Une preuve constructive d'existence d'orbites périodiques spontanées pour les équations de Navier-Stokes

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Spontaneous periodic orbits of the Navier-Stokes equations

- Consider the Navier-Stokes equations on the torus

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f & \text{on } \mathbb{R} \times \mathbb{T}^3 \\ \nabla \cdot u = 0 & \text{on } \mathbb{R} \times \mathbb{T}^3 \end{cases}$$

with a *Taylor-Green* type of forcing

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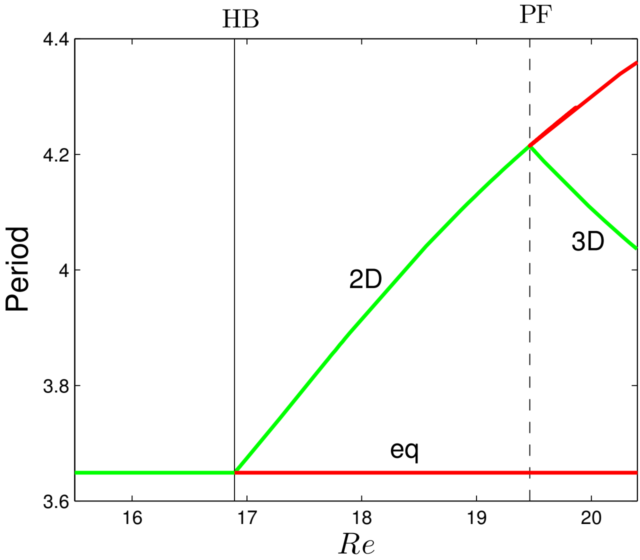
- This steady state is stable if the fluid is viscous enough. When ν decreases, it becomes unstable and the dynamics becomes more and more complex.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A few references

- In the presence of a periodic external influence, periodic motions in fluids have been studied extensively, and are relatively well understood [Serrin '59; Kaniel & Shinbrot '67; Takeshita '69; Maremonti '91; Kozono & Nakao '96; Yamazaki '00; Galdi & Sohr '04; etc.].

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- In the absence of a periodic external influence, periodic motions in fluids are much harder to study, and the existing results are typically of perturbative nature [Iudovich '71; Iooss '72; Joseph & Sattinger '72; Melcher, Schneider & Uecker '08; Galdi '16].

A numerical periodic solution

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Can we say anything rigorous about this specific “solution”?

A general problem

- Assume that we are given a function F defined on a Banach space, together with an approximate zero $\bar{x}$ of F:

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- We want an *a posteriori* error bound, but without knowing *a priori* that the true zero exists.

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A Newton-Kantorovich type of theorem

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $F : \mathcal{X} \to \mathcal{Y}$ a $\mathcal{C}^1$ function. Let $\bar{x} \in \mathcal{X}$ and assume we have the following estimates:

$$\begin{aligned} \left\| F(\bar{x}) \right\|_{\mathcal{Y}} &\leq \varepsilon \\ \left\| DF(\bar{x})^{-1} \right\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} &\leq \kappa \\ \left\| DF(x) - DF(\bar{x}) \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} &\leq \gamma \left(\left\| x - \bar{x} \right\|_{\mathcal{X}} \right). \end{aligned}$$

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If there exists $r > 0$ such that

$$\kappa\varepsilon + \kappa\gamma(r)r < r,$$

then F has a unique zero x satisfying $\|x - \bar{x}\|_{\mathcal{X}} \leq r$.

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Proof.

The operator $T : x \mapsto x - DF(\bar{x})^{-1}F(x)$ is a contraction on $B(\bar{x}, r)$. □

A Newton-Kantorovich type of theorem, version 2

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $F : \mathcal{X} \to \mathcal{Y}$ a $\mathcal{C}^1$ function. Let $\bar{x} \in \mathcal{X}$, $A \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ injective, and assume we have the following estimates:

$$\begin{aligned} \|F(\bar{x})\|_{\mathcal{Y}} &\leq \varepsilon \\ \|A\|_{\mathcal{L}(\mathcal{Y},\mathcal{X})} &\leq \kappa \\ \|I - ADF(\bar{x})\|_{\mathcal{L}(\mathcal{X},\mathcal{X})} &\leq \delta < 1 \\ \|DF(x) - DF(\bar{x})\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} &\leq \gamma\left(\|x - \bar{x}\|_{\mathcal{X}}\right). \end{aligned}$$

If there exists $r > 0$ such that

$$\kappa\varepsilon + (\delta + \kappa\gamma(r))\, r < r,$$

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Proof.

The operator $T : x \mapsto x - AF(x)$ is a contraction on $B(\bar{x}, r)$. □

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- We want to use this procedure to study periodic orbits of the Navier-Stokes equations.
- The first step is to find an appropriate $F = 0$ formulation, on a well-chosen Banach space.

Reformulation using the vorticity

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- Thanks to the continuity equation, we can express u as a function of ω

$$u = -\Delta^{-1} \nabla \times \omega,$$

which yields

$$\partial_t \omega - ((\Delta^{-1} \nabla \times \omega) \cdot \nabla) \omega + (\omega \cdot \nabla)\, (\Delta^{-1} \nabla \times \omega) - \nu \Delta \omega = g.$$

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- We call Ω the (unknown) frequency of the solution, and write

$$\omega(t,x) = \sum_{n=(n_t,\tilde{n}) \in \mathbb{Z}^4} \omega_n e^{i(n_t \Omega t + \tilde{n} \cdot x)}$$

with $\tilde{n} = (n_x, n_y, n_z)$ the space indices and n_t the time index.

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- Denoting $W = \left(\Omega, (\omega_n)_{n \in \mathbb{Z}^4}\right)$ and Fourier-transforming the vorticity equation, we obtain the problem $F(W) = (F_n(W))_{n \in \mathbb{Z}^4} = 0$ with

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  - Corresponds to smooth (analytic) functions
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- The computation of γ satisfying

$$\left\|DF(W) - DF(\bar{W})\right\| \leq \gamma\left(\|W - \bar{W}\|\right)$$

is rather straightforward (Banach algebra property).

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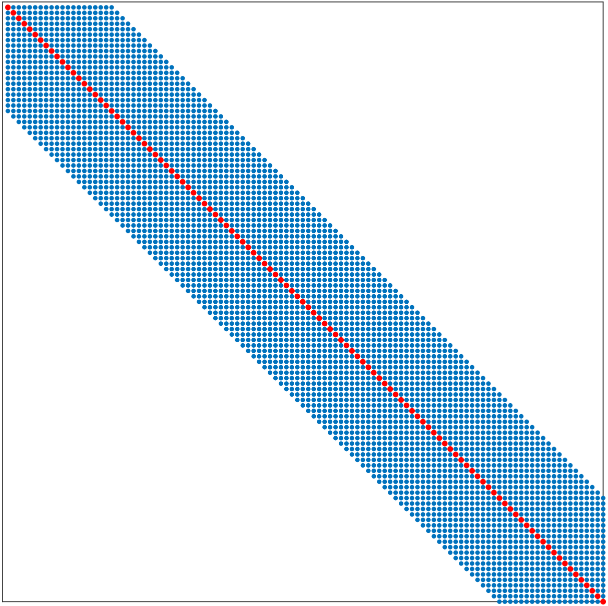
$DF(\bar{W}) =$ 

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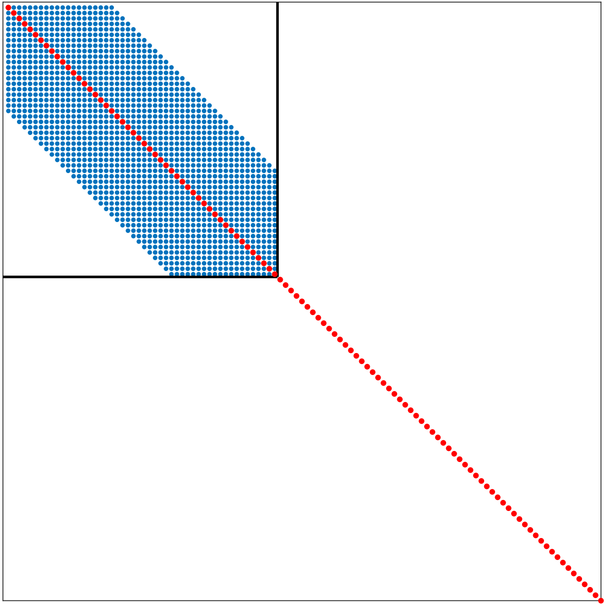

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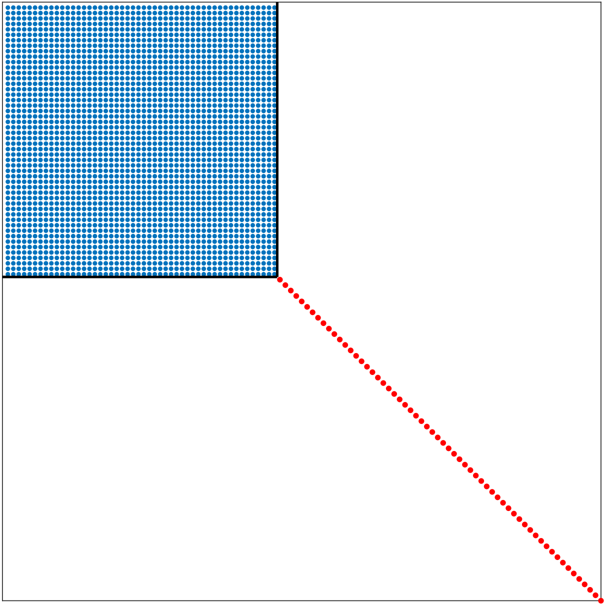
$A :=$ 

The inverse eigenvalues: $\lambda_n^{-1} = \left(i\bar{\Omega} n_t + \nu(n_x^2 + n_y^2 + n_z^2)\right)^{-1}$

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- The finite dimensional part can be obtained by inverting some finite dimensional projection of $DF(\bar{x})$.

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- In practice, it is crucial to reduce as much as possible the dimension of the finite-dimensional space that we keep for the validation (which drastically increases with m).

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- By deriving a posteriori estimates that are *compatible* with the symmetries, we can also reduce the number of modes used for the validation (and in particular to reduce the size of the finite block used in A).

Taking advantage of the symmetries

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- It turns out that the first branch of periodic orbits that we obtain after the bifurcation do not depend on z (the solutions are essentially 2D), which we also use to reduce the number of modes.

A 2D validated solution

Contour lines of the vertical vorticity $\omega^{(z)}$. $\nu = 0.286$.

Theorem [B., Lessard, van den Berg & van Veen '21]

There exists a periodic solution of NS at a distance of at most 10^{-5} (in $\mathcal{C}^0$ norm) of this numerical solution.

THANK YOU FOR YOUR ATTENTION!