

Approximation discrète de la fonctionnelle de Griffith

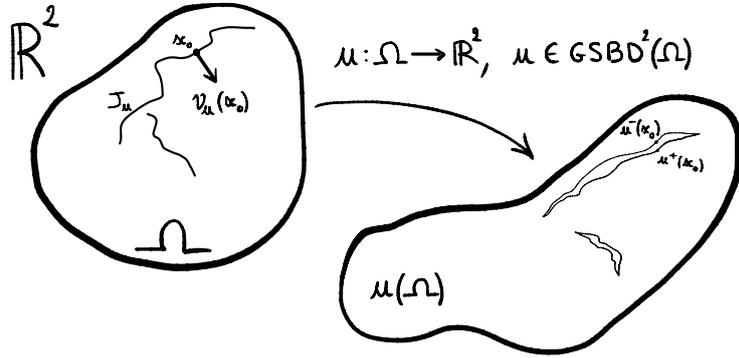
Élise Bonhomme

Travail en collaboration avec Jean-François Babadjian

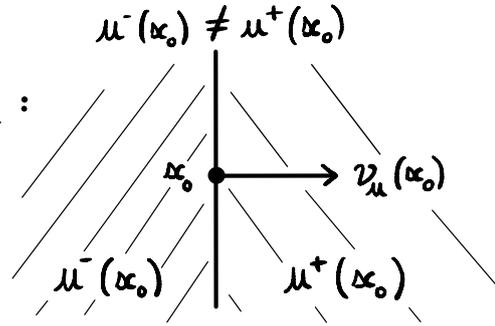
Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay

CANUM « 2020 », Evian-les-Bains, 13-17 juin 2022

Mécanique de la fracture et énergie de Griffith

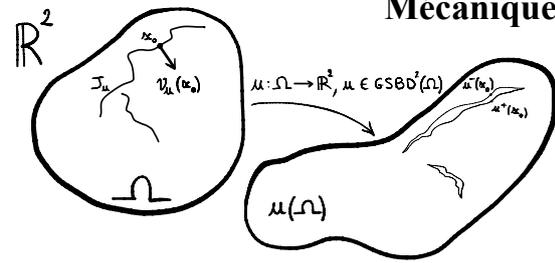


"ZOOM"
 $x_0 \in J_\mu$:

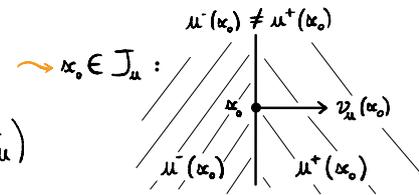


Énergie de Griffith $G(\mu) = \int_{\Omega \setminus J_\mu} |e(u)|^2 dx + \mu H^1(J_\mu)$

Mécanique de la fracture et énergie de Griffith

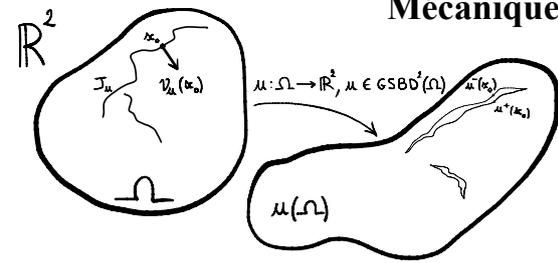


Énergie de Griffith $\int_{\Omega \setminus J_\mu} |e(u)|^2 dx + \mu H^1(J_\mu)$

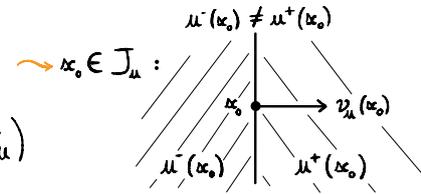


$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int \min(|eu|^2; \frac{1}{\varepsilon}) dx & \text{si } u \in V_\varepsilon(\Omega), \\ -\infty & \\ +\infty & \text{sinon.} \end{cases} \xrightarrow[\varepsilon \searrow 0]{\Gamma\text{-CV}} \mathcal{F}^\circ(u) = \begin{cases} \int |eu|^2 dx + \sin \theta_0 H^1(J_u) & \text{si } u \in \text{GSBD}^2(\Omega), \\ -\infty & \\ +\infty & \text{sinon.} \end{cases}$$

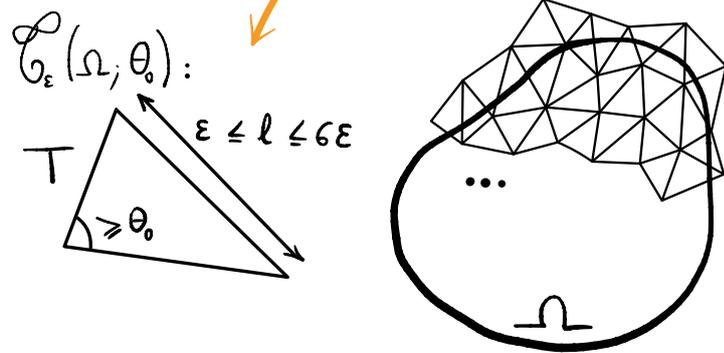
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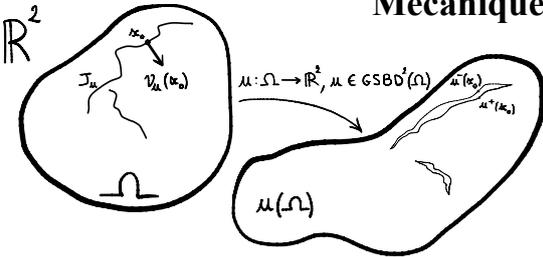


$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_{\Omega} \min(|e(u)|^2; \frac{1}{\varepsilon}) dx & \text{si } u \in V_\varepsilon(\Omega), \\ +\infty & \text{sinon} \end{cases} \xrightarrow[\varepsilon \searrow 0]{\Gamma\text{-cv}} \mathcal{F}(u) = \begin{cases} \int_{\Omega} |e(u)|^2 dx + \sin \theta_0 H^1(J_u) & \text{si } u \in \text{GSBD}^2(\Omega), \\ +\infty & \text{sinon} \end{cases}$$

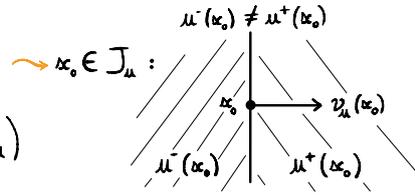


$$V_\varepsilon(\Omega) = \left\{ \begin{array}{l} \mu: \Omega \rightarrow \mathbb{R}^2 \text{ CONTINUE,} \\ \exists T_\varepsilon \in \mathcal{C}_\varepsilon(\Omega), \forall T \in T_\varepsilon, \\ \mu|_T \text{ AFFINE.} \end{array} \right\}$$

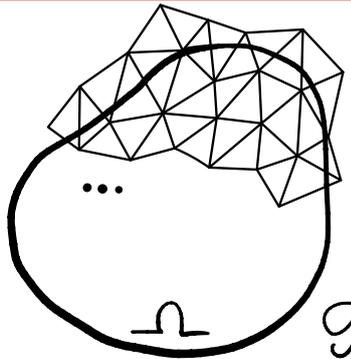
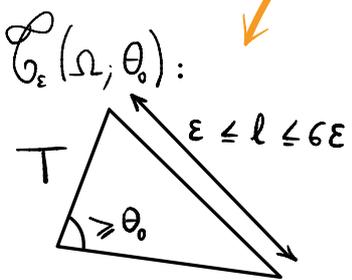
Mécanique de la fracture et énergie de Griffith



Énergie de Griffith $\int_{\Omega \setminus \Gamma_u} |e(u)|^2 dx + \mu H^1(\Gamma_u)$



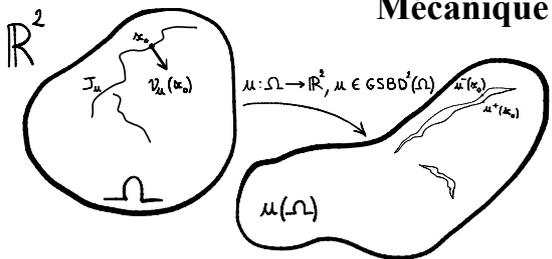
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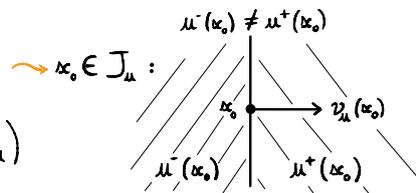
$$V_\varepsilon(\Omega) = \left\{ \begin{array}{l} u: \Omega \rightarrow \mathbb{R}^2 \text{ CONTINUE,} \\ \exists T_\varepsilon \in \mathcal{T}_\varepsilon(\Omega), \forall T \in T_\varepsilon, \\ u|_T \text{ AFFINE.} \end{array} \right\}$$

$$\mathcal{F}_\varepsilon(u) = \sum_{T \in \mathcal{T}_\varepsilon: |eu_T|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} + \sum_{T \in \mathcal{T}_\varepsilon: |eu_T|^2 \leq \frac{1}{\varepsilon}} \mathcal{L}^2(T) |eu_T|^2$$

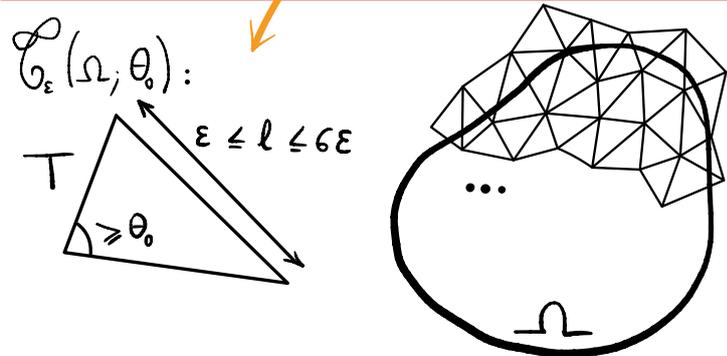
Mécanique de la fracture et énergie de Griffith



Énergie de Griffith $\int_{\Omega \setminus \Gamma_u} |e(u)|^2 dx + \mu H^1(\Gamma_u)$

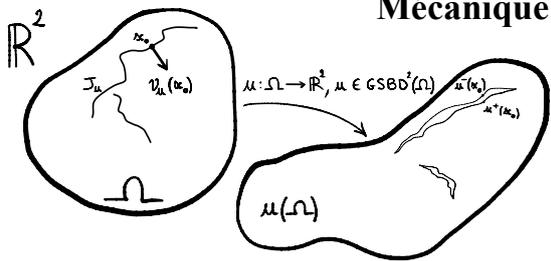


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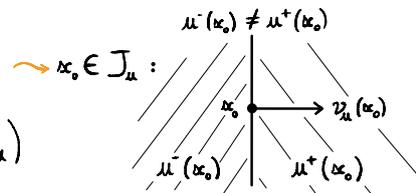


$$V_\varepsilon(\Omega) = \left\{ \begin{array}{l} \mu: \Omega \rightarrow \mathbb{R}^2 \text{ CONTINUE,} \\ \exists T_\varepsilon \in \mathcal{C}_\varepsilon(\Omega), \forall T \in T_\varepsilon, \\ \mu|_T \text{ AFFINE.} \end{array} \right\}$$

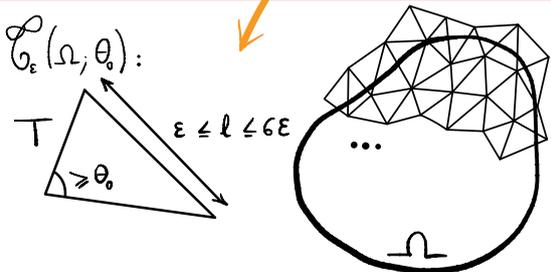
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Énergie de Griffith $\int_{\Omega \setminus J_\mu} |e(u)|^2 dx + \mu H^1(J_\mu)$



$$\mathcal{F}_\varepsilon(u) = \begin{cases} \int_{\Omega} \min(|e(u)|^2; \frac{1}{\varepsilon}) dx & \text{si } \mu \in V_\varepsilon(\Omega), \\ +\infty & \text{sinon.} \end{cases} \xrightarrow[\varepsilon \searrow 0]{\Gamma\text{-CV}} \mathcal{F}^2(u) = \begin{cases} \int_{\Omega} |e(u)|^2 dx + \sin\theta_0 H^1(J_u) & \text{si } \mu \in \text{GSBD}^2(\Omega), \\ +\infty & \text{sinon.} \end{cases}$$



$$V_\varepsilon(\Omega) = \left\{ \begin{array}{l} \mu: \Omega \rightarrow \mathbb{R}^3 \text{ CONTINUE,} \\ \exists T_\varepsilon \in \mathcal{C}_\varepsilon(\Omega), \forall T \in T_\varepsilon, \\ \mu|_T \text{ AFFINE.} \end{array} \right\}$$

Stratégie de preuve :

Compacité (A. Chambolle - V. Crismale, 2021)

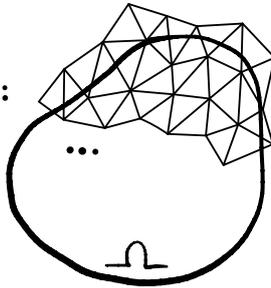
Upper bound (G. Cortesani - R. Toader 99,
Chambolle - Crismale 2019,
Chambolle - G. Dal Maso 99)

Lower bound :

$$\forall \mu \in \text{GSBD}^2(\Omega), \forall \mu_\varepsilon \rightarrow \mu, \\ \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon^2(\mu_\varepsilon) \geq \mathcal{F}^2(\mu)$$

Lower bound pour la partie saut

• $\mu \in \text{GSBD}^2(\Omega)$, $u_\varepsilon \rightarrow u$, $\liminf_{\varepsilon} \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$:



$$u_\varepsilon \in V_\varepsilon(\Omega)$$

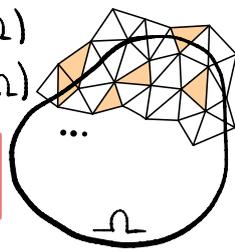
$$T_\varepsilon \in \mathcal{C}_\varepsilon(\Omega)$$

Mq :

$$\liminf_{\varepsilon} \mathcal{F}_\varepsilon(u_\varepsilon) \geq \sin \theta_0 \, H^1(J_\mu)$$

Lower bound pour la partie saut

$$\begin{aligned} u_\varepsilon &\in V_\varepsilon(\Omega) \\ T_\varepsilon &\in \mathcal{T}_\varepsilon(\Omega) \end{aligned}$$



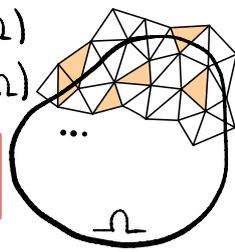
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$$\liminf_{\varepsilon} \mathcal{F}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon} \int_{\Omega} \min(|e(u_\varepsilon)|^2, \frac{1}{\varepsilon}) dx \geq \liminf_{\varepsilon} \left(\sum_{T \in \mathcal{T}_\varepsilon: |e(u_\varepsilon)|_T|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} \right)$$

Lower bound pour la partie saut

$$\begin{aligned} \mu_\varepsilon &\in \mathcal{V}_\varepsilon(\Omega) \\ \mathbb{T}_\varepsilon &\in \mathcal{C}_\varepsilon(\Omega) \end{aligned}$$



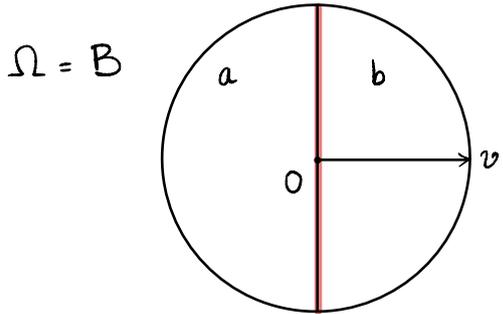
Mq

$$\frac{\lim}{\varepsilon} \mathcal{F}_\varepsilon^p(\mu_\varepsilon) \geq \sin \theta_0 \, H^1(\mathbb{J}_\mu)$$

$\mu \in \text{GSBD}^2(\Omega)$, $\mu_\varepsilon \rightarrow \mu$, $\lim_{\varepsilon} \mathcal{F}_\varepsilon^p(\mu_\varepsilon) < +\infty$:

$$\frac{\lim}{\varepsilon} \mathcal{F}_\varepsilon^p(\mu_\varepsilon) \geq \frac{\lim}{\varepsilon} \left(\sum_{T \in \mathbb{T}_\varepsilon: |e(\mu_\varepsilon)|_T|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} \right)$$

Blow up



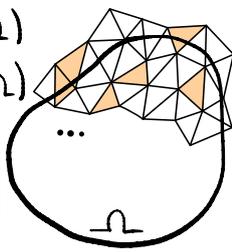
$$\mathbb{J}_\mu = B \cap v^\perp$$

$$a \neq b$$

$$\mu(x) = \begin{cases} a & \text{si } x \cdot v < 0 \\ b & \text{si } x \cdot v > 0 \end{cases}$$

Lower bound pour la partie saut

$$\begin{aligned} \mu_\varepsilon &\in \mathcal{V}_\varepsilon(\Omega) \\ \mathbb{T}_\varepsilon &\in \mathcal{C}_\varepsilon(\Omega) \end{aligned}$$



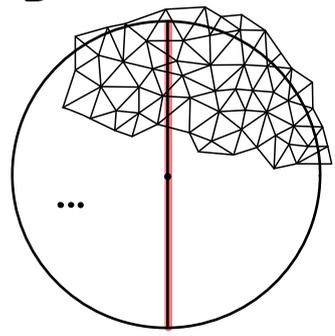
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Mq $\frac{\liminf_{\varepsilon} \mathcal{F}_\varepsilon^p(\mu_\varepsilon)}{\varepsilon} \geq \sin \theta_0 \, H^1(J_\mu)$

Blow up

$\Omega = B$



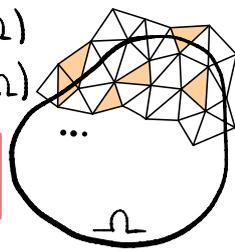
$J_\mu = B \cap v^\perp$
 $a \neq b$

$$\mathbb{T}_\varepsilon, \mu_\varepsilon \rightarrow \mu, \quad \mathbb{T}_{\text{big}}^\varepsilon = \left\{ T \in \mathbb{T}_\varepsilon, |e\mu_\varepsilon|_T : v \otimes v|^2 > \frac{1}{\varepsilon} \right\}$$

Lower bound pour la partie saut

$$\mu_\varepsilon \in \mathcal{V}_\varepsilon(\Omega)$$

$$\mathbb{T}_\varepsilon \in \mathcal{C}_\varepsilon(\Omega)$$

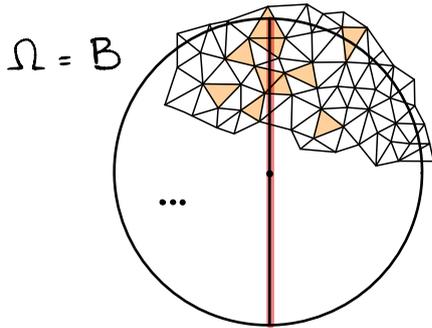


$\mu \in \text{GSBD}^2(\Omega)$, $\mu_\varepsilon \rightarrow \mu$, $\liminf_{\varepsilon} \mathcal{F}_\varepsilon(\mu_\varepsilon) < +\infty$:

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Mq $\liminf_{\varepsilon} \mathcal{F}_\varepsilon(\mu_\varepsilon) \geq \text{sim } \Theta_0 \quad H^1(J_\mu)$

Blow up



$$\mathbb{T}_\varepsilon, \mu_\varepsilon \rightarrow \mu, \quad \mathbb{T}_{\text{big}}^\varepsilon = \left\{ T \in \mathbb{T}_\varepsilon, |e_{\mu_\varepsilon}|_T : \nu \otimes \nu|^2 > \frac{1}{\varepsilon} \right\}$$

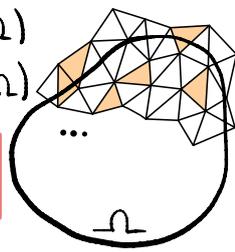
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$$J_\mu = B \cap \nu^\perp$$

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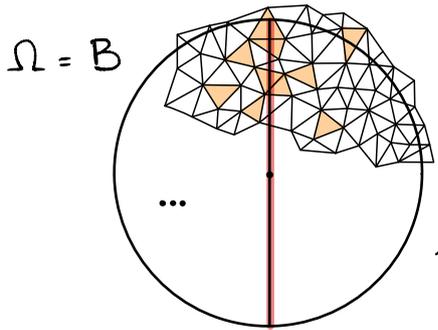
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Mq:

$$\liminf_{\varepsilon} \mathcal{F}_\varepsilon(\mu_\varepsilon) \geq \sin \theta_0 \int_{J_\mu} H^1$$

$$\liminf_{\varepsilon} \mathcal{F}_\varepsilon(\mu_\varepsilon) \geq \liminf_{\varepsilon} \left(\sum_{T \in \mathbb{T}_\varepsilon: |e(\mu_\varepsilon)|_T|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} \right)$$

Blow up

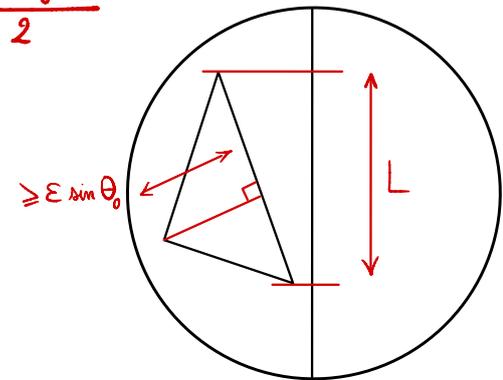


$$\mathbb{T}_\varepsilon, \mu_\varepsilon \rightarrow \mu, \mathbb{T}_{\text{big}}^\varepsilon = \left\{ T \in \mathbb{T}_\varepsilon, |e\mu_\varepsilon|_T : \nu \otimes \nu|^2 > \frac{1}{\varepsilon} \right\}$$

$$\liminf_{\varepsilon} \mathcal{F}_\varepsilon(\mu_\varepsilon) \geq \liminf_{\varepsilon} \frac{1}{\varepsilon} \sum_{T \in \mathbb{T}_{\text{big}}^\varepsilon} \mathcal{L}^2(T) \geq \frac{\varepsilon \sin \theta_0 L}{2}$$

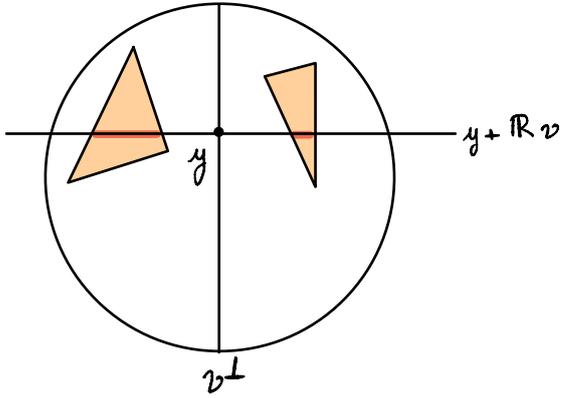
$$\begin{aligned} J_\mu &= B \cap \nu^\perp \\ a &\neq b \end{aligned}$$

$$\geq \frac{\sin \theta_0}{2} \liminf_{\varepsilon} \sum_{\mathbb{T}_{\text{big}}^\varepsilon} H^1(P_{\nu^\perp}(T))$$



$$\geq \varepsilon \sin \theta_0$$

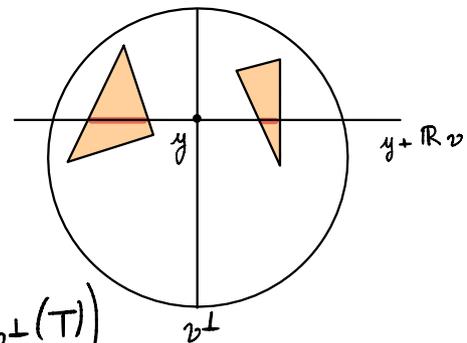
Énumération par slicing et conclusion



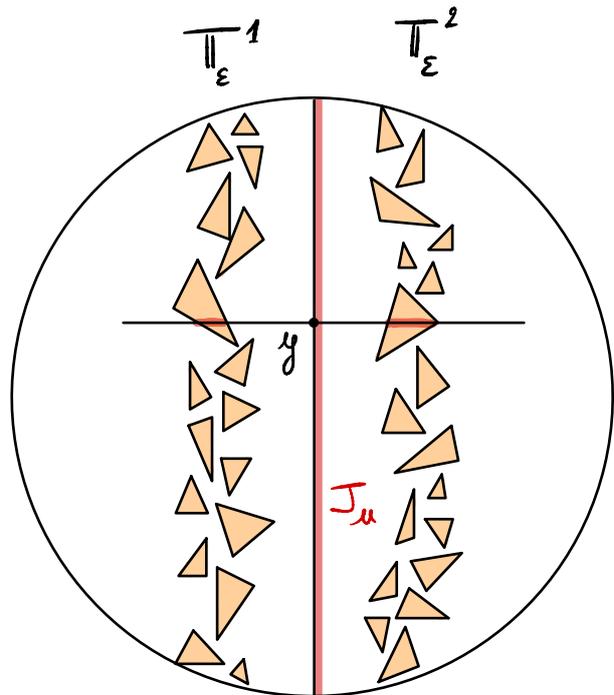
$$\forall y \in J_u, \# \{ \tau \in \Pi_{\text{big}}^\varepsilon : \text{"-"} \neq \emptyset \} \geq 2$$

Énumération par slicing et conclusion

$$\forall y \in \mathcal{J}_u, \# \{ \tau \in \pi_\varepsilon^E : \text{"-"} \neq \emptyset \} \geq 2$$

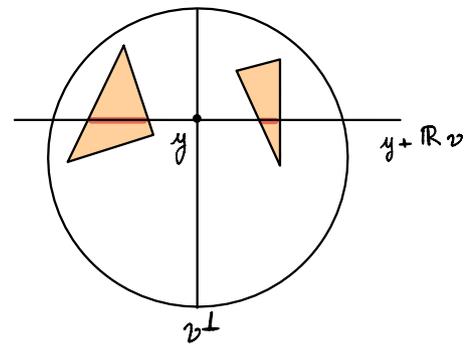


$$\exists \pi_\varepsilon^1, \pi_\varepsilon^2 \subset \pi_\varepsilon^E, \pi_\varepsilon^1 \cap \pi_\varepsilon^2 = \emptyset, \mathcal{J}_u \subset \left(\bigcup_{\pi_\varepsilon^1} P_{v^\perp}(\tau) \right) \cap \left(\bigcup_{\pi_\varepsilon^2} P_{v^\perp}(\tau) \right)$$

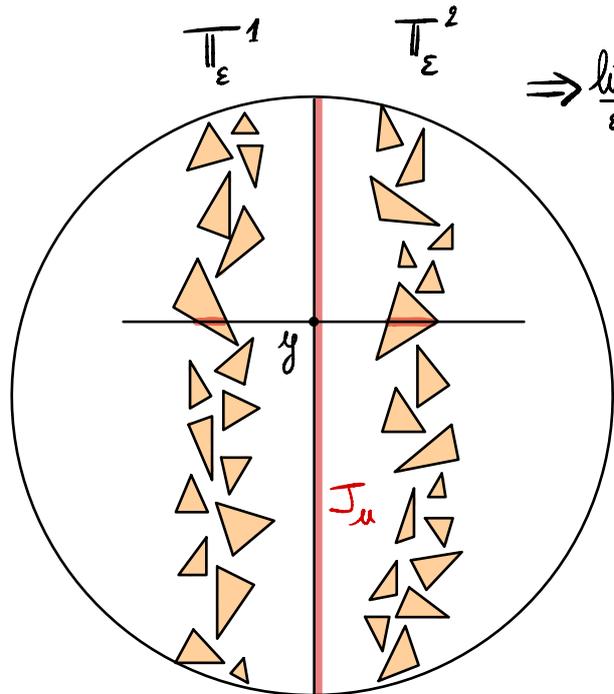


Énumération par slicing et conclusion

$$\forall y \in J_\mu, \# \left\{ \tau \in \Pi_\varepsilon^{\text{big}} : \text{"-"} \neq \emptyset \right\} \geq 2$$



$$\exists \Pi_\varepsilon^1, \Pi_\varepsilon^2 \subset \Pi_\varepsilon^{\text{big}}, \Pi_\varepsilon^1 \cap \Pi_\varepsilon^2 = \emptyset, J_\mu \subset \left(\bigcup_{\Pi_\varepsilon^1} P_{v^\perp}(\tau) \right) \cap \left(\bigcup_{\Pi_\varepsilon^2} P_{v^\perp}(\tau) \right)$$



$$\begin{aligned} \Rightarrow \lim_{\varepsilon} \mathcal{F}_\varepsilon^{\text{po}}(\mu_\varepsilon) &\geq \frac{\sin \theta_0}{2} \lim_{\varepsilon} \sum_{\Pi_\varepsilon^{\text{big}}} H^1(P_{v^\perp}(\tau)) \\ &\geq \frac{\sin \theta_0}{2} \lim_{\varepsilon} \left(\sum_{\Pi_\varepsilon^1} H^1(P_{v^\perp}(\tau)) + \sum_{\Pi_\varepsilon^2} H^1(P_{v^\perp}(\tau)) \right) \\ &\geq \frac{\sin \theta_0}{2} 2 H^1(J_\mu) \\ &= \sin \theta_0 H^1(J_\mu) . \end{aligned}$$



Récapitulatif

• $\mathcal{F}_\varepsilon(\mu) = \sum_{T \in \mathbb{T}_\varepsilon: |e_{u_T}|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} + \sum_{T \in \mathbb{T}_\varepsilon: |e_{u_T}|^2 \leq \frac{1}{\varepsilon}} \mathcal{L}^2(T) |e_{u_T}|^2 \xrightarrow[\varepsilon]{\Gamma\text{-CV}} \text{Griffith: } \mathcal{F}(\mu) = \int_{\Omega \setminus J_\mu} |e_{u_\mu}|^2 dx + \sin \theta_0 H^1(J_\mu)$

• $\mu_\varepsilon \in \operatorname{argmin} \mathcal{F}_\varepsilon \xrightarrow{\text{Compacité}} \mu_\varepsilon \xrightarrow{\varepsilon} \mu$

\Rightarrow Théorème fondamental de la Γ -CV: "CV des minimiseurs"

$$\mu \in \operatorname{argmin} \mathcal{F}, \quad \mathcal{F}_\varepsilon(\mu_\varepsilon) \xrightarrow{\varepsilon} \mathcal{F}(\mu).$$

Récapitulatif

$$\bullet \mathcal{F}_\varepsilon(\mu) = \sum_{T \in \mathbb{T}_\varepsilon: |e_{u_T}|^2 > \frac{1}{\varepsilon}} \frac{\mathcal{L}^2(T)}{\varepsilon} + \sum_{T \in \mathbb{T}_\varepsilon: |e_{u_T}|^2 \leq \frac{1}{\varepsilon}} \mathcal{L}^2(T) |e_{u_T}|^2 \xrightarrow[\varepsilon]{\Gamma\text{-CV}} \text{Griffith: } \mathcal{F}(\mu) = \int_{\Omega \setminus J_\mu} |e_{u_\mu}|^2 dx + \sin \theta_0 H^1(J_\mu)$$

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Merci pour votre attention !