Optimisation of Robin coefficients from the optimal control point of view

I. Mazari, Y. Privat

CEREMADE, Paris Dauphine Université

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What I will talk about

Consider

$$\begin{cases} -\Delta u_\beta = f & \text{in } \Omega\,, \\ \partial_\nu u_\beta + \beta(x) u_\beta = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

and a function

$$J : \beta \mapsto \quad \underbrace{\int_\Omega j\left(x, u_\beta\right)}_{\text{Distributed criterion}} \quad \text{or} \quad \underbrace{\int_{\partial\Omega} j\left(x, u_\beta\right)}_{\text{Boundary criterion}} \quad .$$

Consider

$$\mathcal{B} := \left\{ \beta : 0 \leq \beta \leq 1 \text{ a.e. on } \partial\Omega\,, \int_{\partial\Omega} \beta = V_0 \right\}.$$

We want to understand the qualitative properties of

$$\boxed{\max_{\beta \in \mathcal{B}} J(\beta).} \tag{$\mathbf{P}_{\mathrm{Rob}}$}$$

What do we want?

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$$\text{Do optimal } \beta^* \text{ saturate the } L^\infty \text{ constraints?}$$

What do we want?

1. There are two constraints on β: an equality (L^1) and an inequality (L^∞).
2. Naturally:

   Do optimal β^* saturate the L^∞ constraints?

3. In other words, if β^* is optimal, do we have

$$\beta^* = \mathbb{1}_\Gamma\,, \text{ for some measurable } \Gamma \subset \partial\Omega?$$

   Such functions are called bang-bang functions.

Why do we want it?

1. Natural question in optimisation: are optimisers extreme points of the admissible set?

[1] CF. Kao-Lou-Yanagida, Lamboley-Laurain-Nadin-Privat, Berestycki-Hamel-Roques...

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Why do we want it?

1. Natural question in optimisation: are optimisers extreme points of the admissible set?
2. Corresponds to the optimal design of the border of a natural habitat in mathematical biology[1].
3. From a pure "shape optimisation" point of view: approximation of a mixed boundary condition optimisation problem.

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$(\mathbf{P}_{\mathrm{Rob}})$ as an approximation of a shape. opt. problem

For any $\Gamma \subset \partial\Omega$, $|\Gamma| = V_0$ let v_Γ be the solution of

$$\begin{cases} -\Delta v_\Gamma = f & \text{in } \Omega\,, \\ \partial_\nu v_\Gamma = 0 & \text{on } \partial\Omega \backslash \Gamma\,, \\ v_\Gamma = 0 & \text{on } \Gamma \end{cases} \tag{2}$$

and solve

$$\boxed{\sup_{\Gamma \subset \partial\Omega\,, |\Gamma| = V_0} J(\Gamma) := \int_{\Omega \text{ or } \partial\Omega} j(x, v_\Gamma).} \qquad (\mathbf{P}_{\mathrm{Shape}})$$

($\mathbf{P}_{\mathrm{Rob}}$) as an approximation of a shape. opt. problem

$$\begin{cases} -\Delta v_\Gamma = f & \text{in } \Omega\,, \\ \partial_\nu v_\Gamma = 0 & \text{on } \partial\Omega\backslash\Gamma\,, \\ v_\Gamma = 0 & \text{on } \Gamma \end{cases} \rightsquigarrow \begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega\,, \\ \partial_\nu u_\varepsilon + \frac{\mathbb{1}_\Gamma}{\varepsilon} u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

and solve

$$\boxed{\sup_{\Gamma\subset\partial\Omega\,,|\Gamma|=V_0} J(\Gamma) := \int_{\Omega \text{ or } \partial\Omega} j(x, v_\Gamma).} \rightsquigarrow \boxed{\sup_{\int \beta = V_0\,,0\leq\beta\leq\frac{1}{\varepsilon}} \int_{\Omega \text{ or } \partial\Omega} j(x, u_\beta)} \tag{3}$$

It should be noted that the convergence of u_ε to v_Γ can be proved rigorously. We hope the second problem is easier.

$(\mathbf{P}_{\mathrm{Rob}})$ as a relevant problem on its own

Several recent contributions to the study of Robin optimal control problems in different directions:

1. A shape optimisation approach: β is a fixed constant, find the best Ω to optimise a criterion. (Bossel, Daners, Della Pietra-Gavitone for (an)isotropic Faber-Krahn inequalities, Alvino-Nitsch-Trombetti for Talenti inequalities, Bucur-Nahon-Nitsch-Trombetti for thermal insulation problems...)
2. Optimising $\beta = \beta(x)$: Hömberg-Krumbiegel-Rehberg, Lenhart-Protopopescu-Yong for the optimisation of tracking-type functionals in parabolic models.
3. A closely related problem: Bucur-Buttazzo-Nitsch (x2). Instead of optimising β optimise $\gamma = \gamma(x)$ with

$$\gamma(x)\partial_\nu w_\gamma + w_\gamma = 0.$$

   While similar, this problem belongs more to the "homogenisation" class of problems than to the present "potential optimisation" context.

In many of the references above, energetic criteria are considered. Our goal here is to study non-energetic criteria.

$$\mathcal{B} := \{0 \leq \beta \leq 1\,,\, \int_{\partial\Omega} \beta = V_0\}.$$

$$\max_{\beta \in \mathcal{B}} J(\beta) = \int_{\Omega \text{ or } \partial\Omega} j(x, u_\beta) \text{ subject to } \begin{cases} -\Delta u_\beta = f & \text{in } \Omega\,, \\ \partial_\nu u_\beta + \beta u_\beta = 0 & \text{on } \partial\Omega. \end{cases} \qquad (\mathbf{P}_{\mathrm{Rob}})$$

Theorem (M. Privat, 2022, Submitted)

Assume that $f \geq 0$ and that

$$\frac{\partial j}{\partial u}(x, \cdot) > 0 \text{ on } (0; +\infty).$$

Then any solution of $(\mathbf{P}_{\mathrm{Rob}})$ is a bang-bang function:

$$\beta^* = \mathbb{1}_{\Gamma^*} \text{ for some } \Gamma^* \subset \partial\Omega.$$

Note that we do not require any form of convexity a priori, merely the monotonicity of j.

How would we tackle the shape optimisation problem?

Let's focus on

$$\sup_{\Gamma\subset\partial\Omega\,,|\Gamma|=V_0} J(\Gamma) := \int_{\Omega \text{ or } \partial\Omega} j(x, v_\Gamma) \text{ subject to } \begin{cases} -\Delta v_\Gamma = f & \text{in } \Omega\,, \\ \partial_\nu v_\Gamma = 0 & \text{on } \partial\Omega\backslash\Gamma\,, \\ v_\Gamma = 0 & \text{on } \partial\Omega. \end{cases}$$

A typical tool we could think of is the Buttazzo-DalMaso theorem:

Monotonicity of J for the inclusion+Regularity $\Rightarrow$ Existence of an optimal shape.

How would we tackle the shape optimisation problem?

In [M. Nadin, Privat, Comm. in PDEs,2022] we observed that the same holds true for bilinear control problems:

$$\max_{0\leq m\leq 1\,,\int_\Omega m=m_0} K(m) := \int_\Omega j(x,u_m) \text{ subject to } -\Delta u_m = m u_m + F(x,u_m).$$

We showed that

$$(m \leq m' \Rightarrow K(m) \leq K(m')) \Rightarrow \text{any optimal } m^* \text{ is bang-bang: } m^* = \mathbb{1}_E.$$

The tools are however completely different.

Proof (I): first-order derivative of the criterion

We work with $J = \int_{\partial\Omega} j(u_\beta)$ (the idea is the same with $j(x, u)$ and Ω). The existence is immediate. We use **optimality conditions**; we differentiate the equation with respect to β.

Proof (I): first-order derivative of the criterion

Consider β, h and compute the derivative of $\beta \mapsto u_\beta$ in the direction h. Let us write

$$\dot{u}_\beta := \frac{\partial u_\beta}{\partial \beta}[h]$$

$$\frac{\partial}{\partial \beta} \begin{cases} -\Delta u_\beta = f \\ \partial_\nu u_\beta + \beta u_\beta = 0 \end{cases} \quad \rightsquigarrow \begin{cases} -\Delta \dot{u}_\beta = 0 \\ \partial_\nu \dot{u}_\beta + \beta \dot{u}_\beta = -h u_\beta. \end{cases}$$

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$$J(\beta) = \int_{\partial\Omega} j(u_\beta) \rightsquigarrow \dot{J}(\beta)[h] = \int_{\partial\Omega} \dot{u}_\beta j'(u_\beta).$$

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$$J(\beta) = \int_{\partial\Omega} j(u_\beta) \rightsquigarrow \dot{J}(\beta)[h] = \int_{\partial\Omega} \dot{u}_\beta j'(u_\beta).$$

Introduce the **adjoint state** p_β solution of

$$\begin{cases}-\Delta p_\beta = 0\\ \partial_\nu p_\beta + \beta p_\beta = j'(u_\beta).\end{cases} \qquad \text{As } j' > 0 \text{ we have } p_\beta > 0. \tag{4}$$

Multiplying the equation on $\dot{u}_\beta$ and integrating by parts we obtain

$$\dot{J}(\beta)[h] = -\int_{\partial\Omega} h(u_\beta p_\beta) = -\int_{\partial\Omega} h\Phi_\beta.$$

Proof (I): first-order derivative of the criterion

$$\dot{J}(\beta)[h] = -\int_{\partial\Omega} h(u_\beta p_\beta) = -\int_{\partial\Omega} h\Phi_\beta \text{ with } \Phi_\beta > 0.$$

If β^* is optimal and if by contradiction $\omega^* = \{0 < \beta^* < 1\}$ has positive measure the following holds: for any $h \in L^\infty(\partial\Omega)$ that is supported in ω^*

$$\dot{J}(\beta^*)[h] = 0.$$

We now analyse the **second-order derivative** of J to obtain a contradiction.

Proof (II): second-order derivative

Keep on differentiating: we have

$$\begin{cases} -\Delta \ddot{u}_\beta = 0 \\ \partial_\nu \ddot{u}_\beta + \beta \ddot{u}_\beta = -2h\dot{u}_\beta \end{cases} \quad \text{and } \ddot{J}(\beta)[h,h] = \int_{\partial\Omega} \dot{u}_\beta^2 j''(u_\beta) + \int_{\partial\Omega} \ddot{u}_\beta j'(u_\beta)$$

and

$$\begin{cases} -\Delta p_\beta = 0 \\ \partial_\nu p_\beta + \beta p_\beta = j'(u_\beta). \end{cases}$$

Thus

$$\ddot{J}(\beta)[h,h] = -2\int_{\partial\Omega} h\dot{u}_\beta p_\beta + \int_{\partial\Omega} j''(u_\beta)\dot{u}_\beta^2.$$

Proof (II): second-order derivative

$$\ddot{J}(\beta)[h,h] = -2\int_{\partial\Omega} h\dot{u}_\beta p_\beta + \int_{\partial\Omega} j''(u_\beta)\dot{u}_\beta^2.$$

But now recall that

$$-h = \frac{\partial_\nu \dot{u}_\beta + \beta \dot{u}_\beta}{u_\beta} \Rightarrow$$

$$-2\int_{\partial\Omega} h\dot{u}_\beta p_\beta = \int_{\partial\Omega} \underbrace{\frac{p_\beta}{u_\beta}}_{=:\Psi_\beta} \left(\partial_\nu(\dot{u}_\beta^2) + 2\beta\dot{u}_\beta^2\right).$$

Using

$$-\int_\Omega z\Delta\Psi_\beta + \int_\Omega \Psi_\beta \Delta z = \int_{\partial\Omega} (\partial_\nu z)\Psi_\beta - \int_{\partial\Omega} (\partial_\nu \Psi_\beta) z$$

with $z = \dot{u}_\beta^2$ $(\Delta z = 2|\nabla \dot{u}_\beta|^2 + 2\dot{u}_\beta \Delta \dot{u}_\beta)$ we obtain (Some steps are omitted)

$$-2\int_{\partial\Omega} h\dot{u}_\beta p_\beta = 2\int_\Omega \Psi_\beta |\nabla \dot{u}_\beta|^2 - \int_\Omega (\Delta\Psi_\beta)\dot{u}_\beta^2 + \int_{\partial\Omega} W_2 \dot{u}_\beta^2$$

for an L^∞ potential W_2.

Proof (III): more second-order derivative

Adding the missing terms we get

$$\ddot{J}(\beta)[h,h] = 2\int_\Omega \Psi_\beta |\nabla \dot{u}_\beta|^2 - \int_\Omega (\Delta \Psi_\beta)\dot{u}_\beta^2 + \int_{\partial\Omega} W \dot{u}_\beta^2$$

We already observed that

$$\inf p_\beta > 0 \to \inf \Psi_\beta > 0.$$

After some technical steps we obtain the estimate

$$\ddot{J}(\beta)[h,h] \geq A\int_\Omega |\nabla \dot{u}_\beta|^2 - B\int_\Omega \dot{u}_\beta^2 - C\int_{\partial\Omega} \dot{u}_\beta^2.$$

To obtain that $\omega^* = \{0 < \beta^* < 1\}$ has measure zero, argue by contradiction. Then we need to exhibit one perturbation h located in ω^* such that

$$\int_\Omega |\nabla \dot{u}_\beta|^2 \gg \int_\Omega \dot{u}_\beta^2 + \int_{\partial\Omega} \dot{u}_\beta^2.$$

Proof (IV): $\int_\Omega |\nabla \dot{u}_\beta|^2 \gg \int_{\partial\Omega} \dot{u}_\beta^2$

Back to the analysis of a linear problem: find h located in ω^* such that

$$\begin{cases} -\Delta \dot{u} = 0 \\ \partial_\nu \dot{u} + \beta \dot{u} = -hu \end{cases} \quad \text{and} \int_\Omega |\nabla \dot{u}_\beta|^2 \gg \int_{\partial\Omega} \dot{u}_\beta^2.$$

Introduce the set of eigenfunctions

$$\begin{cases} -\Delta \Psi_k = 0 \\ \partial_\nu \Psi_k + \beta \Psi_k = \lambda_k \Psi_k\,, \\ \int_{\partial\Omega} \Psi_k^2 = 1. \end{cases}$$

If $-hu$ writes

$$-hu = \sum_{k=K}^{\infty} a_k \Psi_k$$

then

$$\dot{u} = \sum_{k=K}^{\infty} \frac{a_k}{\lambda_k} \Psi_k \Rightarrow \int_\Omega |\nabla \dot{u}|^2 + \int_{\partial\Omega} \beta \Psi_k^2 = \sum_{k\geq K} \frac{a_k^2}{\lambda_k} \gg \sum_{k \geq K} \frac{a_k^2}{\lambda_k^2} = \int_{\partial\Omega} \dot{u}^2$$

and we are done!

But how to find such an h?

But how to find h such that

$$-hu = \sum_{k=K}^{\infty} a_k \Psi_k ?$$

This amounts to finding h supported in ω^* such that

$$-hu \in \cap_{k \leq K} \ker(T_k)\,,\; T_j : L^2(\omega^*) \ni f \mapsto \int_{\partial\Omega} f \Psi_j .$$

But this is just intersecting a finite number of hyperplanes in an infinite dimensional space.

Proof (V): $\int_\Omega |\nabla \dot u_\beta|^2 \gg \int_\Omega \dot u_\beta^2$

Back to the analysis of a linear problem: find h located in ω^* such that

$$\begin{cases} -\Delta \dot u = 0 \\ \partial_\nu \dot u + \beta \dot u = -hu \end{cases} \quad \text{and } \int_\Omega |\nabla \dot u_\beta|^2 \gg \int_\Omega \dot u_\beta^2.$$

Argue by contradiction: imagine there is a constant C such that

$$\forall h \in L^2(\omega^*)\,,\, \int_\Omega |\nabla \dot u_\beta|^2 \leq C \int_\Omega \dot u_\beta^2.$$

$L^2(\omega^*)$ is infinite dimensional. Then consider the set

$$X := \{\ \dot u_\beta\,, h \in L^2(\omega)\} \subset W^{1,2}(\Omega).$$

X is infinite dimensional: it has an orthonormal family $\{v_k\}_{k\in\mathbb{N}}$. By our assumption it is bounded in $W^{1,2}$. By Parseval

$$v_k \rightharpoonup 0.$$

By Rellich-Kondrachov, $v_k \to 0$ in L^2, a contradiction.

Conclusion of the proof

If ω^* has positive measure:

1. For any h supported in ω there holds $\dot{J}[h] = 0$.
2. $\ddot{J} \geq A \int_\Omega |\nabla \dot{u}|^2 - B \int_\Omega \dot{u}^2 - C \int_{\partial\Omega} \dot{u}^2$.
3. We can find h that satisfies both

$$\int_\Omega |\nabla \dot{u}|^2 \gg \int_\Omega \dot{u}^2 \, , \int_{\partial\Omega} \dot{u}^2.$$

   (omitted here but easy)

4. The contradiction follows.

Conclusion

1. The monotonicity of the functional implies some form of convexity.
2. Analog of the BDM theorem (see also [M, Nadin, Privat, CPDE, 2022], [M, 2022]). For instance, minimisation problems enjoy a relaxation phenomenon.
3. In [M,Privat, 2022] several other qualitative results about minimisation problems or minimisation of energetic functionals.

Thank You!