Robust intersection algorithms for non-matching grids

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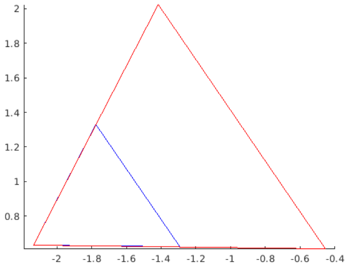
Failures of clipping algorithms



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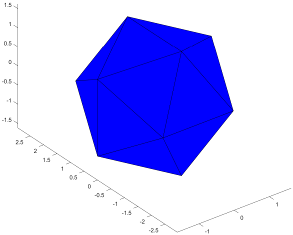

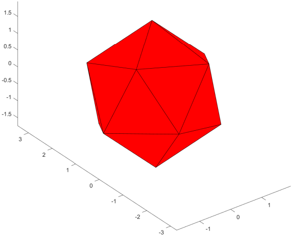

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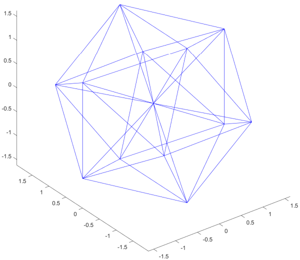
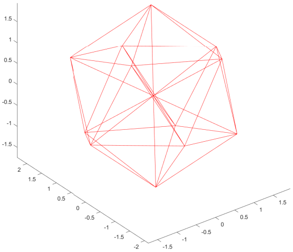
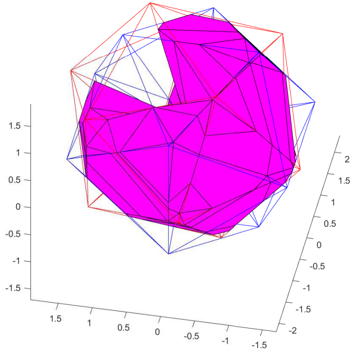




Failures of clipping algorithms



Towards consistency

Robust: resistant to failure.

One way to be robust is to use parsimony:
Parsimony: the principle of using the fewest resources to solve a problem.

If an algorithm is parsimonious it is self-consistent, meaning the result represents a possible accurate outcome, even if it's inaccurate for the given problem.

Change of coordinates

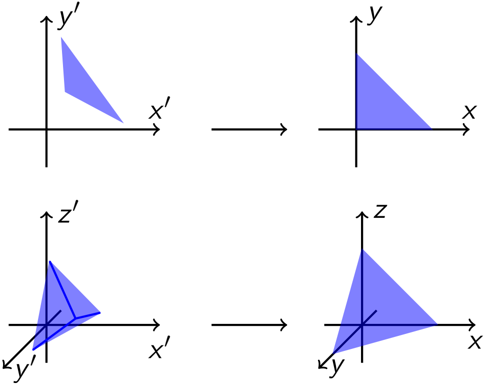
Simplex V is transformed to reference simplex Y:

$$\hat{V} = \mathbf{v_0}\mathbf{1}^\top + \begin{bmatrix} \mathbf{0} & \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \to \begin{bmatrix} \mathbf{0} & \mathbf{e}_1 & \dots & \mathbf{e}_n \end{bmatrix}.$$

Simplex U is transformed to simplex X under the same transformation:

$$\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 & \dots & \mathbf{x}_n \end{bmatrix} = \hat{U} - \mathbf{v}_0\mathbf{1}^\top.$$

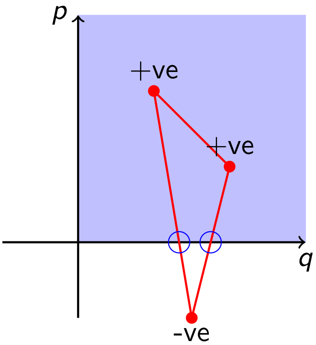
Reference simplices



Binary-valued sign function

$$\operatorname{sign}(p) = \begin{cases} 1 & p \geq 0, \\ 0 & p < 0, \end{cases}$$

We only calculate an intersection between two points if they have different signs in a given direction.

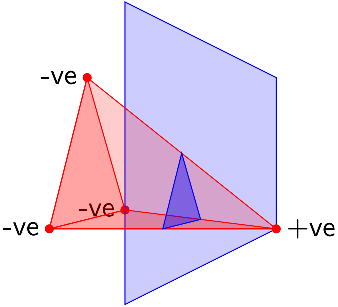

p
+ve
+ve
q
-ve

Hyperplanes of Y

Definition (Simplex 1)

A simplex in $\mathbb{R}^n$ is the intersection of $n+1$ half-spaces bounded by $n+1$ hyperplanes.

Let $P_\gamma = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{e}_\gamma = 0\}$. The positive side of P_γ is the space $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{e}_\gamma \geq 0\}$.


-ve
-ve
-ve
+ve

m-faces of X

Definition (Simplex 2)

A simplex in $\mathbb{R}^n$ is the convex hull between $n+1$ vertices.

An m-face of X is the convex hull between m of its vertices, ie. three vertices form a triangle, four a tetrahedron, etc.

Intersection between an m-face of X and m hyperplanes of Y

$$\mathbf{q}_\Gamma^J \cdot \mathbf{e}_\eta = \frac{\begin{vmatrix} \mathbf{x}_{i_0} \cdot \mathbf{e}_\eta & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_m} \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_{i_m} \cdot \mathbf{e}_\eta & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_m} \end{vmatrix}}{\begin{vmatrix} 1 & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_m} \\ \vdots & \vdots & & \vdots \\ 1 & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_m} \end{vmatrix}}$$

Corollary

The numerator of $\mathbf{q}_\Gamma^J \cdot \mathbf{e}_\eta$ is shared with the numerators of $\mathbf{q}_{\Gamma_j}^J \cdot \mathbf{e}_j$ for m values of j, up to a change in sign, where Γ and Γ_j have cardinality m.

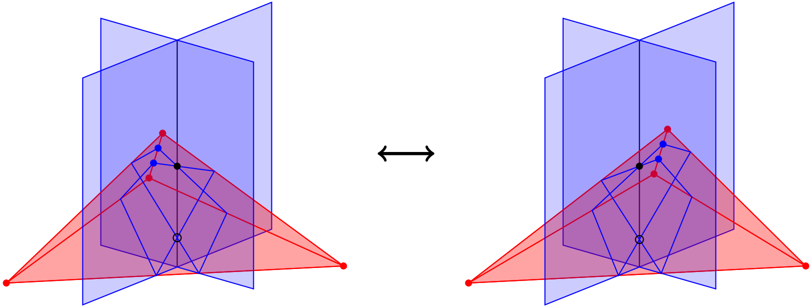
Intersections for triangles

Edge/Line	Distance from 1st vertex	Distance from 2nd vertex
$y = 0$	$\frac{x_i y_j - x_j y_i}{y_j - y_i}$	$\frac{(1-x_i)y_j - (1-x_j)y_i}{y_j - y_i}$
$x = 0$	$\frac{x_i y_j - x_j y_i}{x_i - x_j}$	$\frac{(1-y_i)x_j - (1-y_j)x_i}{x_j - x_i}$
$x + y = 1$	$\frac{(1-x_i)y_j - (1-x_j)y_i}{x_i + y_i - x_j - y_j}$	$\frac{(1-y_i)x_j - (1-y_j)x_i}{x_j + y_j - x_i - y_i}$

1st type of intersections for tetrahedra

Plane P	Num. of q_γ^{ij}	Num. of r_γ^{ij}	Num. of $1 - q_\gamma^{ij} - r_\gamma^{ij}$
$x = 0$	$\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}$	$\begin{vmatrix} x_i & z_i \\ x_j & z_j \end{vmatrix}$	$\begin{vmatrix} x_i & 1 - y_i - z_i \\ x_j & 1 - y_j - z_j \end{vmatrix}$
$y = 0$	$\begin{vmatrix} y_i & z_i \\ y_j & z_j \end{vmatrix}$	$\begin{vmatrix} y_i & x_i \\ y_j & x_j \end{vmatrix}$	$\begin{vmatrix} y_i & 1 - x_i - z_i \\ y_j & 1 - x_j - z_j \end{vmatrix}$
$z = 0$	$\begin{vmatrix} z_i & x_i \\ z_j & x_j \end{vmatrix}$	$\begin{vmatrix} z_i & y_i \\ z_j & y_j \end{vmatrix}$	$\begin{vmatrix} z_i & 1 - x_i - y_i \\ z_j & 1 - x_j - y_j \end{vmatrix}$
$x + y + z = 1$	$\begin{vmatrix} 1 - y_i - z_i & x_i \\ 1 - y_j - z_j & x_j \end{vmatrix}$	$\begin{vmatrix} 1 - x_i - z_i & y_i \\ 1 - x_j - z_j & y_j \end{vmatrix}$	$\begin{vmatrix} 1 - x_i - y_i & z_i \\ 1 - x_j - y_j & z_j \end{vmatrix}$

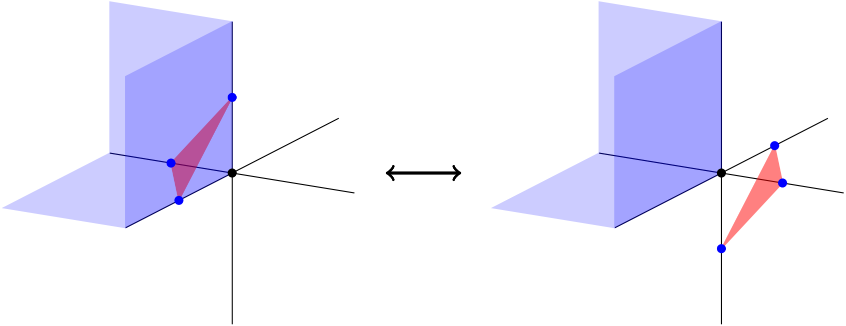
Pairings



2nd type of intersections for tetrahedra

Edge of Y	y, z	x, z	x, y	...
Num. of t^{ijk}	$\begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}$	$\begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}$	$\begin{vmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{vmatrix}$	
Num. of $1 - t^{ijk}$	$\begin{vmatrix} 1 - x_i & y_i & z_i \\ 1 - x_j & y_j & z_j \\ 1 - x_k & y_k & z_k \end{vmatrix}$	$\begin{vmatrix} x_i & 1 - y_i & z_i \\ x_j & 1 - y_j & z_j \\ x_k & 1 - y_k & z_k \end{vmatrix}$	$\begin{vmatrix} x_i & y_i & 1 - z_i \\ x_j & y_j & 1 - z_j \\ x_k & y_k & 1 - z_k \end{vmatrix}$	

Triplings



Simplicial intersection algorithm

Step 1: Change of coordinates

$$\begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 & \dots & \mathbf{x}_n \end{bmatrix} = \hat{U} - \mathbf{v}_0 \mathbf{1}^\top .$$

Step 2: Vertices of X in Y

$$\prod_{\gamma} \text{sign}(\mathbf{x}_i \cdot \mathbf{e}_\gamma) = 1 \implies \mathbf{x}_i \in Y$$

Simplicial intersection algorithm

Step 3: Intersections of X with Y

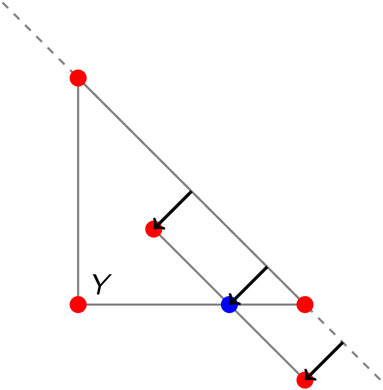
Step 3(i): Intersection magnitude

$$\mathbf{q}_\Gamma^J \cdot \mathbf{e}_\eta = \frac{\begin{vmatrix} \mathbf{x}_{i_0} \cdot \mathbf{e}_\eta & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_m} \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_{i_m} \cdot \mathbf{e}_\eta & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_m} \end{vmatrix}}{\begin{vmatrix} 1 & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_0} \cdot \mathbf{e}_{\gamma_m} \\ \vdots & \vdots & & \vdots \\ 1 & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_1} & \dots & \mathbf{x}_{i_m} \cdot \mathbf{e}_{\gamma_m} \end{vmatrix}}$$

Simplicial intersection algorithm

Step 3(ii): Intersection signs

If there aren't m intersections for a given m-face of X then the signs of their numerators can be determined without calculations. Otherwise, only one needs to be calculated and the rest become known.



Simplicial intersection algorithm

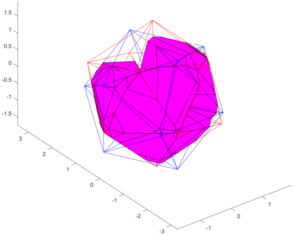
Step 3(iii): Reverse transformation

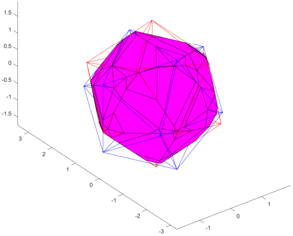
$$\mathbf{w}_\Gamma^J = \mathbf{v}_0 + \sum_{\gamma \notin \Gamma \cup \{0\}} \left(\mathbf{q}_\Gamma^J \cdot \mathbf{e}_\gamma \right) \cdot \mathbf{v}_\gamma$$

Simplicial intersection algorithm

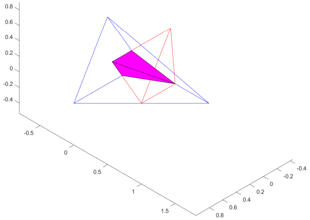
Step 4: Vertices of Y in X
At the highest order, every Γ has 2 planes not inside it. There are only two intersections for this collection. If they have different signs for one of the planes not in Γ then the vertex for the other plane lies in X.
Essentially, there's one more intersection calculation but since we're at the highest order the intersection lies in a 0-dimensional space, the vertex of Y.

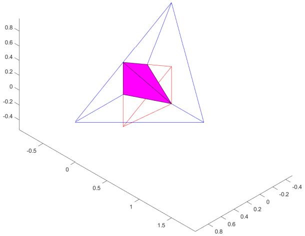
Computational examples





Computational examples





Computational examples

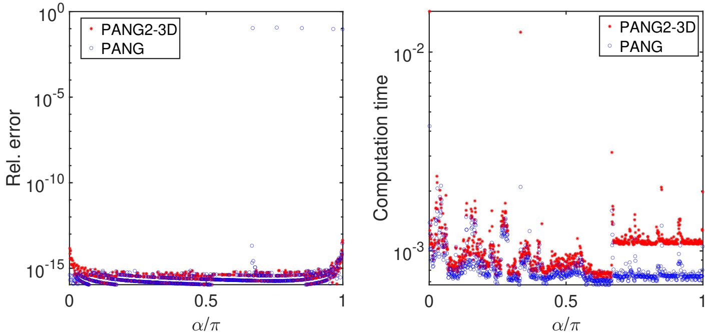


Figure: Relative error (left) and computation time (right)

Adjacent intersections

At intermediate steps in the algorithm we compare the signs of intersections to see if we need to calculate an intersection. But only those pairs of intersections whose edges are on the *outside* of X are important; those *inside* won't be seen inside the convex intersection.

An edge will lie outside if the two index sets J and K of its points differ by one element, ie.

$$|J \cup K| = |J| - 1 = |K| - 1.$$

We say the two sets are adjacent.

Combinadics

Checking which J and K are adjacent can be costly. Luckily, there's a possible shortcut.

Each J and K represents a combination of the numbers $\{0, \dots, n\}$ and so can be ranked using the combinatorial number system:

$$N(J) = \binom{c_m}{m} + \cdots + \binom{c_1}{1} \in \left\{0, \dots, \binom{n}{m} - 1\right\}$$

where $J = \{c_j\}_{j=1}^m$, $c_1 < c_2 < \cdots < c_m$.

Combinadic adjacency matrix

Then, we can construct an adjacency matrix for the nodes $N(J)$:

Lemma

For $n > m > 1$

$$A_m^n = \begin{bmatrix} A_m^{n-1} & \tilde{A}_m^n \\ (\tilde{A}_m^n)^\top & A_{m-1}^{n-1} \end{bmatrix}, \quad \tilde{A}_m^n = \begin{bmatrix} \tilde{A}_m^{n-1} & 0_{\binom{n-2}{m-2}\times\binom{n-2}{m}} \\ I_{\binom{n-2}{m-1}} & \tilde{A}_{m-1}^{n-1} \end{bmatrix}, \quad (1)$$

with starting conditions

$$A_1^n = \begin{bmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix}, \quad \tilde{A}_1^n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \tilde{A}_2^4 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (2)$$

Sign connectivity

We saw how the numerators of $\mathbf{q}_\Gamma^J \cdot \mathbf{e}_\eta$ are connected, but what about the denominators? These can be constructed from previous steps (X_J are the vectors $\mathbf{x}_j$ with $j \in J$, I_Γ the $\mathbf{e}_\gamma$ with $\gamma \in \Gamma$):

Lemma

Suppose $\operatorname{sign}\left(\mathbf{q}_\Gamma^{J\setminus\{i\}} \cdot \mathbf{e}_\eta\right) \neq \operatorname{sign}\left(\mathbf{q}_\Gamma^{J\setminus\{j\}} \cdot \mathbf{e}_\eta\right)$ *and an intersection* $\mathbf{q}_{\Gamma\setminus\{\gamma\}}^{J\setminus\{i,j\}}$ *was calculated for some* $\gamma \in \Gamma$, *then*

$$\operatorname{sign}\left(\left|\mathbf{1} \quad X_J^\top I_{\Gamma\cup\{\eta\}}\right|\right) = \operatorname{sign}\left(\left|X_{J\setminus\{i\}}^\top I_{\Gamma\cup\{\eta\}}\right| \left|X_{J\setminus\{i,j\}}^\top I_\Gamma\right| \left|\mathbf{1} \quad X_{J\setminus\{j\}}^\top I_\Gamma\right|\right)$$

Sign connectivity

If the intersection $\mathbf{q}_{\Gamma\setminus\{\gamma\}}^{J\setminus\{i,j\}}$ from the last lemma wasn't calculated then we need another way to find the sign of $\left|X_{J\setminus\{i,j\}}^\top I_\Gamma\right|$.

Lemma

Suppose $\operatorname{sign}\left(\mathbf{q}_\Gamma^{J\setminus\{i\}}\cdot\mathbf{e}_\gamma\right)=\operatorname{sign}\left(\mathbf{q}_\Gamma^{J\setminus\{j\}}\cdot\mathbf{e}_\gamma\right)$ *for all* $\gamma\notin\Gamma$. *Then, for all* $\eta\notin\Gamma$,

$$\left|X_J^\top I_{\Gamma\cup\{\eta,\gamma\}}\right|=\begin{vmatrix}\mathbf{q}_\Gamma^{J\setminus\{i\}}\cdot\mathbf{e}_\eta & \mathbf{q}_\Gamma^{J\setminus\{i\}}\cdot\mathbf{e}_\gamma\\ \mathbf{q}_\Gamma^{J\setminus\{j\}}\cdot\mathbf{e}_\eta & \mathbf{q}_\Gamma^{J\setminus\{j\}}\cdot\mathbf{e}_\gamma\end{vmatrix}\frac{\left|\mathbf{1}\quad X_{J\setminus\{i\}}^\top I_\Gamma\right|\left|\mathbf{1}\quad X_{J\setminus\{j\}}^\top I_\Gamma\right|}{\left|X_{J\setminus\{i,j\}}^\top I_\Gamma\right|}$$