SDEs with distributional drift and path-by-path solutions

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joint work with Alexandre Richard[1] and Etienne Tanré[2]

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Outline

1. Motivation and definitions

2. Existence of a solution via nonlinear Young integration

3. Path-by-path vs. adapted solutions

Table of Contents

Regularization by noise

- Let B^H be a fractional Brownian motion with Hurst parameter $H \leqslant 1/2$ ($\implies$ $W := B^{1/2}$ denotes Brownian motion).
- Well known: Reflected Brownian motion solves the following SDE:

$$X_t = L_t^X(0) + B_t^{1/2}, \text{ where } \text{``}L_t^X(0) = \int_0^t \delta_0(X_r)dr\text{''}.$$

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- Do we have existence and uniqueness to the SDE

$$X_t = \int_0^t \phi(X_r)dr + B_t^H, \tag{$\bullet$}$$

where ϕ has a singularity, is a finite measure, or is even in a more general class of distributions? (see also Nualart and Ouknine (2002), if $H = 1/2$ see e.g. Krylov and Röckner (2005), Davie (2007))

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- We can rephrase ($\bullet$) by $\tilde{X} = X - B^H$, as

$$\tilde{X}_t = \int_0^t \phi(\tilde{X}_r + B_r^H)dr.$$

- Hope: $x \mapsto T_t^{B^H}\phi(x) := \int_0^t \phi(x + B_r^H)dr$ is regular.

Averaging operator

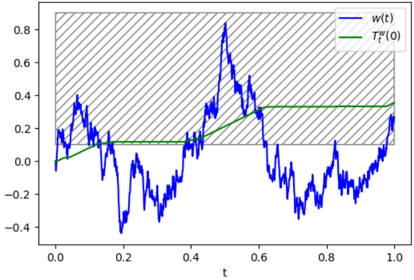
Definition

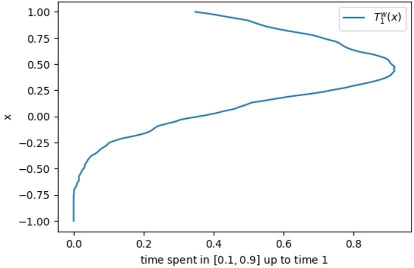
Let $w \in \mathcal{C}([0,T],\mathbb{R})$. For bounded measurable $\phi : \mathbb{R} \to \mathbb{R}$ let

$$T_t^w \phi(x) := \int_0^t \phi(x + w_r) dr \textit{ for } (t,x) \in [0,T] \times \mathbb{R}.$$

Averaging operator for $\phi = \mathbb{1}_{[0.1,0.9]}$

$$T_t^w \mathbb{1}_{[0.1,0.9]}(x) := \int_0^t \mathbb{1}_{[0.1,0.9]}(x + w_r) dr \ \ for \ (t, x) \in [0, T] \times \mathbb{R}$$





Different kinds of solutions

Definition (Adapted vs. path-by-path solutions)

For a bounded measurable function ϕ, we consider

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- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and $\{B_t^H\}_{t\in[0,T]}$ an $\mathbb{F}$-fBm defined on it. We call a continuous $\mathbb{F}$-adapted process $\{X_t\}_{t\in[0,T]}$ fulfilling (2) a weak solution.

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- If X is $\mathbb{F}^{B^H}$-adapted, we call it a strong solution.
- We call $X : \Omega \to \mathcal{C}([0,T], \mathbb{R})$ a path-by-path solution if it fulfills (2) for $\omega \in \tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$.

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$\boxed{\text{strong existence}} \implies \boxed{\text{weak existence}} \implies \boxed{\text{path-by-path existence}}$

Local time

Definition of local time at $x \in \mathbb{R}$

$$L_t^W(x) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{B_\varepsilon(x)\}}(W_r)dr$$

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- $\int_0^t \mathbb{1}_A(W_r)dr = \int_A L_t^W(x)dx$ ($\implies \int_0^t f(W_r)dr = \int_{\mathbb{R}} f(x)L_t^W(x)dx$ for bounded measurable f).

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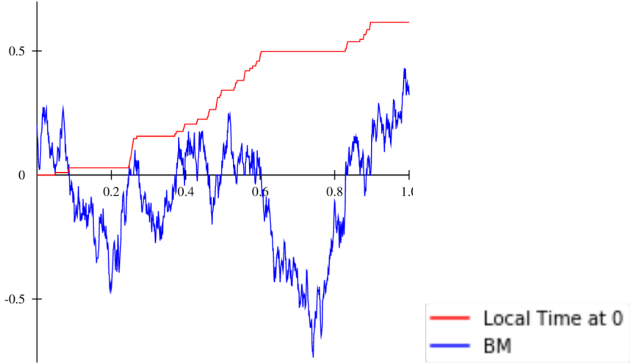

Fractional Brownian motion

Definition

Let $H \in (0, 1)$. We call the centered Gaussian process $\{B_t^H\}_{t \geqslant 0}$ with covariance function

$$\mathbb{E}[B_s^H B_t^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

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- $B^{1/2}$ is a Brownian motion.
- For $H \neq 1/2$, B^H is neither Markov nor a (semi)martingale.
- B^H is almost surely γ-Hölder continuous for $\gamma \in (0,H)$ on any compact interval.

Regularising with fBm

Idea: "rougher noise $\implies$ more regularisation"

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Theorem (Catellier and Gubinelli 2016)

Let $\phi \in \mathcal{C}^\alpha$ for $\alpha > 1 - \frac{1}{2H}$. Then there exists a set of full measure w.r.t. the law of fBm such that there exists a unique solution $X \in \mathcal{C}([0,T])$ to

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In case of $\phi = \delta_0$ this gives a unique solution for $H < 1/4$. Main objective: Establish sharpness of this inequality or extend to larger values of H!

Table of Contents

Averaging operator defined via the local time

Equation of interest

$$X_t = \int_0^t \phi(X_r)dr + B_t^H \iff \tilde{X}_t = \int_0^t \phi(\tilde{X}_r + B_r^H)dr, \text{ for } \tilde{X} = X - B^H.$$

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Let $\phi \in \mathcal{C}_b(\mathbb{R})$ and $T_t^{B^H}\phi(x) := \int_0^t \phi(x + B_r^H)dr$. Then

$$\begin{aligned}\int_0^t \phi(\tilde{X}_r + B_r^H)dr &= \lim_{N\to\infty} \sum_{i=1}^N \int_{t_i}^{t_{i+1}} \phi(\tilde{X}_{t_i} + B_r^H)dr \\ &= \lim_{N\to\infty} \sum_{i=1}^N T_{t_i,t_{i+1}}^{B^H}\phi(\tilde{X}_{t_i}) \text{ “}=\text{” } \int_0^t T_{dr}^{B^H}\phi(\tilde{X}_r).\end{aligned}$$

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Let L denote the local time of B^H and $\phi \in \mathcal{C}_b(\mathbb{R})$. Notice that

$$T_{s,t}^{B^H}\phi(x) = \int_{\mathbb{R}} \phi(x+z)L_{s,t}(z)dz = (\phi \star \check{L}_{s,t})(x), \ \forall x \in \mathbb{R},$$

where $\check{L}_t(x) = L_t(-x)$

Reformulation as nonlinear Young differential equation

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Definition

Let ϕ be a distribution such that $T^{B^H}\phi$ is **sufficiently regular**. We call $X : \Omega \times [0, T] \to \mathbb{R}$ a path-by-path solution to $(\bullet)$ if, for $\tilde{X} = X - B^H$,

$$\tilde{X}_t = \int_0^t T_{dr}^{B^H}\phi(\tilde{X}_r)$$

holds for almost every realisation of B^H.

Key steps for existence of path-by-path solution

$$\tilde{X}_t = \int_0^t T_{dr}^{B^H} \phi(\tilde{X}_r)$$

- Determine regularity of $T^{B^H}\phi$ via the regularity of the local time of a fBm ($\approx L \in \mathcal{C}^{1/2(1-H)}(\mathcal{C}^{1/(2H)-1/2})$). (also see Harang and Perkowski (2021))

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Please give me a function with regular local time

Note that the above approach requires no probability theory, having a function with sufficiently regular local time is sufficient.

Theorem

Let $\eta \in (0,1]$ *and* $p \geqslant 1$ *with* $1/p + \eta > 1$*. Let* $A \in \mathcal{C}^{p\text{-var}}_{[0,T]}(\mathcal{C}^{\eta})$ *with* $A_{s,t}(y) \geqslant 0$ *for all* $y \in \mathbb{R}$ *and all* $(s,t) \in \Delta_{[0,T]}$*. Then there exists a solution* $x \in \mathcal{C}^{1\text{-var}}_{[0,T]}$ *to the nonlinear Young equation*

$$x_t = \int_0^t A_{dr}(x_r), \ \forall t \in [0, T]. \tag{4}$$

Sketch of the proof

W.l.o.g., let $T = 1$. For $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$, let $t_k^n := k/n$, $\bar{x}_0^n := 0$ and define recursively

$$\bar{x}_{k+1}^n = \bar{x}_k^n + A_{t_k^n, t_{k+1}^n}(\bar{x}_k^n).$$

We embed $(\bar{x}_k^n)_{k=0}^n$ into $\mathcal{C}_{[0,1]}$ by

$$\begin{aligned} x_t^n &= \sum_{0 \leqslant k \leqslant \lfloor nt \rfloor} A_{t_k^n, t \wedge t_{k+1}^n}(\bar{x}_k^n) && (5) \\ &= \int_0^t A_{dr}(x_r^n) + \sum_{0 \leqslant k \leqslant \lfloor nt \rfloor} \left(\int_{t_k^n}^{t \wedge t_{k+1}^n} A_{dr}(x_{t_k^n}^n) - A_{dr}(x_r^n) \right). && (6) \end{aligned}$$

Let $\varepsilon > 0$. Then for n large and $0 \leqslant s \leqslant u \leqslant 1$ with $|u - s|$ small

$$\begin{aligned} x_{s,t}^n = [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} &\leqslant C(p,\eta) \left([A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} + [A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}}^\eta + \varepsilon \right) \\ \implies [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} &\leqslant C(p,\eta) \frac{[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)} + \varepsilon}{1 - C(p,\eta)[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}(\mathcal{C}^\eta)}}. \end{aligned}$$

Regularising with fBm

Theorem (A., Richard, and Tanré 2021)

For $H < \sqrt{2} - 1$ *and* ϕ *a* nonnegative *finite measure, there exists a* path-by-path *solution to* $(\bullet)$*. Furthermore, building on this construction, we can also construct a* weak solution*.*

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Corollary

As a corollary of the general Theorem one gets, for a nonnegative bump function ξ *and* $\varepsilon > 0$, *a weak solution to*

$$X_t = \int_0^t |X_r|^{-3/4+\varepsilon} \xi(X_r) dr + W_t.$$

Table of Contents

Different notions of uniqueness

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- We say that pathwise uniqueness holds if for any weak solutions X, Y defined on the same filtered probability space, $X \equiv Y$.
- We say that path-by-path uniqueness for $(\bullet)$ holds if for any probability space on which a Brownian motion W is defined, there exists a null-set $\mathcal{N}$ such that for $\omega \notin \mathcal{N}$, there exists a unique solution to

$$X_t(\omega) = \int_0^t \phi(r, X_r(\omega)) dr + W_t(\omega).$$

$$\boxed{\text{path-by-path uniqueness}} \implies \boxed{\text{pathwise uniqueness}}$$

Uniqueness and counterexamples

$$X_t = X_0 + \int_0^t \phi(r, X_r)dr + W_t, \quad X_0 = x_0 \in \mathbb{R}^d \tag{$\bullet$}$$

Theorem (Krylov and Röckner 2005)

Let $\phi \in L^q([0,T], L^p(\mathbb{R}^d))$ for $p \geqslant 2, q > 2$ with $d/p + 2/q < 1$. Then there exists a pathwise unique strong solution to $(\bullet)$.

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Theorem (Davie 2007)

Let $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded and measurable. Then path-by-path uniqueness holds.

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Counterexamples (Shaposhnikov and Wresch 2020, A. 2022)

One can construct drifts $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

1. there is existence of path-by-path solutions to $(\bullet)$, but there exists no weak solution;
2. there exists a pathwise unique weak solution to $(\bullet)$, but path-by-path uniqueness is lost.

Uniqueness for unbounded functions

$$X_t = \int_0^t \phi(X_r)dr + W_t \iff \tilde{X}_t = \int_0^t \phi(\tilde{X}_r + W_r)dr \qquad (\bullet)$$

Theorem (A., ongoing work)

Let $\alpha > -1/2$ and let $\phi(x) := \mathbb{1}_{\{x \neq 0\}}|x|^\alpha \xi(x)$ for a bump function ξ. Then there exists a unique path-by-path solution to equation $(\bullet)$.

Proof.

- Let $\tilde{X}^1 \neq \tilde{X}^2$ be solutions to $(\bullet)$
- W.l.o.g. $\exists$ time interval $[s,t]$ on which $|\tilde{X}^1 + W| > |\tilde{X}^2 + W|$
- By monotonicity of ϕ, on $[s,t]$, $\phi(\tilde{X}^1 + W) < \phi(\tilde{X}^2 + W)$ (if $\tilde{X}^2 + W \neq 0$)
- $\implies$ $\tilde{X}^1 - \tilde{X}^2$ is monotone on $[s,t]$. Finding a nonlinear Young integral equation that is solved by $\tilde{X}^1 - \tilde{X}^2$, we get

$$[\tilde{X}^1 - \tilde{X}^2]_{\mathcal{C}^{1\text{-var}}_{[s,t]}} = |(\tilde{X}^1 - \tilde{X}^2)_{s,t}| \leqslant C(t-s)^\gamma [\tilde{X}^1 - \tilde{X}^2]_{\mathcal{C}^{1\text{-var}}_{[s,t]}}.$$

Obstacles in the rigorous proof

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3. Open problem: How to ensure 2 in the fractional Brownian motion case?

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Questions please!