Convergence of an implicit scheme for hyperbolic systems

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Outline

Presentation of the system

In this work, we consider a diagonal hyperpolic system of transport equations given by:

$$\begin{cases} \partial_t v^\alpha + \lambda^\alpha(v)\partial_x v^\alpha = 0 \quad \text{on } (0,+\infty)\times\mathbb{R}, \quad \text{for} \quad \alpha = 1,\dots,d, \\ v^\alpha(0,x) = v_0^\alpha(x) \qquad\qquad x\in\mathbb{R}, \quad \text{for} \quad \alpha = 1,\dots,d, \end{cases}$$

where $v(t,x) = (v^\alpha(t,x))_{\alpha=1,\dots,d}$, with $d \geq 1$ is an integer.

Conditions on the velocity and the initial data

The velocity is assumed to verify the following regularity:

$$
(K1)\begin{cases} \lambda^\alpha \in C^1(\mathbb{R}^d), \text{ for } \alpha = 1, \dots, d, \\[2ex] \text{there exists} \quad M_1 > 0 \quad \text{such that for } \alpha = 1, \dots, d, \\ |\lambda^\alpha(u) - \lambda^\alpha(v)| \le M_1 |u - v| \quad \text{ for all } \quad u, v \in \mathbb{R}^d. \end{cases}
$$

The initial data is assumed to satisfy the following property:

$$
(K2)\left\{\begin{array}{l|l} -M^\alpha \le v_0^\alpha \le M^\alpha, \text{ where } M^\alpha > 0 & \\[2ex] v_0^\alpha \quad \text{is nondecreasing,} & \text{for} \quad \alpha = 1, \dots, d. \\[2ex] \partial_x v_0^\alpha \in L^\infty(\mathbb{R}). & \end{array}\right.
$$

Conditions on the velocity

(Nonnegative matrices with nonpositive off-diagonal terms)

$$(A1) \quad \begin{cases} \lambda^{\alpha}_{,\beta}(v) \leq 0 \quad \text{for all} \quad v \in \mathcal{U} \quad \text{and} \quad \alpha \neq \beta \quad \text{with} \quad \alpha, \beta \in \{1, \ldots, d\}, \\ A_{\alpha,\beta} = \inf_{v \in \mathcal{U}} \left(\lambda^{\alpha}_{,\beta}(v) \right) \\ \text{and} \quad \sum_{\alpha,\beta=1,\ldots,d} A_{\alpha,\beta} \xi_\alpha \xi_\beta \geq 0 \quad \text{for every} \quad \xi = (\xi_1, \ldots, \xi_d) \in [0, +\infty)^d. \end{cases}$$

(Diagonally dominant)

$$(A2) \quad \lambda^{\alpha}_{,\alpha}(v) \geq \sum_{\alpha \neq \beta} (\lambda^{\alpha}_{,\beta}(v))^- \quad \text{for all} \quad v \in \mathcal{U} \quad \text{and} \quad \alpha = 1, ..., d,$$

where we note $x^- = \max(0, -x)$ and

$$\mathcal{U} = \prod_{\alpha=1}^{d} [-M^\alpha, M^\alpha].$$

Recall of useful results

Theorem (Existence and uniqueness)

Assume that $(K1)$ and $(K2)$ are satisfied. Suppose also that one of the assumptions $(A1)$ or $(A2)$ is verified. Then, there exists a unique nondecreasing function $v \in \bigcap_{T>0} [W^{1,\infty}([0,T) \times \mathbb{R})]^d$ solution of the given system, in distributional sense. Moreover we have for any $t \in (0,+\infty)$:

$$\sum_{\alpha=1,..,d} \|\partial_x v^\alpha(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \sum_{\alpha=1,..,d} \|\partial_x v_0^\alpha\|_{L^\infty(\mathbb{R})} \quad \text{if} \quad (A1) \quad \text{holds},$$

$$\max_{\alpha=1,..,d} \|\partial_x v^\alpha(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \max_{\alpha=1,..,d} \|\partial_x v_0^\alpha\|_{L^\infty(\mathbb{R})} \quad \text{if} \quad (A2) \quad \text{holds}.$$

Uniqueness results for diagonal hyperbolic systems with large and monotone data (El Hajj, Monneau, Journal of Hyperbolic Differential Equations-2013)

The implicit scheme

In order to find a solution which preserves the Lipschitz estimate, we discretize the system by finite difference implicit scheme as follows:

$$\forall \alpha \in \{1, \ldots, d\}, \quad \begin{cases} \dfrac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(u_i^{n+1}) \left(\dfrac{u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}}{\Delta x} \right) = 0 \quad \text{if} \quad \lambda^\alpha(u_i^{n+1}) \leq 0, \\ \dfrac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(u_i^{n+1}) \left(\dfrac{u_i^{\alpha,n+1} - u_{i-1}^{\alpha,n+1}}{\Delta x} \right) = 0 \quad \text{if} \quad \lambda^\alpha(u_i^{n+1}) \geq 0, \\ u_i^{\alpha,0} = u_0^\alpha(x_i). \end{cases}$$

In a more compact form, this can be written as follows:

$$\begin{cases} \dfrac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} - \left(\lambda_i^{\alpha,n+1}\right)_- \left(\dfrac{u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}}{\Delta x} \right) + \left(\lambda_i^{\alpha,n+1}\right)_+ \left(\dfrac{u_i^{\alpha,n+1} - u_{i-1}^{\alpha,n+1}}{\Delta x} \right) = 0 \\ u_i^{\alpha,0} = u_0^\alpha(x_i). \end{cases}$$

Resolution of the scheme

Theorem (Boudjerada, El Hajj, Oussaily (2020))

Assume that assumptions $(K1)$, $(K2)$ and a CFL condition are satisfied. Let $|u_i^{\alpha,n}| \leq M^\alpha$ be given. Then, we get

(i) **(Existence)**

There exists a unique solution $|u_i^{\alpha,n+1}| \leq M^\alpha$ to the implicit scheme.

(ii) **(Monotonicity)**

Moreover if $u_i^{\alpha,n}$ is nondecreasing, i.e. satisfies

$$u_{i+1}^{\alpha,n} \geq u_i^{\alpha,n} \quad \text{for all} \quad i \in \mathbb{Z}, \quad \text{and} \quad \alpha = 1, \dots, d,$$

then $u_i^{\alpha,n+1}$ is also nondecreasing.

Discrete estimate

Theorem (Boudjerada, El Hajj, Oussaily (2020))

Under the same assumptions considered before and supposing that $(A1)$ or $(A2)$ is verified, if $u_i^{\alpha,n}$ is the solution of the implicit scheme, then, $\theta_{i+\frac{1}{2}}^{\alpha,n}$ is nonnegative for all $n \in \mathbb{N}$ and verifies the following estimates:

$$\sum_{\alpha=1}^{d} \sup_{i\in\mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} \leq \sum_{\alpha=1}^{d} \sup_{i\in\mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,0}, \qquad \text{if} \quad (A1) \quad \text{holds},$$

$$\max_{\alpha=1,\dots,d} \left(\sup_{i\in\mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} \right) \leq \max_{\alpha=1,\dots,d} \left(\sup_{i\in\mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,0} \right), \qquad \text{if} \quad (A2) \quad \text{holds}.$$

where $\theta_{i+\frac{1}{2}}^{\alpha,n} = \dfrac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}$.

Convergence to the continuous solution

Theorem (Boudjerada, El Hajj, Oussaily (2020))

Under the same assumptions, let us consider the solution $u_i^{\alpha,n}$ of the scheme given by the existence theorem. Let us call $\epsilon = (\Delta t, \Delta x)$ and $u^{\epsilon,\alpha}$ the function defined by

$$u^{\epsilon,\alpha}(n\Delta t, i\Delta x) = u_i^{\alpha,n} \quad \text{for} \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}.$$

Then, as ϵ goes to zero, the whole sequence $(u^{\epsilon,\alpha})_\epsilon$ converges to the unique Lipschitz solution v^α of the continuous problem. Moreover, for any compact $K \subset [0, +\infty) \times \mathbb{R}$, we have

$$\sup_{\substack{K \cap ((\Delta t \mathbb{N}) \times (\Delta x \mathbb{Z})) \\ \alpha = 1, \dots, d}} |u^{\epsilon,\alpha} - v^\alpha| \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow (0,0).$$

Our work with respect to the literature

- (Monneau, Monasse (2014)): Convergence result of a semi-explicit scheme for diagonal non-conservative hyperbolic system assuming it strictly hyperbolic. This result was established in a restricted class of solutions defined by vanishing viscosity solutions.
- (El Hajj, Forcadel (2008)): Convergence result of an explicit scheme to the Lipschitz continuous solution for a particular (2×2) Hamilton-Jacobi system was proved in the framework of dislocation densities.

Usefull definitions for the proof of the existence

- $\mathcal{U} = \prod_{\alpha=1}^{d} [-M^\alpha, M^\alpha]$
- $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\ldots,d}$, $u^n = (u_i^n)_{i\in\mathbb{Z}}$
- $u^n \in \mathcal{U}^{\mathbb{Z}}$ if $u_i^n \in \mathcal{U}$, for all $i \in \mathbb{Z}$
- Define on $\mathcal{U}^{\mathbb{Z}}$ the function $F_{u_i^n} = \left(F_{u_i^n}^\alpha\right)_{\alpha=1,\ldots,d}$ such that

$$F_{u_i^n}^\alpha(w) = u_i^{\alpha,n} + \frac{\Delta t}{\Delta x}\left((\lambda^\alpha(w_i))_-\left(w_{i+1}^\alpha - w_i^\alpha\right) - (\lambda^\alpha(w_i))_+\left(w_i^\alpha - w_{i-1}^\alpha\right)\right)$$

Fixed point argument

- The scheme is written as: $u_i^{\alpha,n+1} = F^{\alpha}_{u_i^n}(u^{n+1})$
- $F_{u_i^n}$ is well-defined contraction on $\mathcal{U}^{\mathbb{Z}}$
- Fixed point argument implies the existence and uniqueness of solution of the scheme in $\mathcal{U}^{\mathbb{Z}}$

Preliminaries

Evolution of the discrete gradient

$$\begin{aligned}\theta^{\alpha,n+1}_{i+\frac{1}{2}} &= \theta^{\alpha,n}_{i+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\left[(\lambda^{\alpha,n+1}_{i+1})_+ + (\lambda^{\alpha,n+1}_{i})_-\right]\theta^{\alpha,n+1}_{i+\frac{1}{2}} \\ &\quad + \frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i+1})_-\theta^{\alpha,n+1}_{i+\frac{3}{2}} + \frac{\Delta t}{\Delta x}(\lambda^{\alpha,n+1}_{i})_+\theta^{\alpha,n+1}_{i-\frac{1}{2}}\end{aligned}$$

The key inequality

$$\begin{aligned}\theta^{\alpha,n+1}_{j_\alpha+\frac{1}{2}} &= \max_{i\in\mathbb{Z}} \theta^{\alpha,n+1}_{i+\frac{1}{2}} \\ &\leq \max_{i\in\mathbb{Z}} \theta^{\alpha,n}_{i+\frac{1}{2}} - \frac{\Delta t}{\Delta x}\left(\lambda^{\alpha,n+1}_{j_\alpha+1} - \lambda^{\alpha,n+1}_{j_\alpha}\right)\theta^{\alpha,n+1}_{j_\alpha+\frac{1}{2}}\end{aligned}$$

Proof of the discrete Lipschitz estimate

- $$\max_{\alpha=1,\ldots,d}\left[\max_{i\in\mathbb{Z}}\left(\theta^{\alpha,n+1}_{i+\frac{1}{2}}\right)\right]=\theta^{\alpha_0,n+1}_{j_{\alpha_0}+\frac{1}{2}}$$
  $$\leq \max_{\alpha=1,\ldots,d}\left[\max_{i\in\mathbb{Z}}\left(\theta^{\alpha,n}_{i+\frac{1}{2}}\right)\right]-\frac{\Delta t}{\Delta x}\left(\lambda^{\alpha_0,n+1}_{j_{\alpha_0}+1}-\lambda^{\alpha_0,n+1}_{j_{\alpha_0}}\right)\theta^{\alpha_0,n+1}_{j_{\alpha_0}+\frac{1}{2}}$$
- Use $(A2)$ to prove that
  $$-\frac{\Delta t}{\Delta x}\left(\lambda^{\alpha_0,n+1}_{j_{\alpha_0}+1}-\lambda^{\alpha_0,n+1}_{j_{\alpha_0}}\right)\theta^{\alpha_0,n+1}_{j_{\alpha_0}+\frac{1}{2}}\leq 0.$$

Construction of the discrete solution u^ϵ

The Q^1 extension u^ϵ

$$\begin{aligned} u^\epsilon(t,x) \quad &= \left(\frac{t-t_n}{\Delta t}\right)\left\{\left(\frac{x-x_i}{\Delta x}\right)u_{i+1}^{n+1} + \left(1-\frac{x-x_i}{\Delta x}\right)u_i^{n+1}\right\} \\ &+ \left(1-\frac{t-t_n}{\Delta t}\right)\left\{\left(\frac{x-x_i}{\Delta x}\right)u_{i+1}^{n} + \left(1-\frac{x-x_i}{\Delta x}\right)u_i^{n}\right\}. \end{aligned}$$

Estimates on u^ϵ

$$\|u^{\epsilon,\alpha}\|_{L^\infty([0,T]\times\mathbb{R})} \le M^\alpha$$

$$\|\partial_x u^{\epsilon,\alpha}\|_{L^\infty([0,T]\times\mathbb{R})} \le \mathcal{G}(T, \|\partial_x u_0\|_{(L^\infty(\mathbb{R}))^d})$$

$$\|\partial_t u^{\epsilon,\alpha}\|_{L^\infty([0,T]\times\mathbb{R})} \le 2\Lambda^\alpha \mathcal{G}$$

Proof of the convergence of the discrete solution

- Extraction of a subsequence $u^{\epsilon,\alpha}$ that converges to u^{α}
- Passing to the limit in the PDE in distributional sense
- Showing that u^{α} verifies $\partial_t u^{\alpha} + \lambda^{\alpha}(u)\partial_x u^{\alpha} = 0$
- Convergence of the whole sequence based on the uniqueness of solution in the continuous case.

Numerical solution for system modeling dislocations

Considering a one-dimensional model describing the dynamics of dislocations given by

$$\left\{\begin{array}{lcll}\partial_t u^1(t,x) & = & -\big((u^1-u^2)(t,x)\big)\,\partial_x u^1(t,x) & \text{in} \quad (0,T)\times\mathbb{R},\\[2ex] \partial_t u^2(t,x) & = & \big((u^1-u^2)(t,x)\big)\,\partial_x u^2(t,x) & \text{in} \quad (0,T)\times\mathbb{R}.\end{array}\right.$$

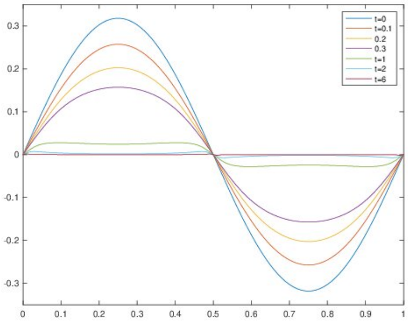
We calculate the numerical solution of the previous system taking the following initial data:

$$u^1(0,x)=\frac{1}{2\pi}\sin(2\pi x)+x, \quad u^2(0,x)=-\frac{1}{2\pi}\sin(2\pi x)+x, \qquad \forall x\in\mathbb{R}.$$

Behavior of the viscoplasticity

We simulate below the long-time behavior of the function $(u^1 - u^2)$, which reflects the viscoplastic deformation.

Figure 1: Evolution of $u^1 - u^2$ over time.



Work to be started

- Establish the error estimate between the numerical solution and the continuous one.
- Numerical study in the case where the solutions are continuous, inspired by the work of Monneau, Monasse (2014).
- Proposing numerical schemes in the case of BV solutions.

Thank you for your attention !