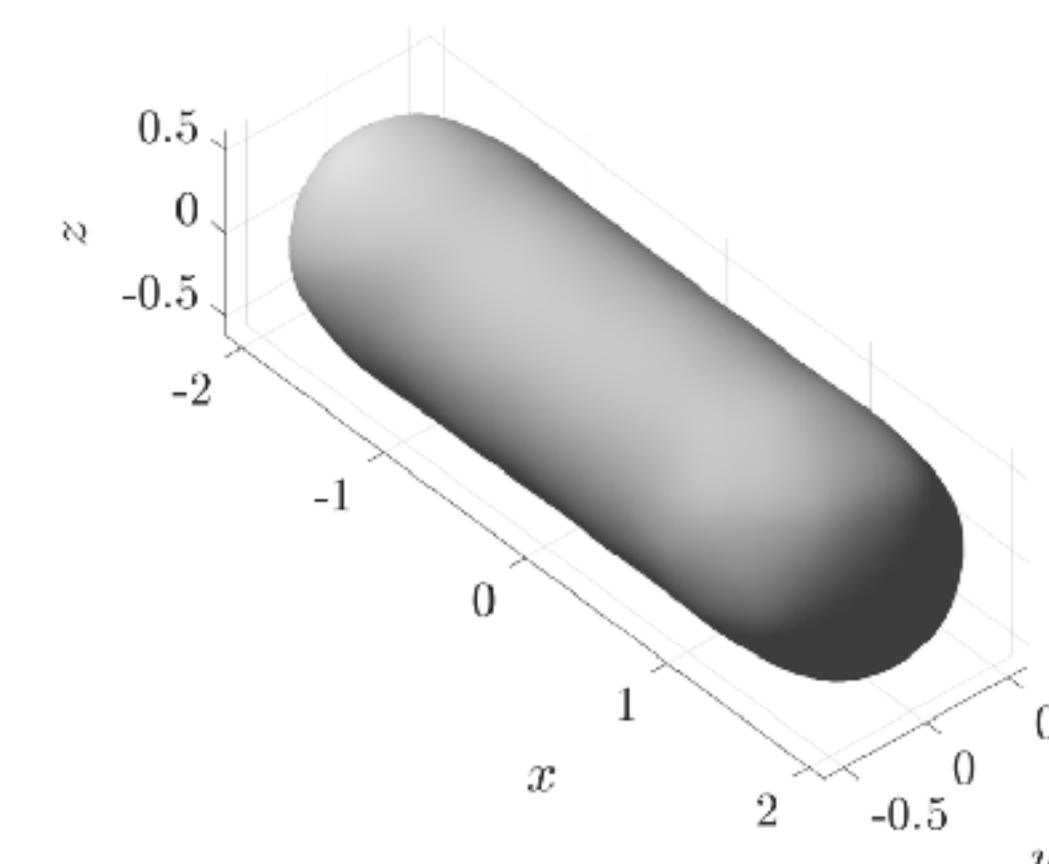


Numerical results

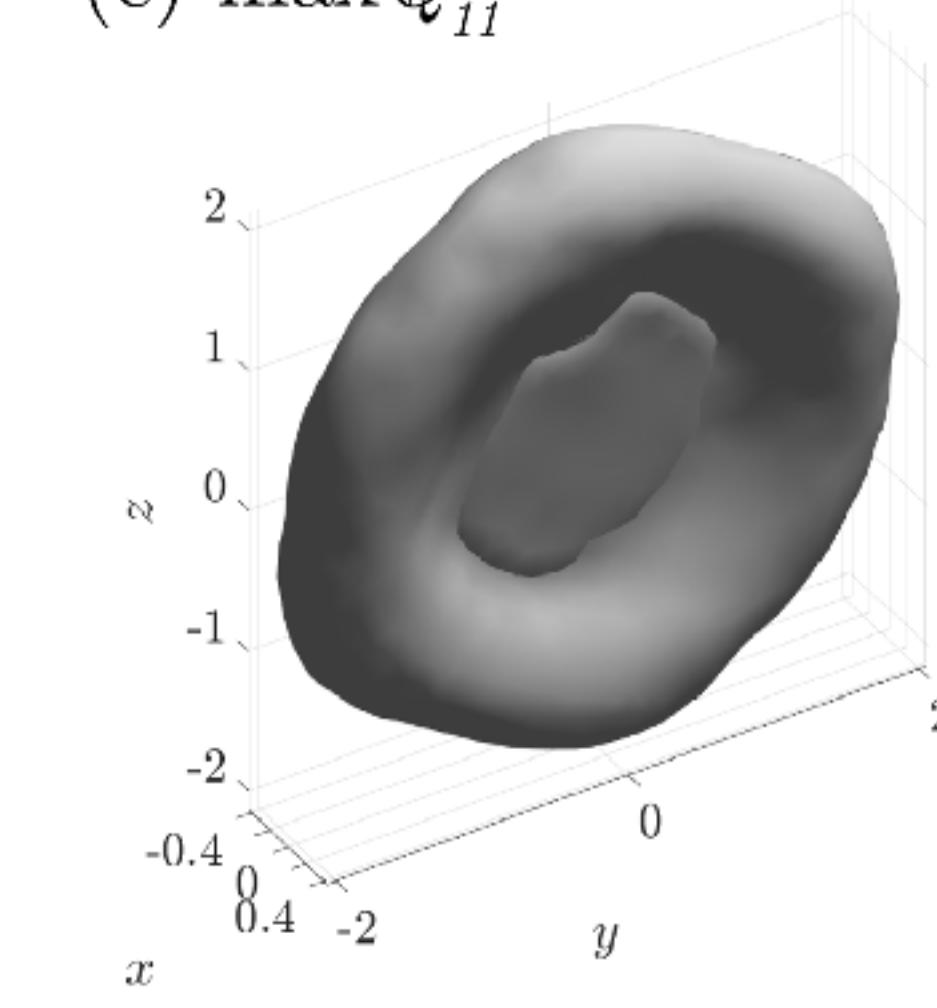
(a) $\max K_{11}$



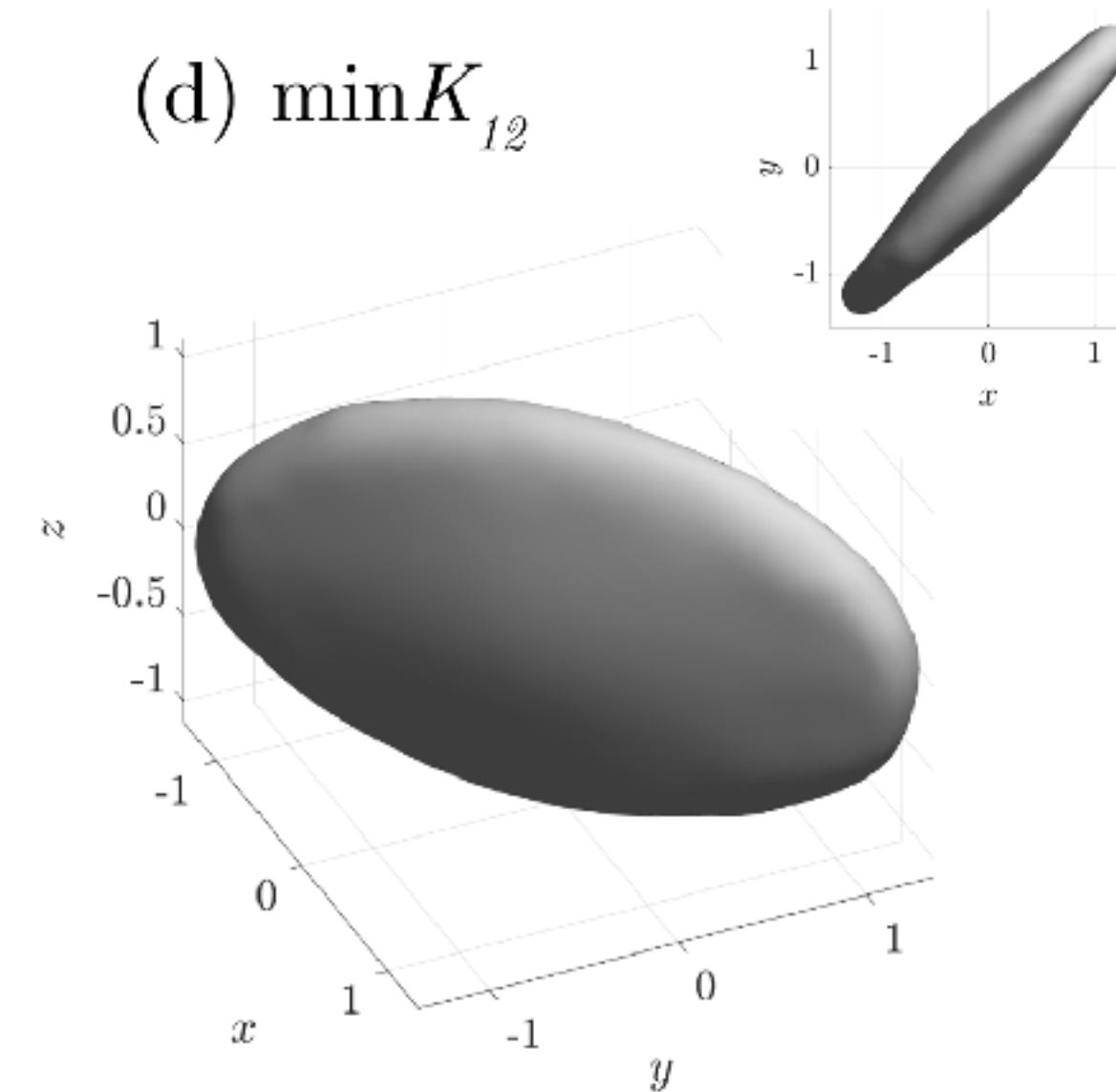
(b) $\min Q_{11}$



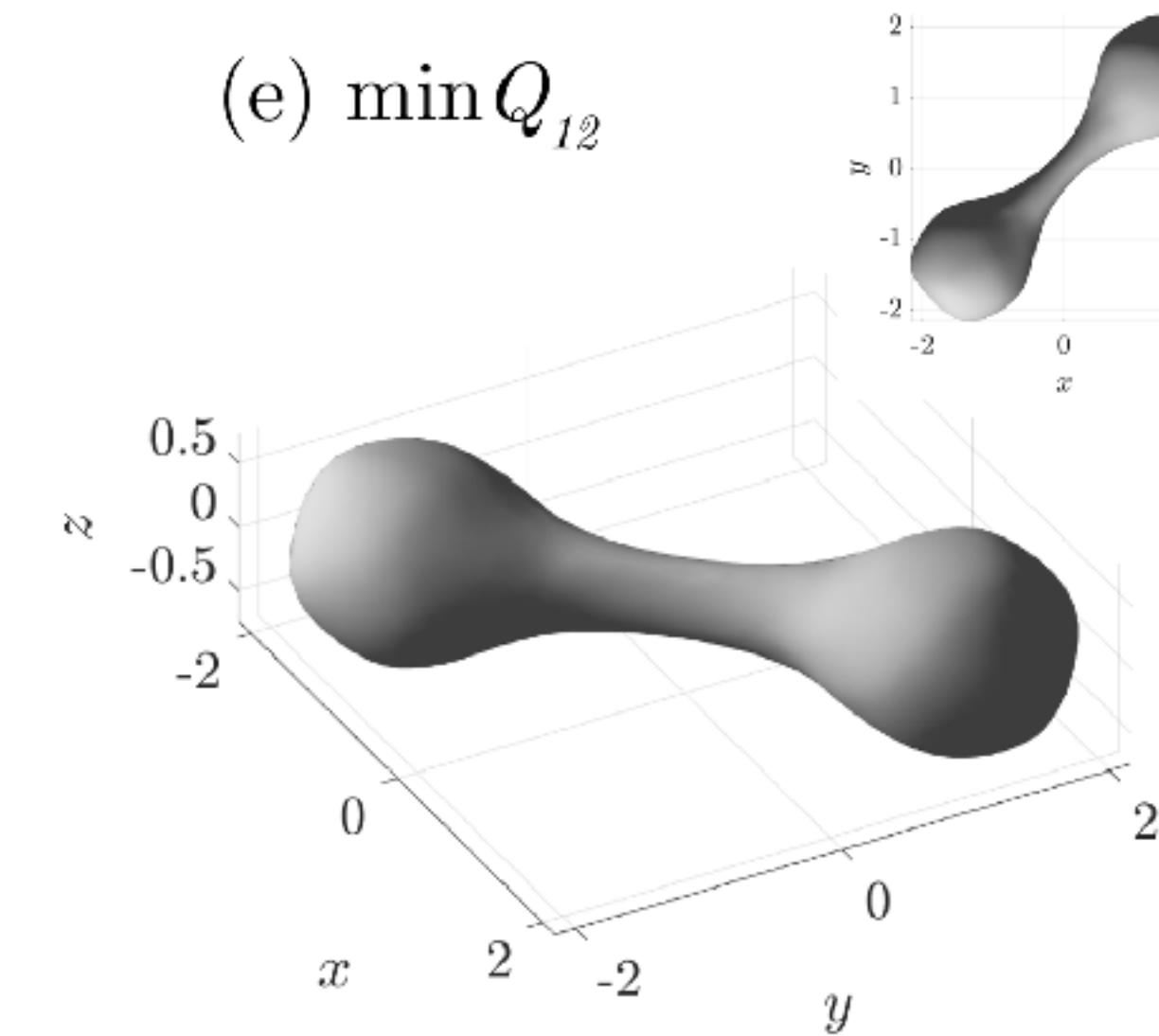
(c) $\max Q_{11}$



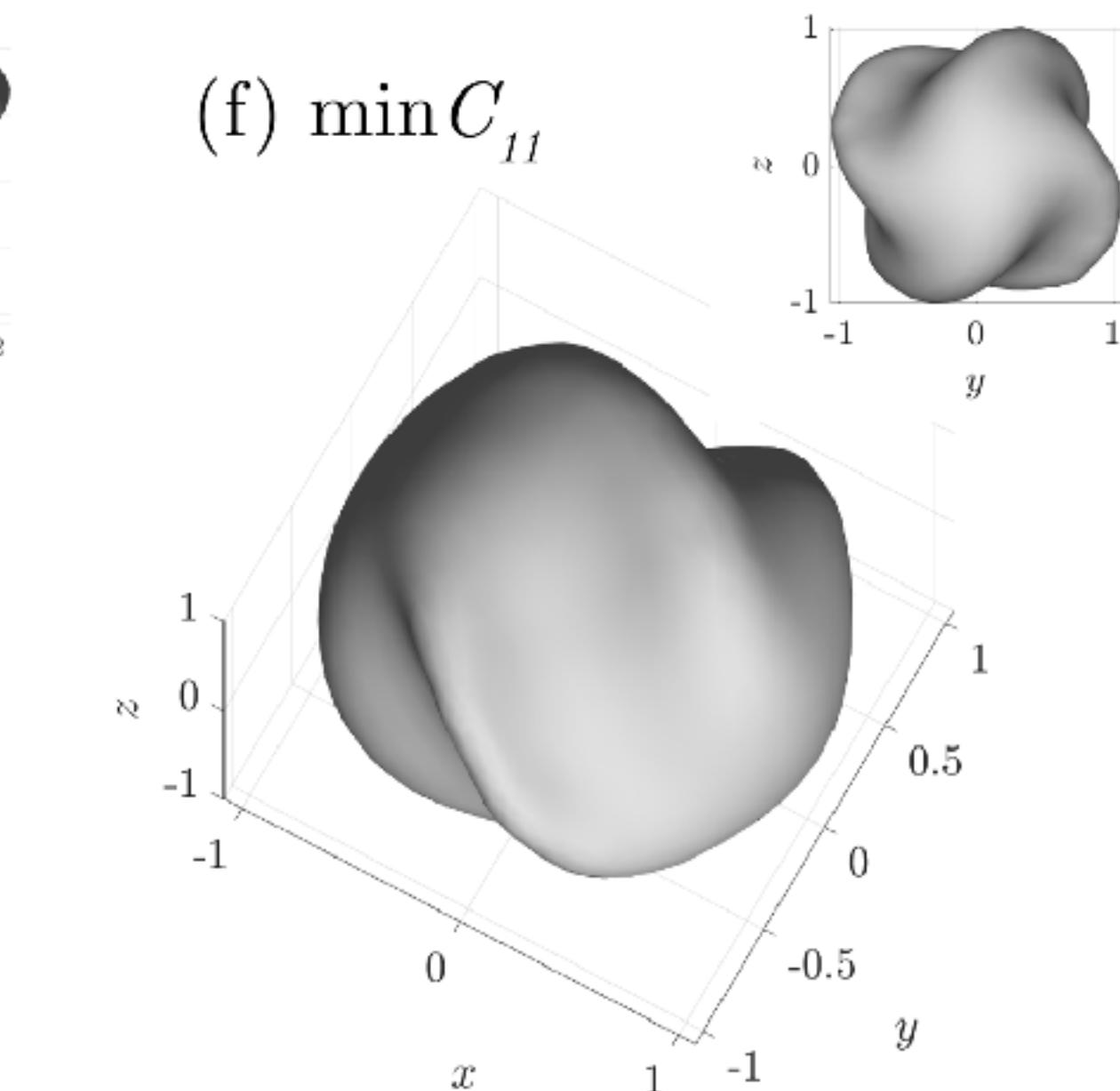
(d) $\min K_{12}$



(e) $\min Q_{12}$



(f) $\min C_{11}$



Perspectives !

- **objets déformables**
- **fonctionnelles plus complexes**
- **présence d'obstacles**
- **conditions au bord non standard**
- **preuve de l'existence des minimiseurs**
- ...