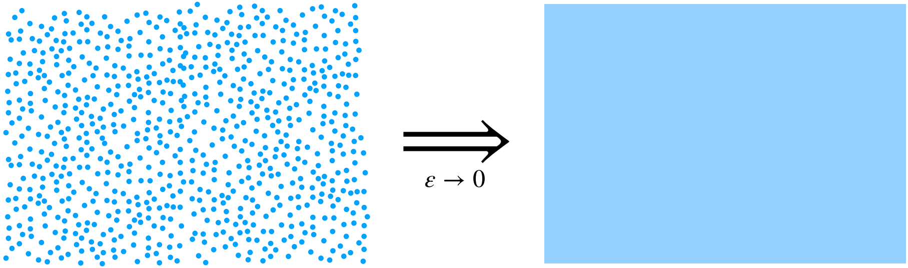
# A fixed point approach to Clausius-Mossotti formulas

Jules Pertinand

June, 16th 2022

# All you need to know about homogenisation (for this talk)



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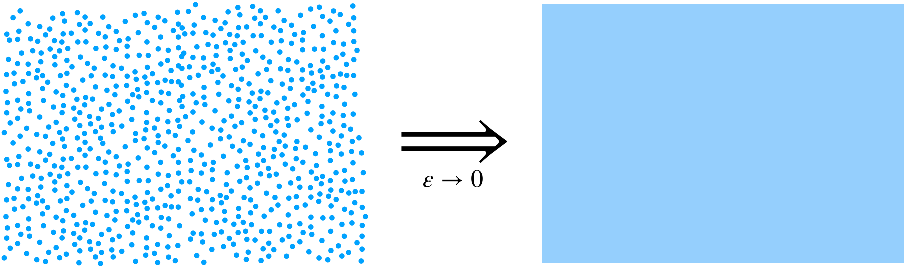


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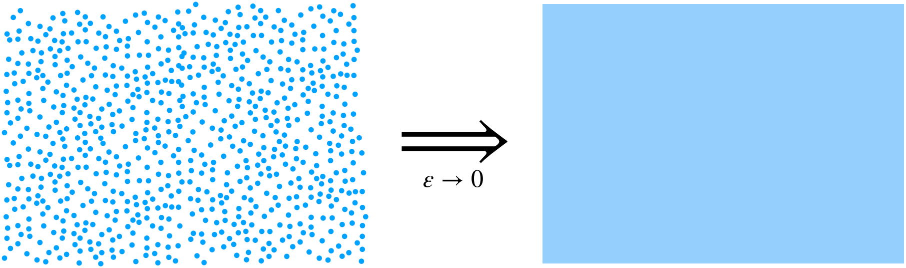


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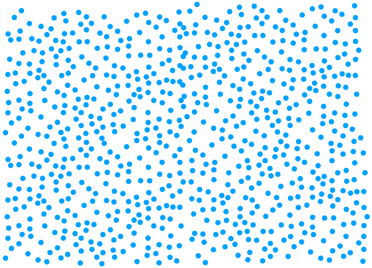

$$\Longrightarrow_{\varepsilon \to 0}$$

$$-\nabla \cdot a(\frac{\cdot}{\varepsilon})\nabla u_\varepsilon = f \qquad -\nabla \cdot \overline{a}\nabla\overline{u} = f$$

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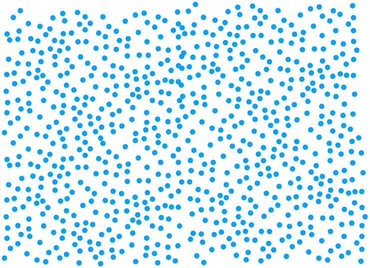

$$\xrightarrow[\varepsilon \to 0]{} $$

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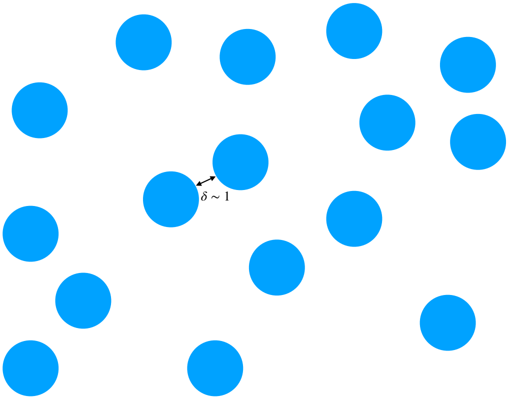
$$-\nabla \cdot a\left(\nabla \varphi_i + e_i\right) = 0 \text{ in } \mathbb{R}^d \Longrightarrow \nabla \varphi_i = \nabla(-\nabla \cdot a\nabla)^{-1}\nabla \cdot a e_i$$

# Two-phase material




$\mathcal{P} = \{x\}_{x \in \mathcal{P}}$ : random point process

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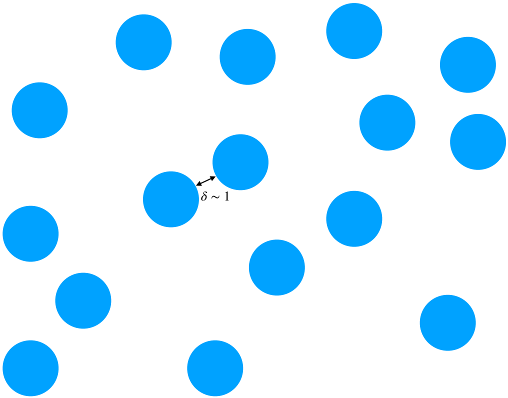


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- Stationnary ergodic

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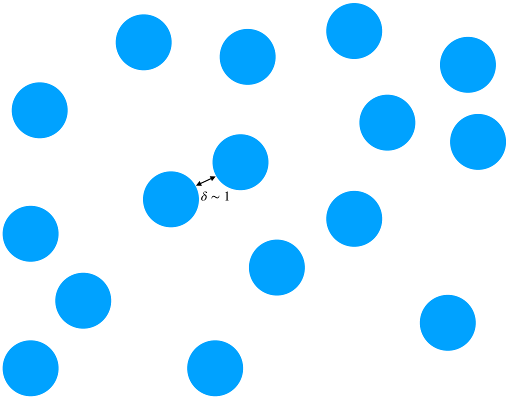


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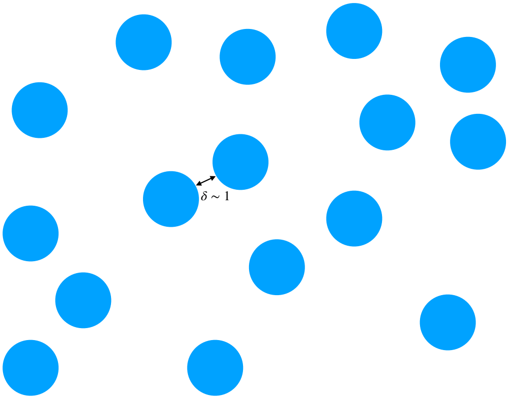


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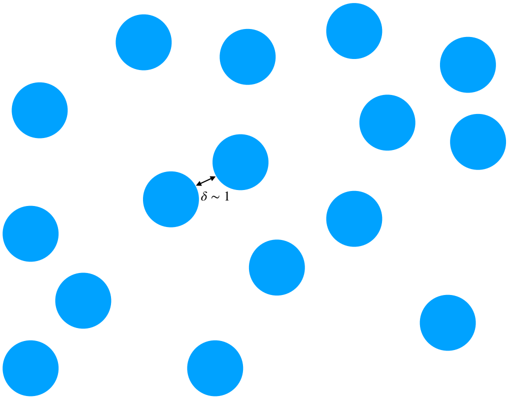


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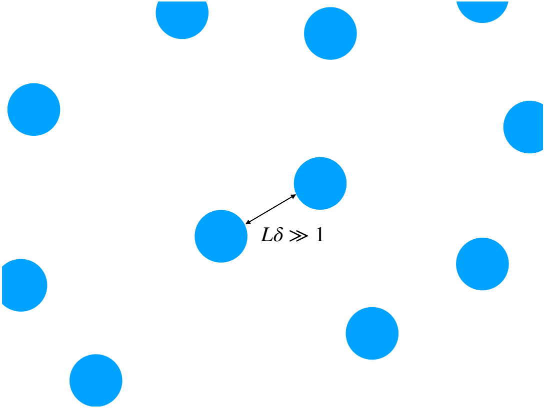
$$a(\mathcal{P}) = \alpha + \textstyle\sum_{x \in \mathcal{P}} \mathbb{1}_{B_1(x)} \beta \quad \overset{\text{Homog.}}{\Longrightarrow} \quad \bar{a}(\mathcal{P})$$

# Dilated two-phase material




$L\mathcal{P} = \{Lx\}_{x\in\mathcal{P}}$ : dilated random point process

- $L \gg 1$

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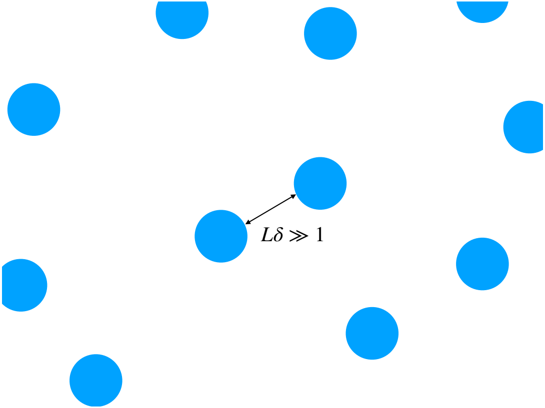


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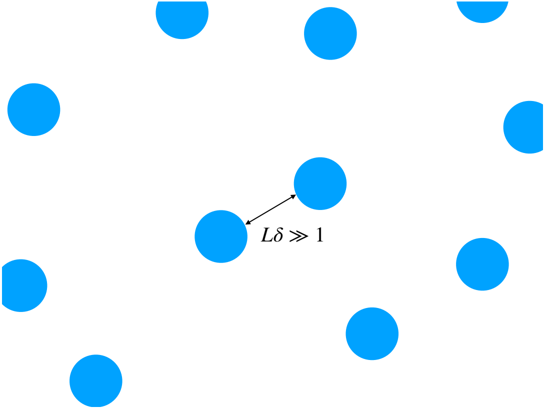


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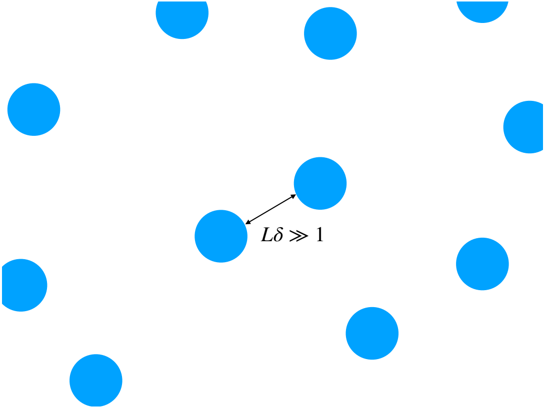


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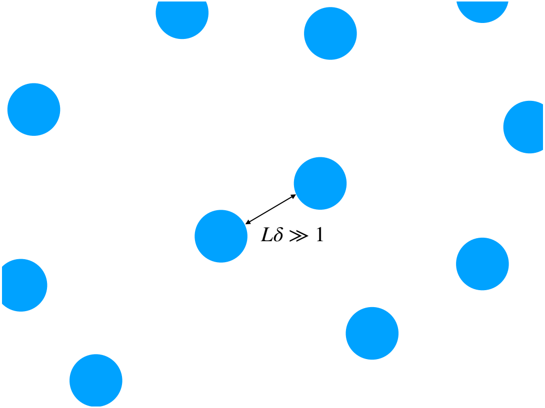


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$L\mathcal{P} = \{Lx\}_{x\in\mathcal{P}}$ : dilated random point process

$$a(L\mathcal{P}) = \alpha + \sum_{x\in\mathcal{P}} \mathbb{1}_{B_1(Lx)}\beta \quad \stackrel{\text{Homog.}}{\Longrightarrow} \quad \overline{a}^L(\mathcal{P})$$

# Main question

What can we say about $L \mapsto \overline{a}^L(\mathcal{P})$ ?

# Outline

Main result

Fixed point approach

Conclusion

## Theorem (P. '21)

*For $\mathcal{P}$ hardcore stationnary ergodic, $L^{-1} \mapsto \overline{a}^L$ analytic at $0$.*
*There exists, $(\mathcal{A}^{(i)})_{i\in\mathbb{N}} \in \left(\mathbb{R}^{d\times d}\right)^{\mathbb{N}}$ s.t,*

$$\overline{a}^L = \alpha + \sum_{i=d}^{\infty} \mathcal{A}^{(i)} L^{-i} \ .$$

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$$\overline{a}^L = \alpha + \sum_{i=d}^{\infty} \mathcal{A}^{(i)} L^{-i} \ .$$

*For $e \in \mathbb{R}^d$,*

$$\overline{a}^L e = \alpha e + L^{-d} \int_{B_1} \beta \mathbb{E}^{\circ} \left[ \left( \mathrm{Id} - L^{-d} \mathcal{B}^{\circ} K^L \right)^{-1} (\nabla \varphi^{\circ} + e) \right]$$

*with $L^{-1} \mapsto K^L$ analytic.*

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- $0^{\text{th}}$ order :

$$\overline{a}^L e = \alpha e + L^{-d} \int_{B_1} \beta \mathbb{E}^\circ \left[ (\nabla \varphi + e)(L\mathcal{P}^\circ, x) \right] \mathrm{d}x$$

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- Fixed-point formulation:

$$(\nabla\varphi + e)(L\mathcal{P}^\circ) = \left( \mathrm{Id} - L^{-d} \mathcal{B}^\circ K(L\mathcal{P}^\circ) \right)^{-1} (\nabla\varphi^\circ + e) \quad \text{in } B_1$$

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- Fixed-point operator : $|||K||| \lesssim 1$

# $0^{\text{th}}$ order reformulation

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Indeed,

$$\overline{a}^L e = \mathbb{E}\left[ \left( \alpha + \sum_{x \in \mathcal{P}} \beta \mathbb{1}_{B_1}(0 - Lx) \right) (\nabla\varphi + e)\,(L\mathcal{P}, 0) \right]$$

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&= \alpha e + \mathbb{E}\left[ \sum_{x \in \mathcal{P}} \beta \mathbb{1}_{B_1}(-Lx)\,(\nabla \varphi + e)\,(L(\mathcal{P} - x), 0 - Lx) \right]
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&= \alpha e + \int_{\mathbb{R}^d} \beta \mathbb{1}_{B_1}(-Lx) \mathbb{E}^\circ \left[ (\nabla \varphi + e)(L\mathcal{P}^\circ, -Lx) \right] \theta \, \mathrm{d}x
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Ref : Rayleigh-Maxwell [JKO], Berdichevsky, Anantharaman-Le Bris, Duerinckx-Gloria, . . .

# Formal computations

- PDE for the $\varphi(L\mathcal{P}^\circ) - \varphi^\circ$

$$-\nabla \cdot a^\circ \nabla \left(\varphi(L\mathcal{P}^\circ) - \varphi^\circ\right) = \nabla \cdot \sum_{\substack{x \in \mathcal{P}^\circ \\ x \neq 0}} \beta \mathbb{1}_{B_1(Lx)} \left(\nabla \varphi(L\mathcal{P}^\circ) + e\right)$$

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&= (\nabla \varphi^\circ + e) \\
&\quad + L^{-d} \sum_{\substack{x \in \mathcal{P}^\circ \\ x \neq 0}} \nabla^2 G \left(\frac{\cdot}{L} - x\right) * \beta \mathbb{1}_{B_1} \left(\nabla \varphi + e\right) (L(\mathcal{P}^\circ - x))
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- Fixed Point : $K(L\mathcal{P}^\circ)f \coloneqq \sum_{\substack{x\in\mathcal{P}^\circ \\ x\neq 0}} \nabla^2 G\left(\frac{\cdot}{L} - x\right) * \beta\mathbb{1}_{B_1} f(L(\mathcal{P}^\circ - x))$

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- $-\Delta \rightsquigarrow -\nabla\cdot a^\circ\nabla$ : $\mathcal{H}_{a^\circ} f \coloneqq \nabla u$ with $-\nabla\cdot a^\circ\nabla u = \nabla\cdot f$

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- $|\nabla^2 G(z)| \sim |z|^{-d} \implies \sum_{\substack{x\in\mathcal{P}^\circ \\ x\neq 0}} \nabla^2 G(x)$ not summable !

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where

$$\begin{aligned} -\Delta u &= \nabla \cdot \sum_{x\in\mathcal{P}^\circ} \beta \left[\mathbb{1}_{B_1} f\right](L(\mathcal{P}^\circ - x), \cdot - Lx) \quad \text{in } \mathbb{R}^d \\ -\Delta u^\circ &= \nabla \cdot \beta \mathbb{1}_{B_1} f(L\mathcal{P}^\circ) \quad \text{in } \mathbb{R}^d \end{aligned}$$

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using $|\nabla u^\circ(x)| \lesssim (1+|x|)^{-d}$

# Summary

- $0^{\text{th}}$ order :

$$\overline{a}^L e = \alpha e + L^{-d} \int_{B_1} \beta \mathbb{E}^\circ \left[ (\nabla \varphi + e)(L\mathcal{P}^\circ, x) \right] \mathrm{d}x$$

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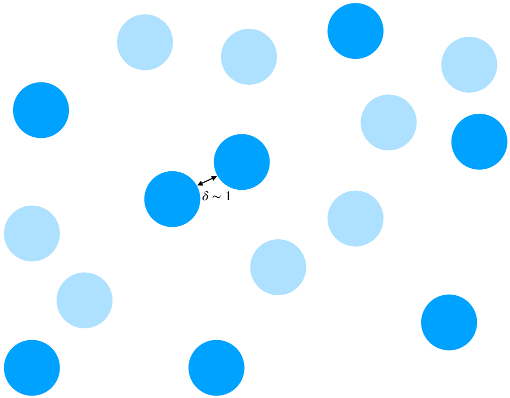


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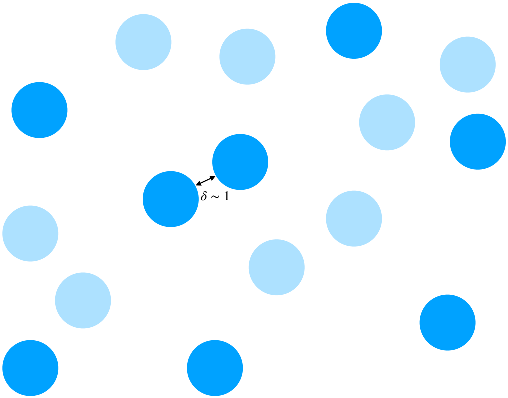


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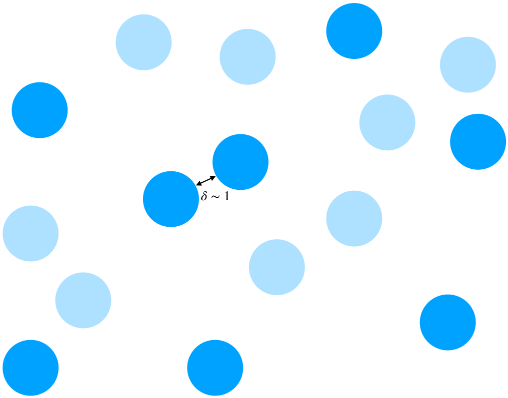


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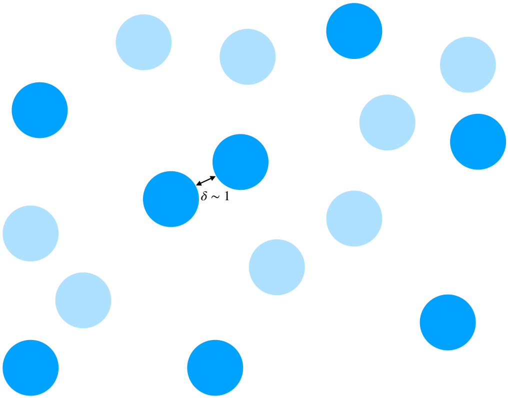


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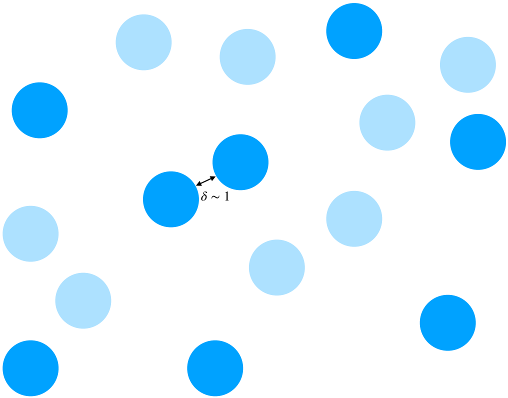


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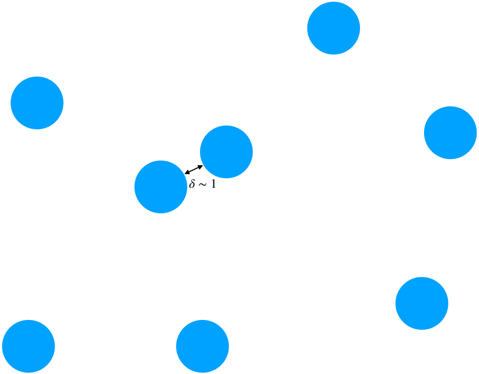


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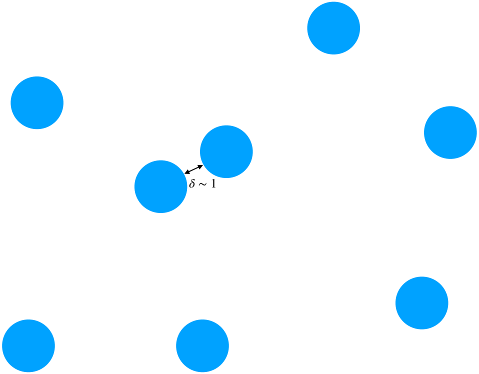


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