On a structure-preserving numerical method for fractional Fokker-Planck

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The fractional kinetic Fokker-Planck equation

The Lévy-Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f \;=\; \nabla_v \cdot (vf) - (-\Delta_v)^{\alpha/2} f =: L_\alpha f \,,$$

$t \geqslant 0$, $x \in \mathbb{T}^d$ and $v \in \mathbb{R}^d$; $\alpha \in (0,2)$.

For any nice function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{F}((-\Delta_v)^{\alpha/2} g)(\xi) \;=\; |\xi|^\alpha \mathcal{F}(g)(\xi)$$

where $\mathcal{F}(\cdot)$ the Fourier transform.

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Another equivalent definition:

$$(-\Delta_v)^{\alpha/2} g(v) \;=\; C_{d,\alpha}\, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{g(v) - g(w)}{|v-w|^{d+\alpha}}\, \mathrm{d}w\,,$$

where $\mathrm{P.V.}$ the principal value.

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$$\|f(t) - \left\langle f^0 \right\rangle \mu_\alpha\|_X \;\leqslant\; C\, \|f^0 - \left\langle f^0 \right\rangle \mu_\alpha\|_X \, e^{-\lambda t}$$

with $\left\langle f^0 \right\rangle := \displaystyle\int\!\!\int_{\mathbb{T}^d \times \mathbb{R}^d} f^0 \mathrm{d}x \mathrm{d}v$.

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Our approach: The H^1 method. For the L^2 method, see Bouin et al (2019).

(Hypo)coercive schemes

Aim: **A numerical approach**.

The classical case: Dujardin et al (2020) [H^1 method], Bessemoulin et al (2020) [L^2 method].

Goal: Design of a **consistent**, **stable** and **structure preserving** numerical method for $d = 1$.

The continuous case

The Sobolev space $H^1_{x,v}(\nu)$ is associated with the norm

$$\|g\|^2_{H^1_{x,v}(\nu)} = \|g\|^2_{L^2_{x,v}(\nu)} + \|\nabla_x g\|^2_{L^2_{x,v}(\nu)} + \|\nabla_v g\|^2_{L^2_{x,v}(\nu)}.$$

Theorem (Ayi, Herda, Hivert, Tristani (2020))

Let f solve **the kinetic Lévy-Fokker-Planck equation** *with initial data $f^{in} \in H^1_{x,v}(\mu_\alpha^{-1})$. Then, for all $t \geqslant 0$ one has*

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for some constant $C \geqslant 1$ and $\lambda > 0$ depending only on d and α.

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Hypocoercivity strategy : Equivalent functional

$$\mathcal{H}(f) = \|f\|^2_{L^2(\mu_\alpha^{-1})} + a\|\nabla_x f\|^2_{L^2(\mu_\alpha^{-1})} + b\,\|\nabla_v f\|^2_{L^2(\mu_\alpha^{-1})} + 2\,c\,\langle \nabla_x f, \nabla_v f \rangle_{L^2(\mu_\alpha^{-1})}$$

Challenges of the discrete setting

Aim: Design of a **consistent**, **stable** and **structure preserving** numerical method for $d = 1$.

Preservation of the structure:

- conservation of mass;
- preservation of the heavy-tailed local equilibrium μ_α;
- preservation of coercivity properties in the homogeneous case;
- preservation of the hypocoercivity properties in the inhomogeneous case;
- approximation of the fractional Fokker-Planck operator L_α on the whole line with a discretization on a truncated domain;
- preservation of the asymptotics $\alpha \to 2^-$,
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Result: Rigorous **coercivity and hypocoercivity properties** $\Rightarrow$ exponential stability of the discrete solution.

Presentation of the numerical method (unbounded velocity domain)

Discretization of $\mathbb{R}$: $(v_j = jh)_{j\in\mathbb{Z}}$ with $h > 0$.

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$$f_j \approx f(v_j).$$

Slight abuse of notation: $f = (f_j)_{j\in\mathbb{Z}}$.

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Discretization of the fractional Laplacian: $\Lambda^h_\alpha : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ such that

$$(\Lambda^h_\alpha f)_j \approx -(-\Delta)^{\alpha/2} f(v_j).$$

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The Huang-Oberman method (2014): the discrete fractional Laplace operator is

$$(\Lambda_\alpha^h f)_j \;=\; \sum_{k=1}^{\infty} \beta_k^h \,(f_{j+k} + f_{j-k} - 2f_j)\, h \;=\; \sum_{k\in\mathbb{Z}} \beta_k^h \,(f_{j-k} - f_j)\, h\,,$$

Lemma

There exist positive constants b_α and B_α depending only on $\alpha \in (0,2)$ such that

$$\frac{b_\alpha}{|hk|^{1+\alpha}} \;\leqslant\; \beta_k^h \;\leqslant\; \frac{B_\alpha}{|hk|^{1+\alpha}}\,, \quad \forall k \in \mathbb{Z}\backslash\{0\}\,.$$

Lemma (Huang-Oberman (2014))

Conservation of mass:

$$\sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h u)_j \;=\; 0\,.$$

Self-adjoint *in the space of square summable sequences:*

$$\sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h u)_j v_j \;=\; \sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h v)_j u_j\,.$$

Consistency *with the usual centered finite difference* **approximation of the Laplacian***:*

$$\lim_{\alpha\to 2^-} (\Lambda_\alpha^h u)_j \;=\; \frac{u_{j+1}+u_{j-1}-2u_j}{h^2}\,,$$

for all $j \in \mathbb{Z}$.

Consistency *at order $3-\alpha$. When $h \to 0$, one has for any $u \in \mathcal{C}_b^4(\mathbb{R})$ that*

$$\sup_{j\in\mathbb{Z}} \left| -(-\Delta)^{\alpha/2} u(hj) - (\Lambda_\alpha^h u)_j \right| \;\leqslant\; K_\alpha \, \|u\|_{\mathcal{C}_b^4(\mathbb{R})} \, h^{3-\alpha}\,,$$

with K_α a positive constant depending only on α.

Discretization of the Lévy-Fokker-Planck operator: $L_\alpha^h \; = \; \Gamma_\alpha^h + \Lambda_\alpha^h$
where Γ_α^h discrete equivalent of $\partial_v(v\,\cdot)$.

Goal: define a **consistent approximation** that **preserves** exactly **the discrete equilibrium** $(M_j)_{j\in\mathbb{Z}}$ defined by

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Idea: Using that $L_\alpha \mu_\alpha = 0$ and that μ_α is symmetric, we get

$$\partial_v(v\,f) \;=\; \partial_v\left(v\,\mu_\alpha\,f/\mu_\alpha\right)$$

and

$$v\mu_\alpha(v) \;:=\; \frac{1}{2}\int_{-v}^{v}(-\Delta_w)^{\alpha/2}\mu_\alpha(w)\mathrm{d}w\,,$$

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The operator Γ_α^h is

$$(\Gamma_\alpha^h f)_j := \frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{h}$$

with the numerical flux defined by the centered approximation

$$\mathcal{F}_{j+\frac{1}{2}} := (VM)_{j+\frac{1}{2}}\left(\frac{f_j}{2M_j} + \frac{f_{j+1}}{2M_{j+1}}\right),$$

and

$$(VM)_{j+\frac{1}{2}} = -(VM)_{-j-\frac{1}{2}} := -\frac{1}{2}\sum_{k=-j}^{j}(\Lambda_\alpha^h M)_k\,h\,, \quad \text{for } j \geqslant 0\,.$$

Lemma (Basic properties)

The operator L_α^h satisfies the following properties.

i) **Mass conservation:** *for any suitably summable sequence u, one has*

$$\sum_{j\in\mathbb{Z}} (L_\alpha^h u)_j \, h \;=\; 0 \, .$$

ii) **Preservation of local equilibrium:**

$$(L_\alpha^h M)_j \;=\; 0 \, , \quad \forall j \in \mathbb{Z} \, .$$

iii) **Consistency:** *for any $u \in \mathcal{C}_b^4(\mathbb{R})$, one has that*

$$\sup_{j\in\mathbb{Z}} \left| (L_\alpha u)(hj) - (L_\alpha^h u)_j \right| \;\leqslant\; K_\alpha \, \|u\|_{\mathcal{C}_b^4(\mathbb{R})} \, h^{\min(3-\alpha,2)} \, ,$$

for some $K_\alpha > 0$.

iv) **Non-negative symmetric part:** *in natural weighted ℓ^2 space*

$$\mathcal{S}_\alpha^h(f,f) = -\sum_{j\in\mathbb{Z}} (L_\alpha^h f)_j f_j M_j^{-1} h = \frac{1}{2} \sum_{(j,k)\in\mathbb{Z}^2} \beta_k^h \left(\frac{f_j}{M_j} - \frac{f_{j+k}}{M_{j+k}} \right)^2 M_j \, h^2 \, ,$$

Numerical schemes

The homogeneous case: For a time discretization $t_n = n\Delta t$ with time step $\Delta t > 0$, $f_j^n \approx f(t_n, v_j)$ is computed by solving the **implicit in time scheme**

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = (L_\alpha^h f)_j^{n+1}, \quad \forall (n,j) \in \mathbb{N} \times \mathbb{Z},$$

and starts at some given initial data $(f_j^0)_j$.

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The inhomogeneous case: For any $(n, i, j) \in \mathbb{N} \times \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}$, $t_n = n\Delta t$, $x_i = i\Delta x$ and $v_j = j\Delta v$ where $\Delta x = N_x^{-1}$ with N_x an odd positive integer is **the space step** and $\Delta v > 0$ **the velocity step**.
$f_{ij}^n \approx f(t_n, x_i, v_j)$ is computed by solving the **implicit in time scheme**

$$\frac{f_{ij}^{n+1} - f_{ij}^n}{\Delta t} + (T^{\Delta x} f)_{ij}^{n+1} = (L_\alpha^{\Delta v} f)_{ij}^{n+1}, \quad \forall (n, i, j) \in \mathbb{N} \times \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z},$$

with given initial data $(f_{ij}^0)_{ij}$. The **discrete transport operator** writes

$$(T^{\Delta x} f)_{ij}^n = v_j \frac{f_{i+1,j}^n - f_{i-1,j}^n}{2\Delta x}.$$

Exponential stability

Theorem (Ayi, Herda, Hivert, Tristani (2021))

Suppose that N_x is odd. There exists $\Delta v_0 > 0$ such that if f is a solution of **the discrete kinetic Lévy-Fokker-Planck equation** *with initial data $(f^0_{i,j})_{i,j} \in H^1_{\Delta x, \Delta v}(M^{-1})$, then for all $\Delta v < \Delta v_0$ and for all $n \in \mathbb{N}$, one has*

$$\|f^n - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})} \leqslant C\|f^0 - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})} \, (1 + 2\lambda\Delta t)^{-\frac{n}{2}},$$

for some constants $C \geqslant 1$ and $\lambda > 0$ depending only on α and
$f^\infty := \frac{\langle f^0 \rangle_{\Delta x,\Delta v}}{\langle M \rangle_{\Delta x,\Delta v}} M$ *with* $\langle f \rangle_{\Delta x,\Delta v} := \sum_{(i,j)\in\mathbb{Z}/N_x\mathbb{Z}\times\mathbb{Z}} f_{i,j}\, \Delta x \Delta v.$

Discrete functional setting: discrete Sobolev norms

$$\|f\|^2_{H^1_{\Delta x,\Delta v}(M^{-1})} = \|f\|^2_{\ell^2_{\Delta x,\Delta v}(M^{-1})} + \|D_{\Delta x} f\|^2_{\ell^2_{\Delta x,\Delta v}(M^{-1})} + \|D_{\Delta v} f\|^2_{\ell^2_{\Delta x,\Delta v}(M^{-1})}$$

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with the **scalar product**

$$\langle f, g\rangle_{\ell^2_{\Delta x,\Delta v}(M^{-1})} = \sum_{i\in\mathbb{Z}/N_x\mathbb{Z}} \sum_{j\in\mathbb{Z}} \frac{f_{i,j} g_{i,j}}{M_j} \Delta x \Delta v\,.$$

and the **associated norm** $\|\cdot\|_{\ell^2_{\Delta x,\Delta v}(M^{-1})}$ and where

$$\forall (i,j) \in \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z},\ (D_{\Delta x} f)_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x},\ \ (D_{\Delta v} f)_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta v}\,.$$

Ideas and challenges in the proof

- Same roadmap as in the continuous setting: **close estimates** on **twisted hypocoercivity norm** of solution

$$\|f\|^2 + a\|D_{\Delta x} f\|^2 + b\|D_{\Delta v} f\|^2 + 2c\left\langle D_{\Delta x} f, D_{\Delta v} f\right\rangle$$

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- **Functional analysis tools** in the discrete setting

Discrete fractional Sobolev norms :

$$|g|^2_{\dot{H}^s_{\Delta v}(M^{-1})} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(g_j - g_{j+k})^2}{|\Delta v k|^{1+2s}} M_j^{-1}\,\Delta v^2, \quad s>0$$

Interpolation inequalities

$$\|D^+_{\Delta v}g\|^2_{\ell^2_{\Delta v}} \lesssim \varepsilon\,|D_{\Delta v}g|^2_{\dot{H}^s_{\Delta v}} + K(\varepsilon)\,\|g\|^2_{H^s_{\Delta v}}$$

Non-local Poincaré inequality

$$\|f - \Pi_{\Delta v}f\|^2_{\ell^2_{\Delta v}(M^{-1})} \lesssim \mathcal{S}^{\Delta v}_{\alpha}(f,f)$$

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- **Technical challenges:** Nasty commutators in the discrete setting

$$[\partial_v, L_\alpha] = \partial_v \quad \text{vs.} \quad [D_{\Delta v}, L^h_\alpha] = D_{\Delta v} + \text{remainders}$$

Numerical simulations

Test case 1: convergence of the scheme in the homogeneous case.

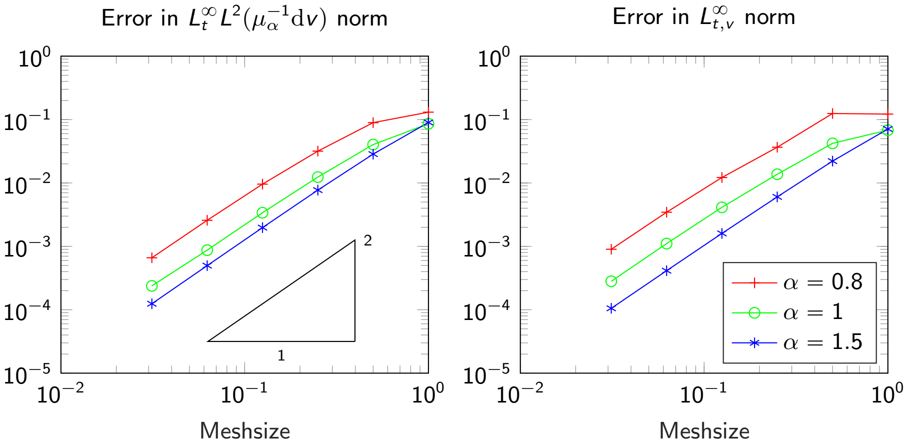

Figure: **Test case 1.** Error in $L^\infty_t L^2(\mu_\alpha^{-1}\mathrm{d}v)$ (left) and $L^\infty_{t,v}$ (right) norm between approximate and analytical solution.

Test case 2: preservation of the heavy tails (homogeneous case).
Truncated velocity domain $[-L, L]$ with $L = 20$, 1025 mesh points, time step $\Delta t = 10^{-2}$.
The initial data:

$$f(0, v) = \frac{1}{2}\chi_{[-3,-1]}(v) + \frac{1}{4}\chi_{[0,4]}(v),$$

where χ_I is the indicator function of the set I.

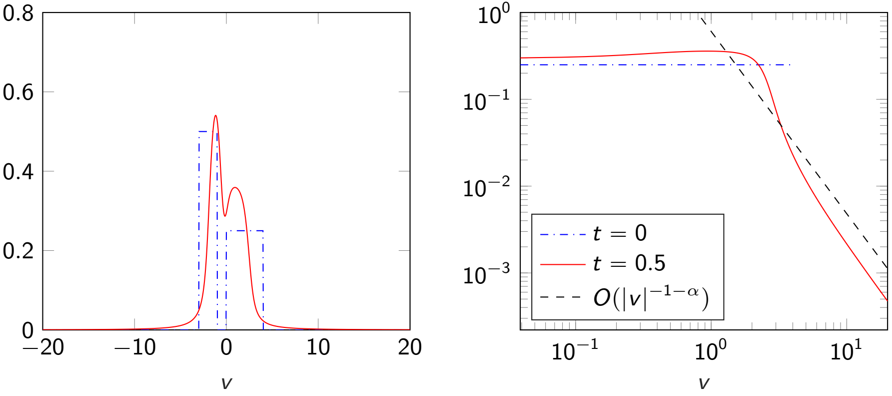

Figure: **Test case 2.** Approximate densities at $t = 0.5$. On the right the logarithmic scale allows to see the heavy-tail decay. Here $\alpha = 1.1$.

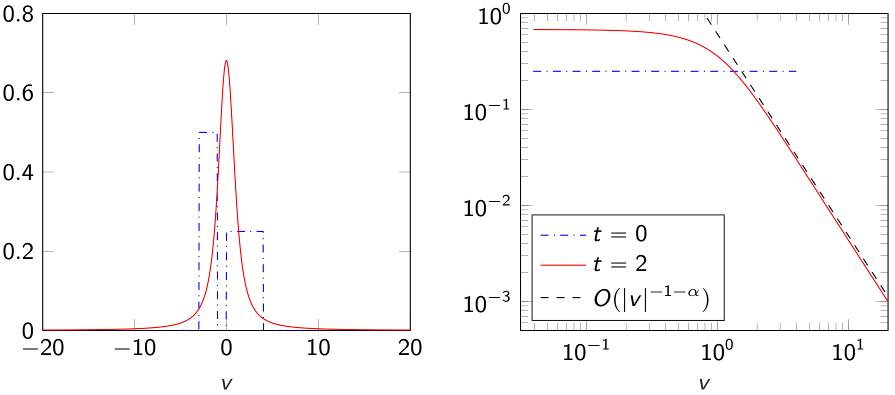

Figure: **Test case 2.** Approximate densities at $t = 2$. On the right the logarithmic scale allows to see the heavy-tail decay. Here $\alpha = 1.1$.

Test case 3: numerical hypocoercivity (long time behavior).

$$\partial_t f + v \cdot \nabla_x f = L_1 f$$

with $x \in \mathbb{R}/(2\pi\mathbb{Z})$ and $v \in \mathbb{R}$. Velocity domain truncated at $L = 16$, discretized 65 points ($J = 32$). Space domain, of size 2π, discretized with 128 points. Time step $\Delta t = 10^{-2}$, final time $T = 35$. Error of $4.5 \cdot 10^{-2}$ in $L^\infty_{t,x,v}$ norm between the computed solution and the reference solution.

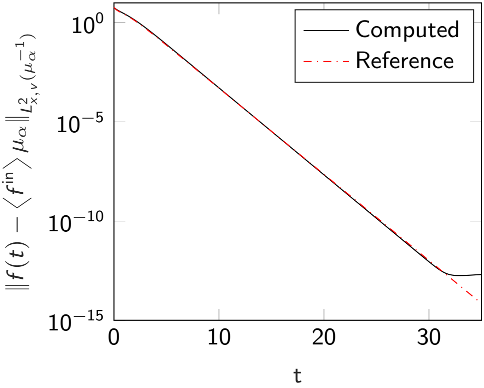

Figure: **Test case 3.** Time evolution of the distance between the steady state and the approximate and reference densities in $L^2_{x,v}(\mu_\alpha^{-1}\mathrm{d}v\mathrm{d}x)$ norm.

Thank you for your attention.