Méthodes numériques sans grille pour la régularisation de problèmes inverses par la variation totale

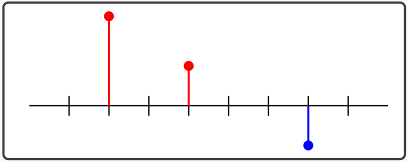
Yohann De Castro, Vincent Duval, Romain Petit

CANUM, 13 Juin 2022

Reconstruction de signaux parcimonieux

Vecteurs parcimonieux

- $x_0 = \sum_i a_i \, (0, \ldots, 1, \ldots, 0)$



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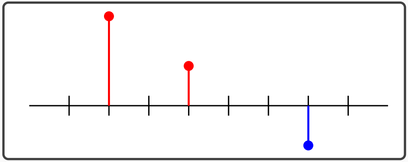

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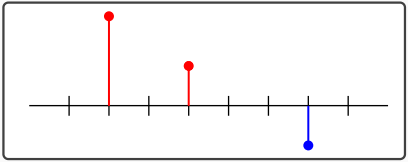

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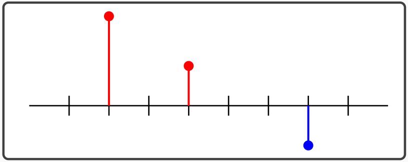

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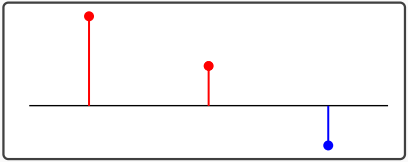

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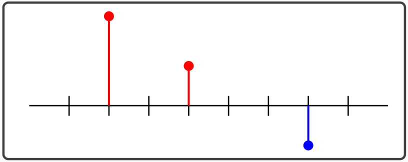

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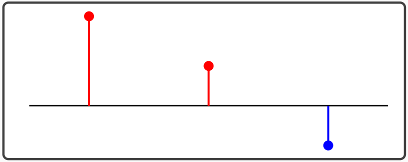

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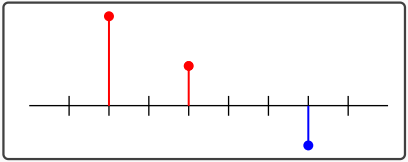

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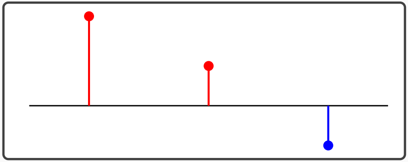

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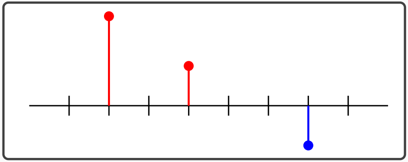

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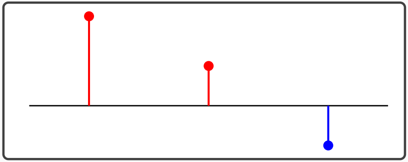
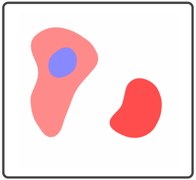


Fonctions “simples”

- $u_0 = \sum_i a_i \, \mathbf{1}_{E_i}$



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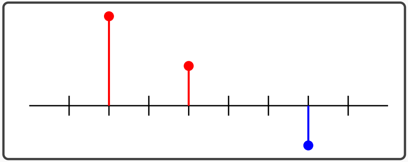

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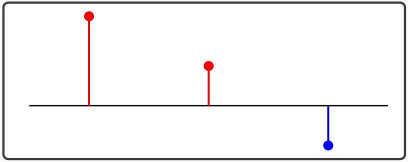

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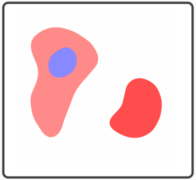

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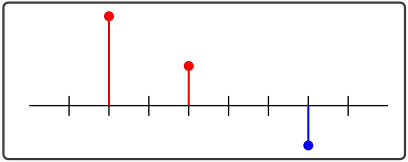

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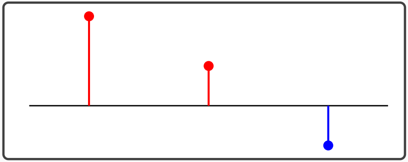

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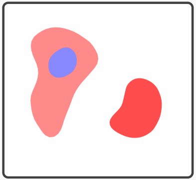

Problème variationnel [Rudin et al., 1992, Chambolle and Lions, 1997]

Résoudre

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \quad T_\lambda(u) \overset{\text{def.}}{=} \frac{1}{2}\|\Phi u - y\|^2 + \lambda\,\mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

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Variation totale (du gradient)

- $\mathrm{TV}(u) = \sup\left\{ -\int_{\mathbb{R}^2} u \,\mathrm{div}\phi \;\middle|\; \phi \in \mathrm{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2),\ \|\phi\|_\infty \leq 1 \right\}$

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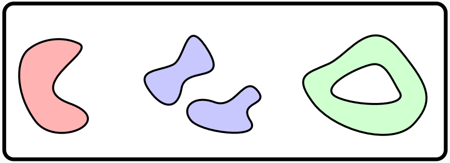


Discrétisations de la variation totale (images : [Tabti et al., 2018])

Anisotrope

- $\sum_{ij} |(D_x u)_{ij}| + |(D_y u)_{ij}|$
- Bords nets, **biais de grille**

Isotrope

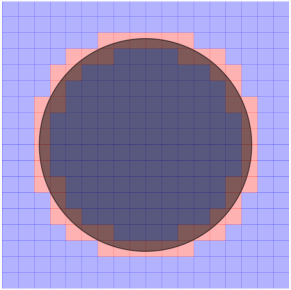
- $\sum_{ij} \sqrt{(D_x u)_{ij}^2 + (D_y u)_{ij}^2}$
- **Flou**

Revue de l'état de l'art : [Chambolle and Pock, 2021]

Représentation d'images simples

Grille fixe

- $\mathcal{O}\left(1/h^2\right)$ pixels
- $\mathcal{O}\left(1/h\right)$ pixels "utiles"
- $u \mapsto \mathrm{TV}(u)$ convexe



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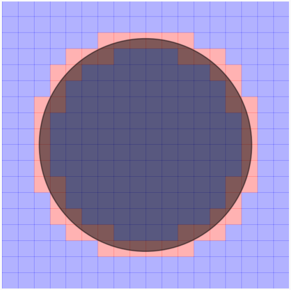
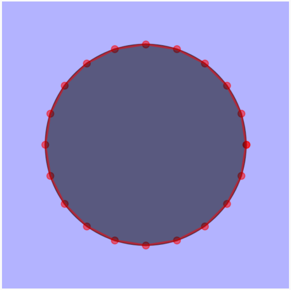
Discrétisation des bords

- Plus compact pour img. simple
- Plus complexe numériquement
- $E \mapsto \mathrm{TV}(\mathbf{1}_E)$ "non convexe"





Frank-Wolfe / gradient conditionnel et variantes

Frank-Wolfe

$$\min_{x \in C} f(x)$$

- $s_k \ni \operatorname{Argmin}_{s \in C} f(x_k) + df(x_k) \,.\, (s - x_k)$
- $x_{k+1} = x_k + \gamma_k (s_k - x_k)$

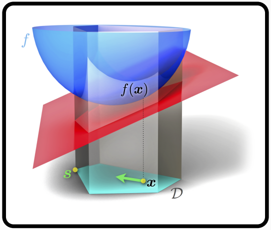


Image : Stephanie Stutz et Martin Jaggi

Itérées combinaisons convexes de quelques points extrémaux de C

Frank-Wolfe / gradient conditionnel et variantes

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Notre cadre

$$\min_{\substack{(u,t) \text{ s.t.} \\ \mathrm{TV}(u) \leq t \leq t_0}} \frac{1}{2} ||\Phi u - y||^2 + \lambda \, t$$

$$\eta_k \stackrel{\text{def.}}{=} \Phi^* (\Phi u_k - y)$$

- $E_{k+1} \ni \underset{E \subset \mathbb{R}^2}{\operatorname{Argmax}} \; \left| \int_E \eta_k \right| / P(E)$
- $u_{k+1} = \alpha_k \, u_k + \beta_k \, \mathbf{1}_{E_{k+1}}$

Itérées combinaisons linéaires de quelques indicatrices d'ensembles simples

Frank-Wolfe / gradient conditionnel et variantes

Frank-Wolfe (variante)

$$\min_{x \in C} f(x)$$

- $s_k \ni \underset{s \in C}{\mathrm{Argmin}}\ f(x_k) + df(x_k)\,.\,(s - x_k)$
- $\tilde{x}_{k+1} = x_k + \gamma_k (s_k - x_k)$
- Trouver x_{k+1} s.t. $f(x_{k+1}) \leq f(\tilde{x}_{k+1})$

Notre cadre

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- Optim. loc. de la combin. convexe

[Bredies and Pikkarainen, 2013]
[Denoyelle et al., 2019]

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- Opt. loc. $(a, E) \mapsto T_\lambda (\sum_i a_i \mathbf{1}_{E_i})$

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Le problème de Cheeger

Résoudre

$$\max_{E \subset \mathbb{R}^2} \frac{\left|\int_E \eta\right|}{P(E)} \text{ s.t. } |E| < +\infty,\ 0 < P(E) < +\infty$$

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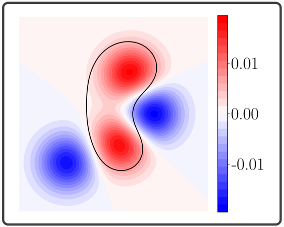
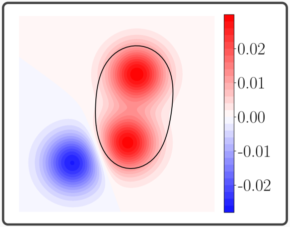
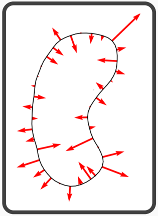


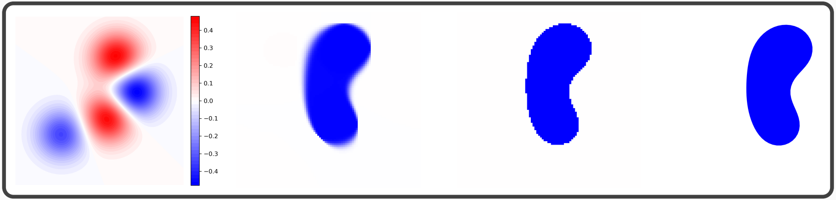


Résolution numérique en deux étapes

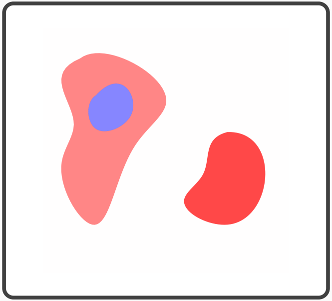
"Descente de gradient de forme"

- $E_{n+1} = (Id + \epsilon_n\, \theta_n)(E_n)$
- $\theta_n \in \underset{\theta \in \Theta_{ad}}{\text{Argmax}} \lim_{\epsilon \to 0^+} \frac{J((Id+\epsilon\theta)(E_n)) - J(E_n)}{\epsilon}$

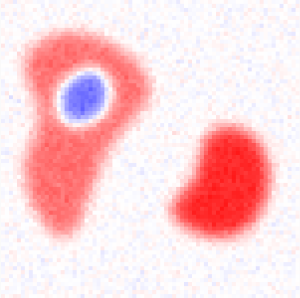




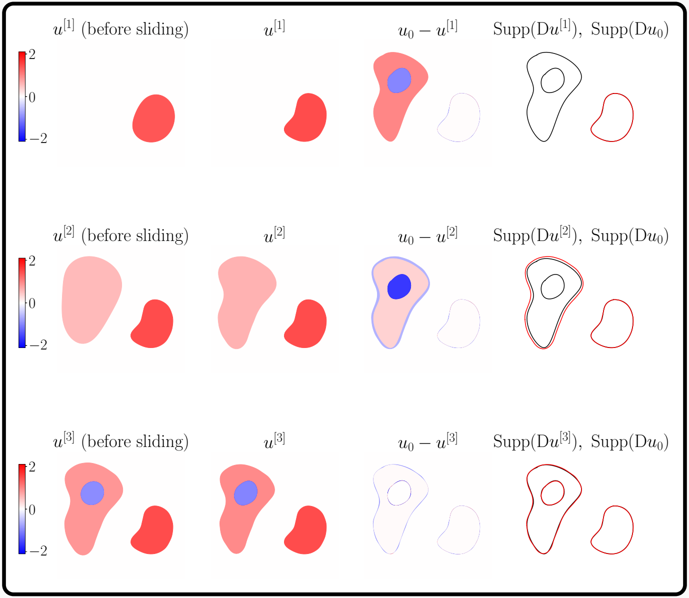
Implémentation : `github.com/rpetit/PyCheeger`



Signal



Observations


$u^{[1]}$ (before sliding)
$u^{[1]}$
$u_0 - u^{[1]}$
$\mathrm{Supp}(\mathrm{D}u^{[1]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2
$u^{[2]}$ (before sliding)
$u^{[2]}$
$u_0 - u^{[2]}$
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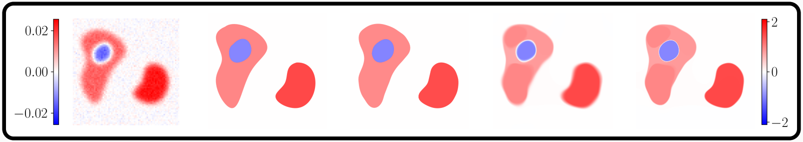
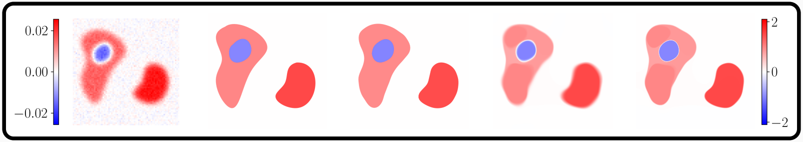
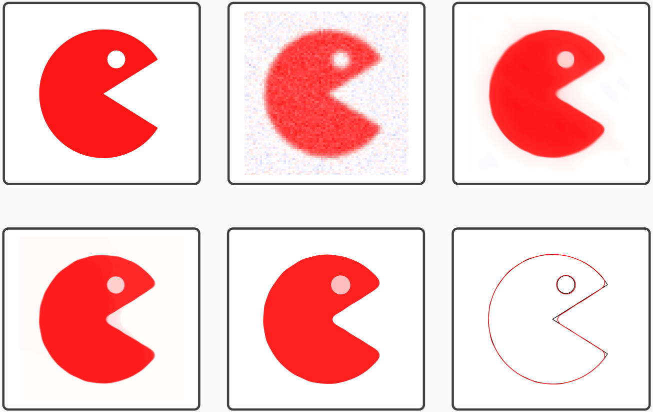
0
−2
$u^{[3]}$ (before sliding)
$u^{[3]}$
$u_0 - u^{[3]}$
$\mathrm{Supp}(\mathrm{D}u^{[3]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2

Résultats numériques




0.02
0.00
−0.02
2
0
−2



Conclusion

Résumé

- Algorithme convergent
- Méthode num. sans grille
- Tol. approx. [Jaggi, 2013]

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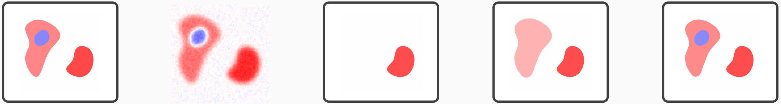

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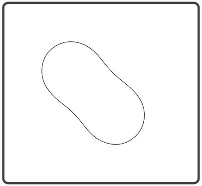
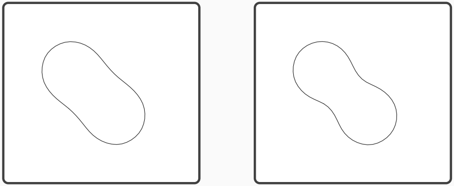
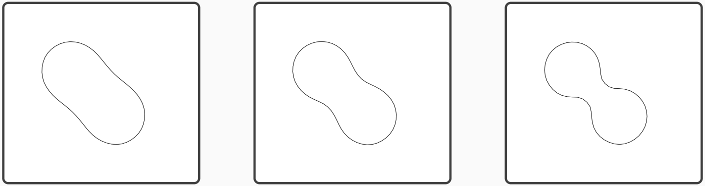
Topology changes during the local descent



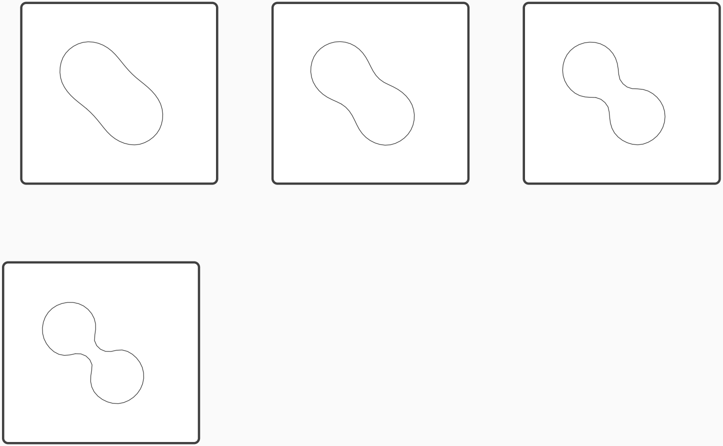
Topology changes during the local descent



Topology changes during the local descent



Topology changes during the local descent



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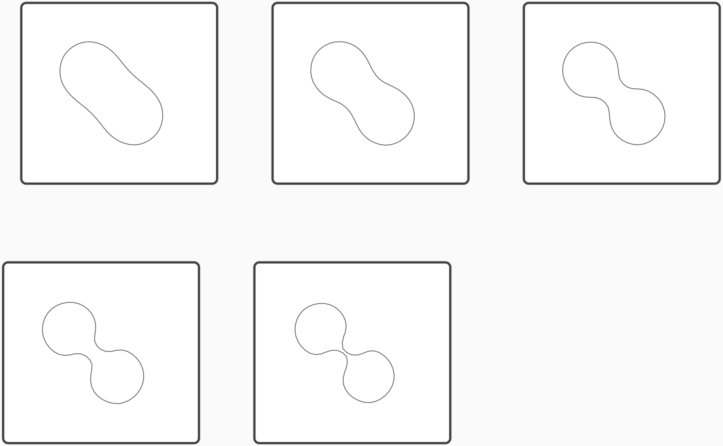

Topology changes during the local descent