CANUM 2020+2 Évian les Bains
Minisymposium *Lagrange Projection*

Un schéma volume fini en une étape, tout régime, équilibré et entropique pour la dynamique des gas avec gravité.

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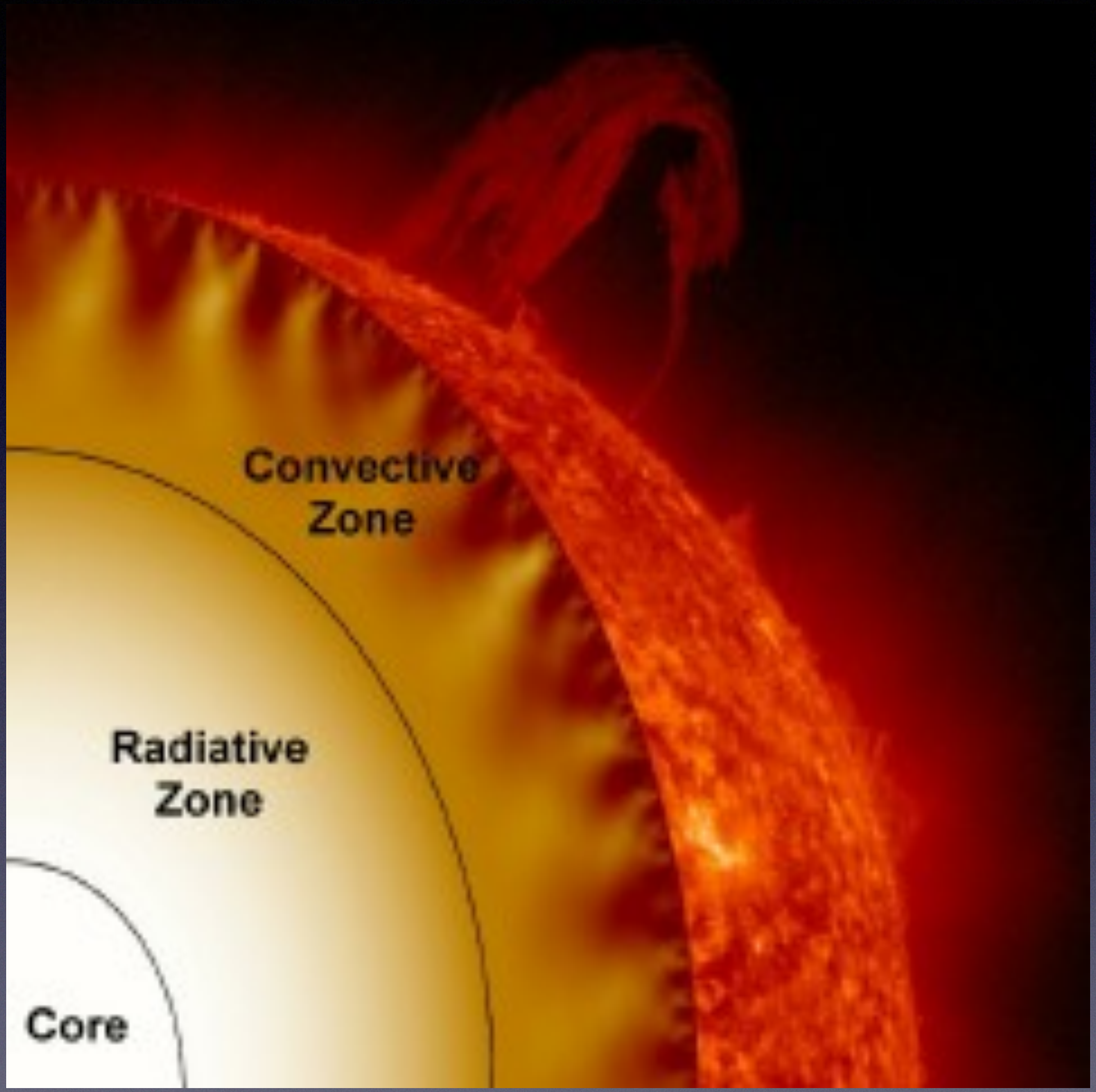
Summary

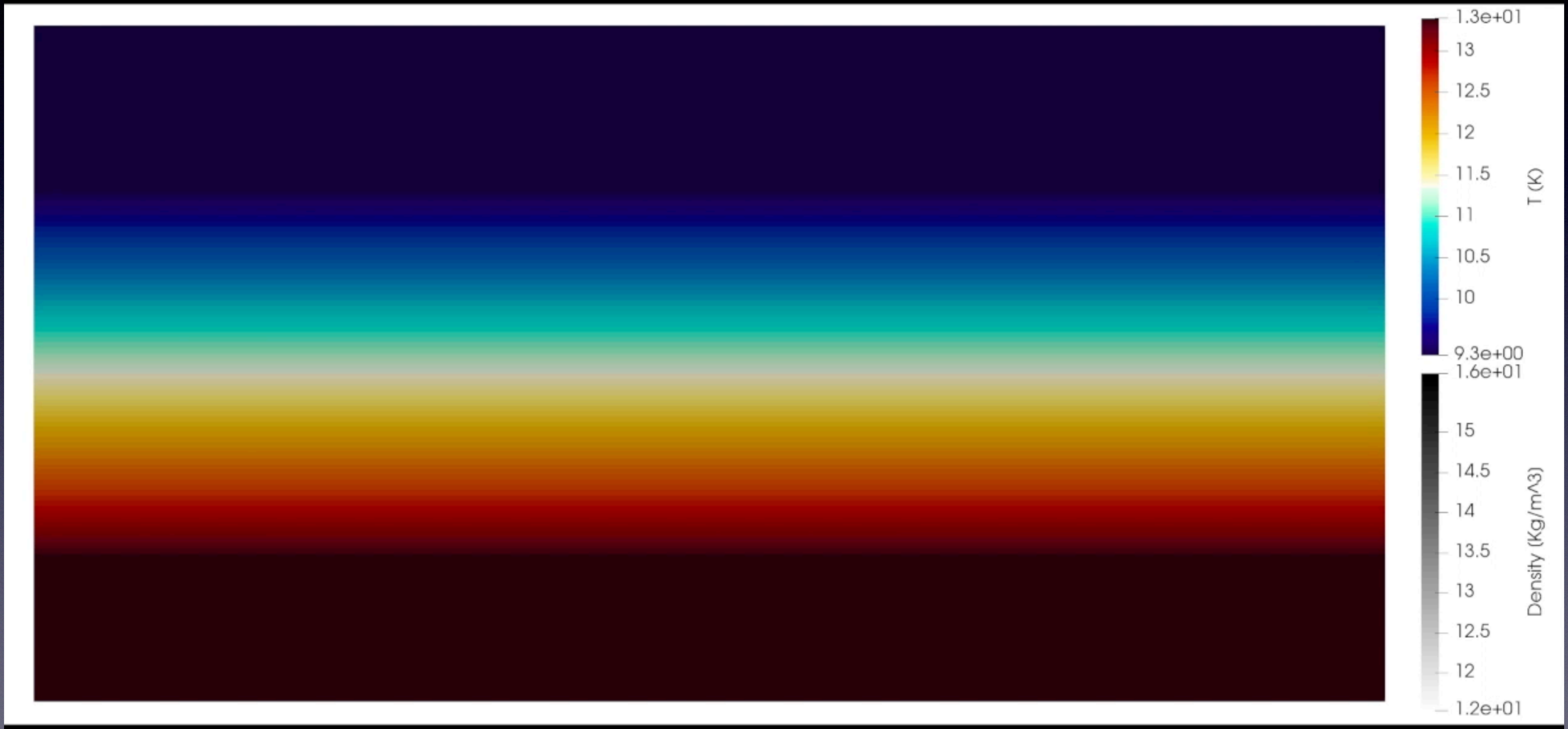
- Why do we need all-regime well-balanced numerical methods to study convection ?
- The new solver as a algorithmic simplification of an existing method*
- Numerical results
- Why does it works ? -Flux splitting approach-

*Chalons et al. 2016 An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes

Star’s structure: the convective zone



Thermally driven hydrodynamical convection



Physical problem: Condition for the convection instability in star's atmospheres

Under which conditions on the vertical profiles (temperature, pressure etc..) will the convective motion start after we apply a perturbation ?

Two approaches to compare and cross validate:

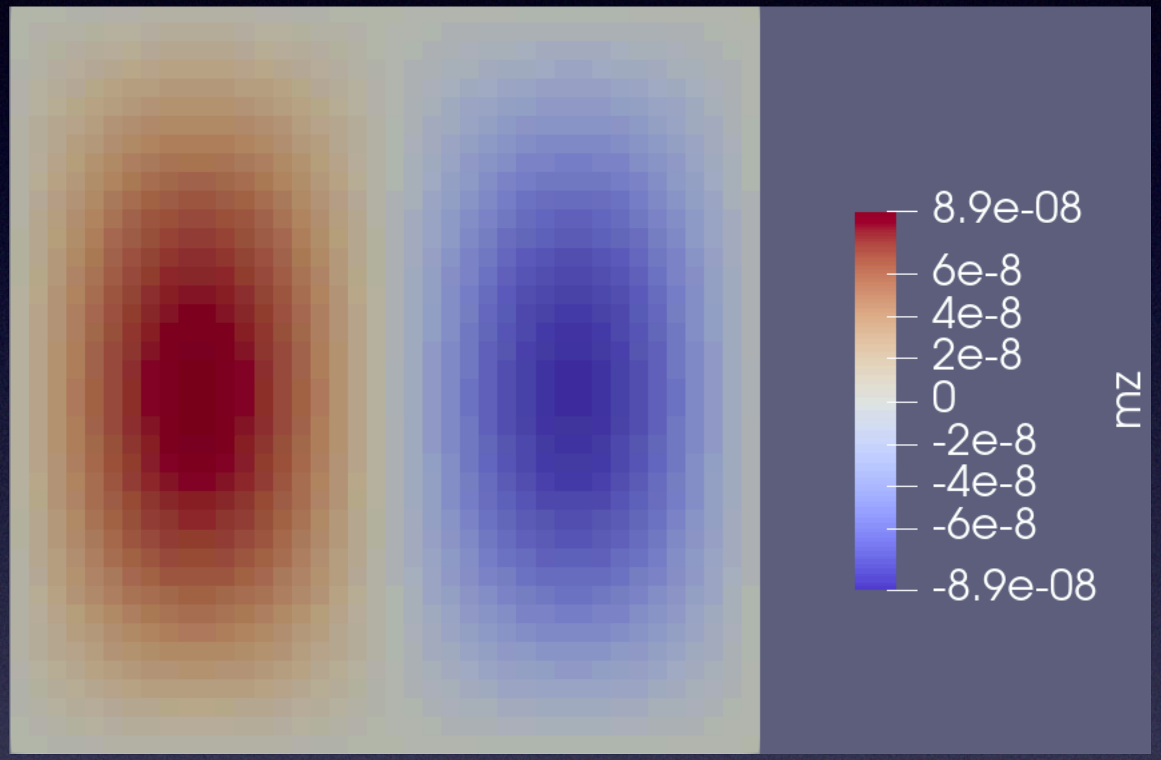
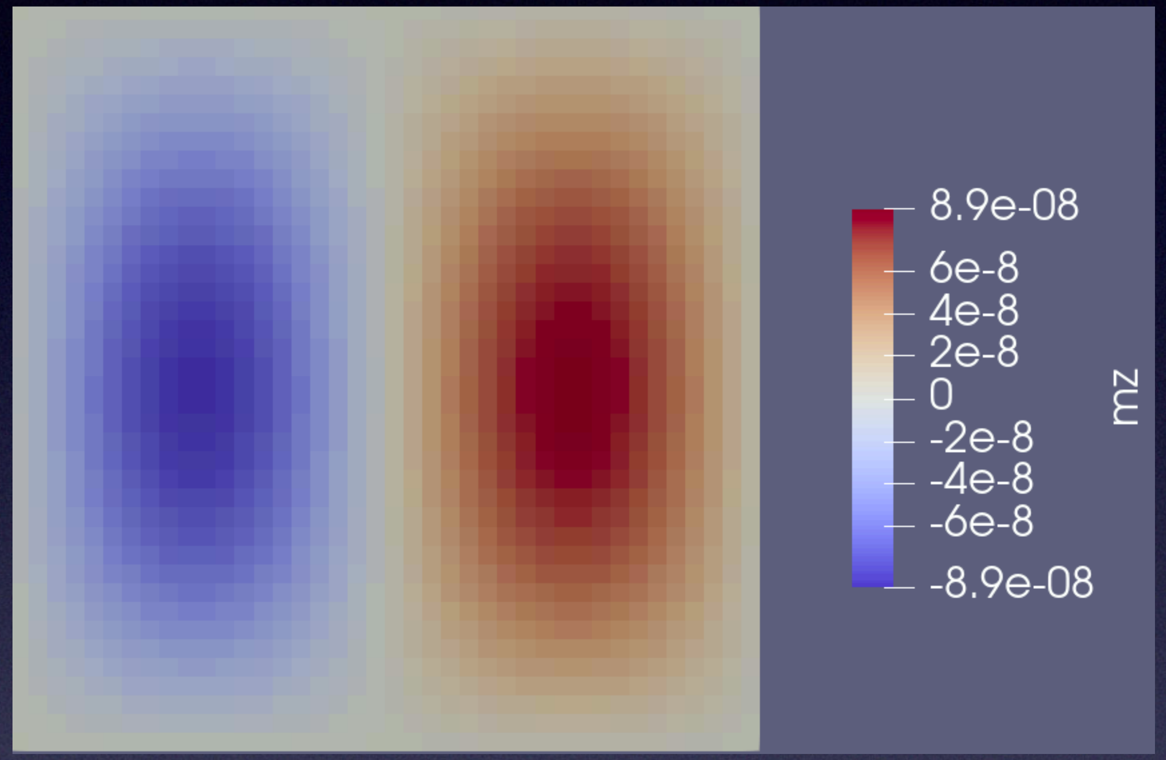
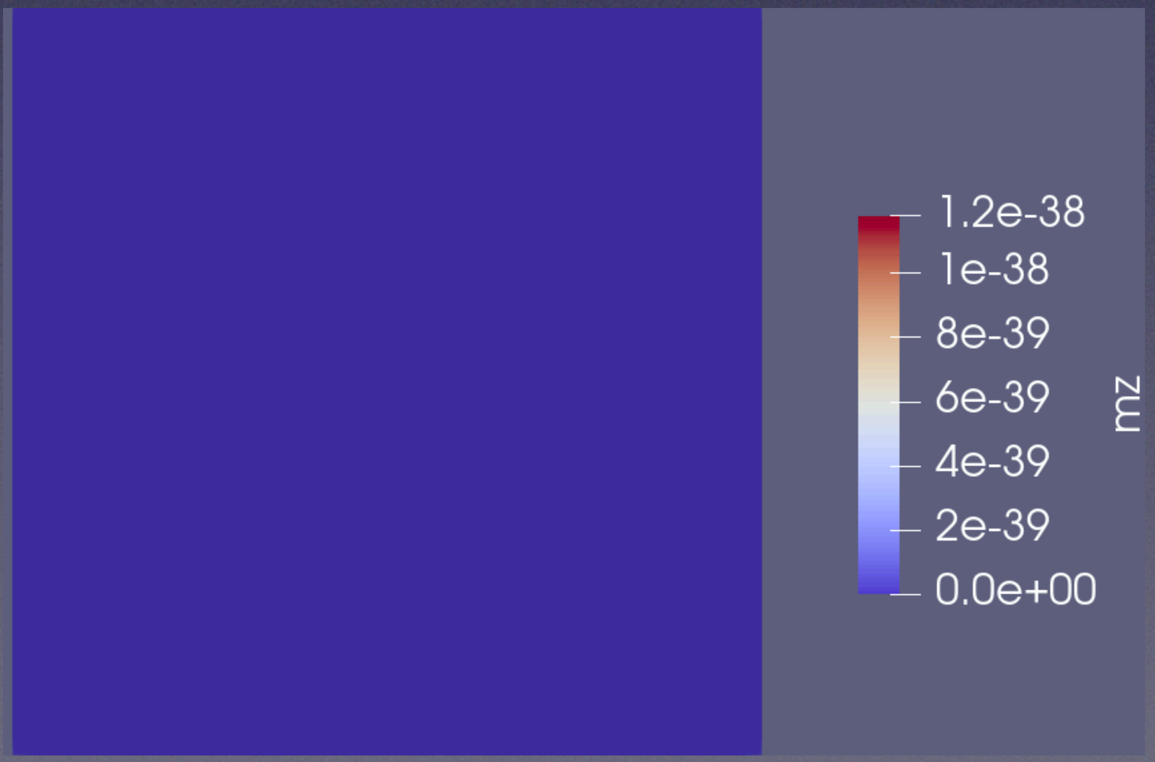
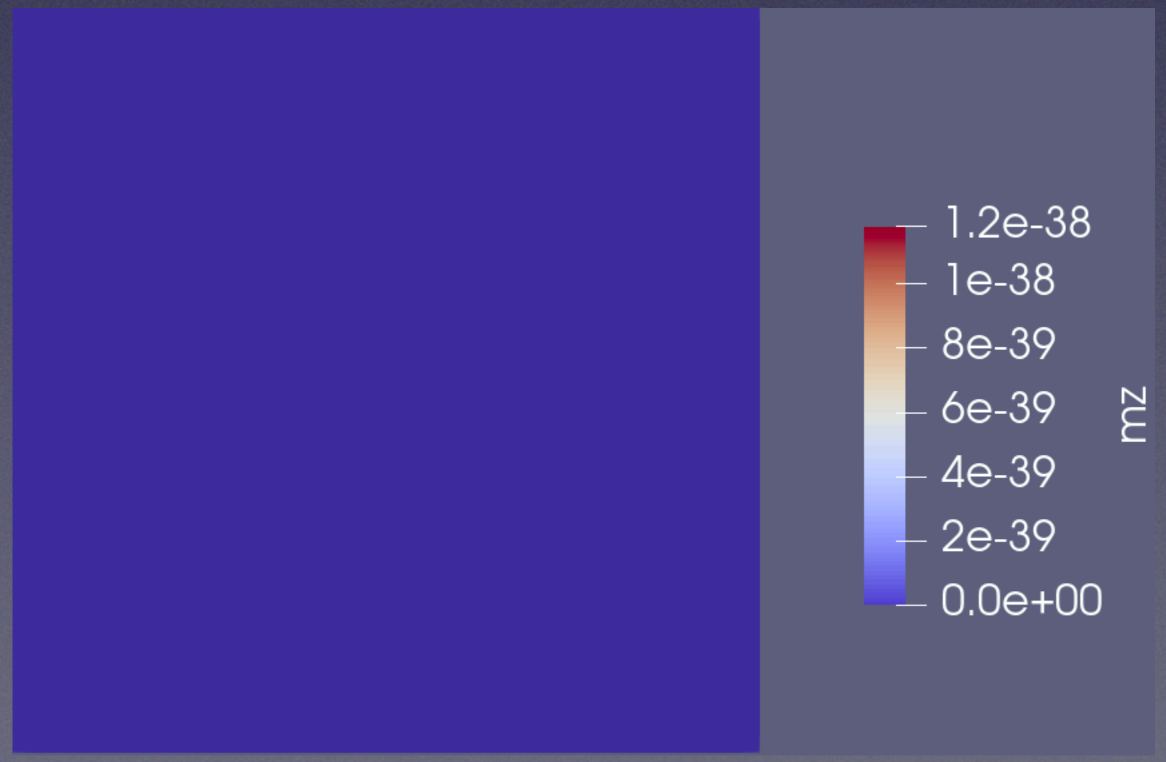
-Analytical: Using the linear stability analysis of Tremblin et al. 2019

-> Encapsulates most known convective instabilities and predicts new ones,
-> *Easy* to extend to additional physics (MHD).

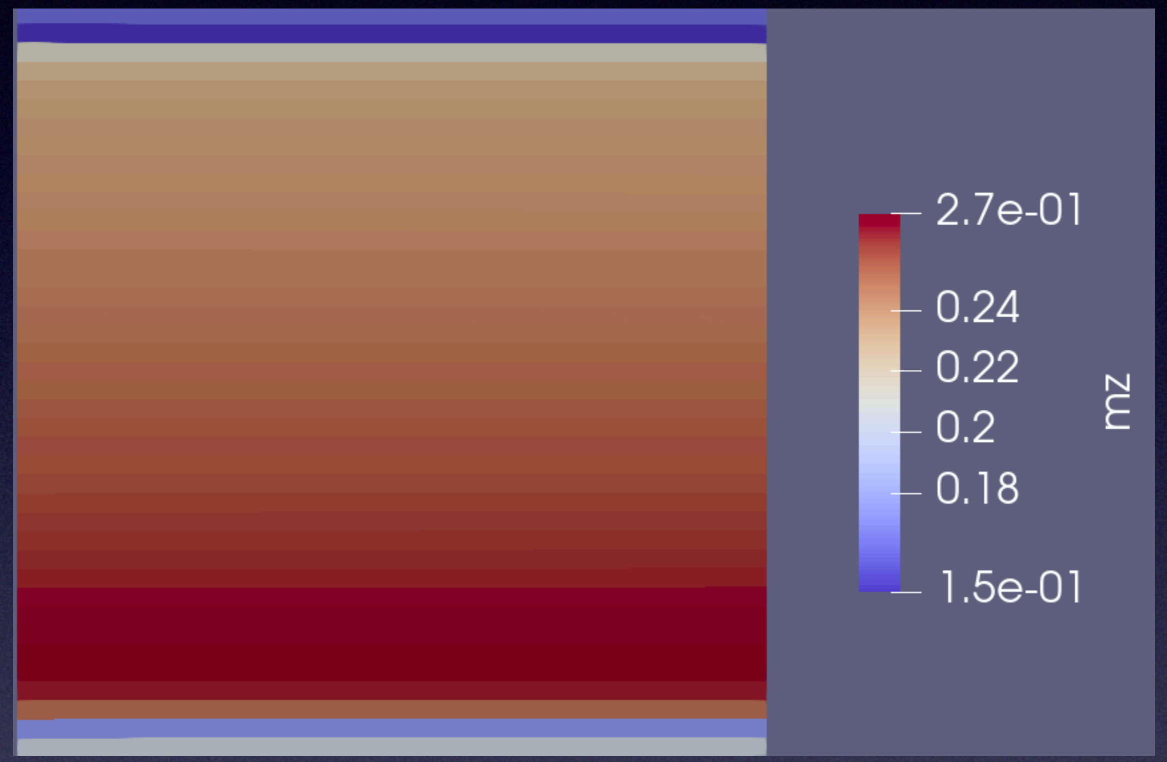
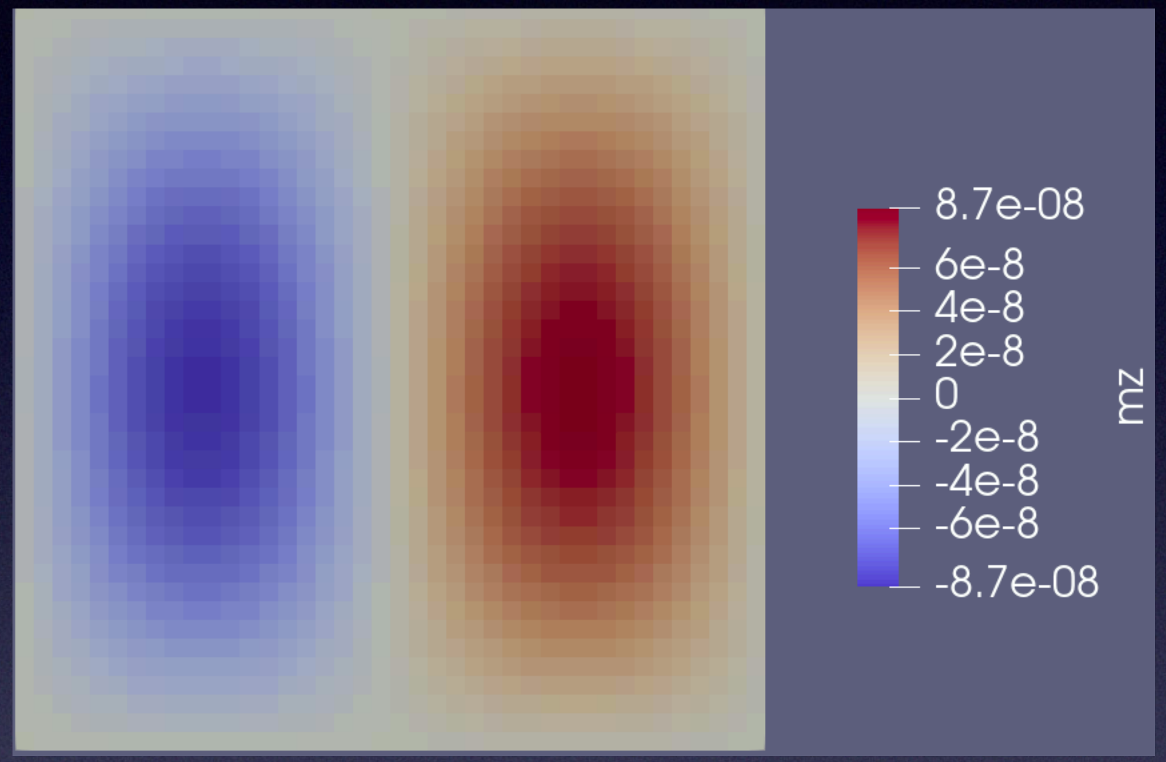
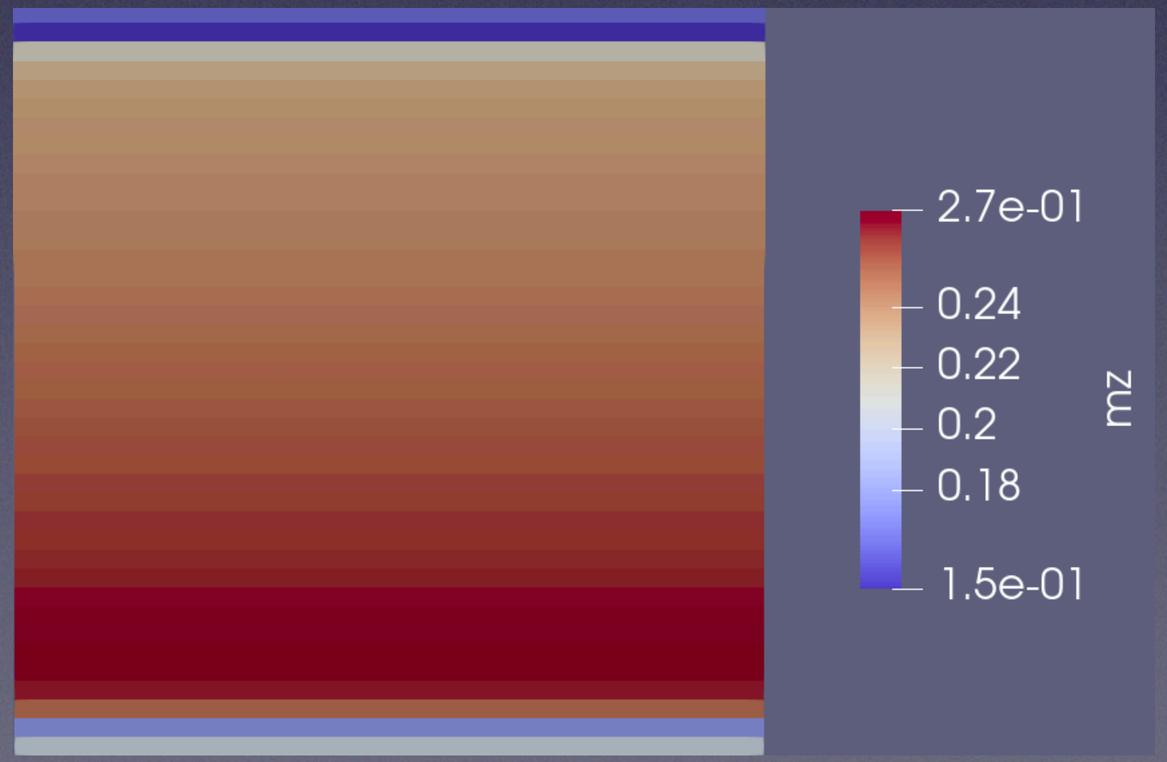
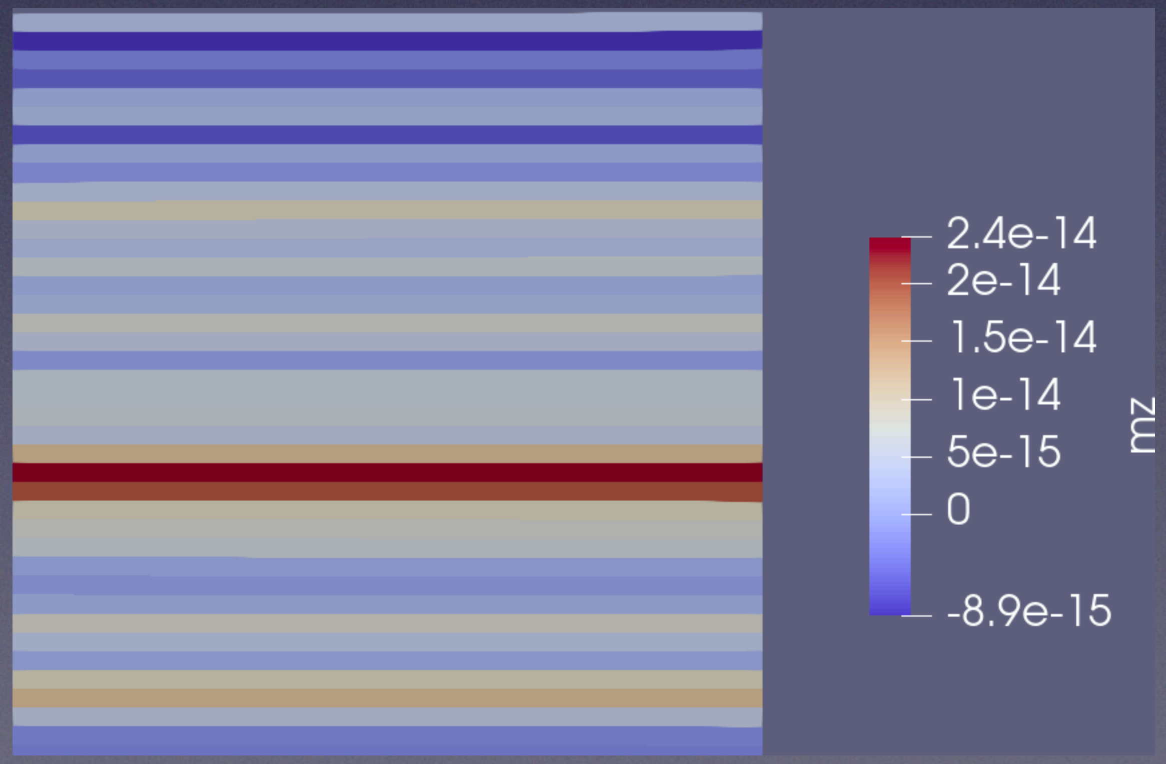
-Numerical: Via finite volume simulation using a stable well-balanced, all-regime method

-> Allows to conserve the conservative quantities up to machine precision.
-> Good behavior at low speed, at the beginning of the instability (all-regime),
-> Does not generate numerical perturbation around the initial equilibrium (well-balanced),

Physical problem: Condition for the convection instability in star's atmospheres. Importance of the well-balanced property

$t = 0$	Not Well balanced	Well balanced
perturbed		
not perturbed		

Physical problem: Condition for the convection instability in star's atmospheres. Importance of the well-balanced property

$t > 0$	Not Well balanced	Well balanced
perturbed		
not perturbed		

Summary

- Why do we need all-regime well-balanced numerical methods to study convection ?
- The new solver as a algorithmic simplification of an existing method*
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All Regime and/or Well balanced methods -some previous works-

-J. M. Greenberg et al. Analysis and approximation of conservation laws with source terms.

-L. Gosse. A well-balanced flux-vector splitting scheme designed for hyperbolic systems of conservation laws with source terms.

-F. Bouchut. Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources.

-P. Chandrashekar et al. A second order well-balanced finite volume scheme for Euler equations with gravity.

-C. Chalons et al. A large time-step and well-balanced Lagrange-projection type scheme for the shallow-water equations

-C. Chalons et al. An all-regime Lagrange-projection-like scheme for the gas dynamics equations on unstructured meshes.

-T. Padioleau et al. A high-performance and portable all-Mach regime flow solver code with well-balanced gravity,application to compressible convection.

Euler's equations for fluid dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x \left(u\rho u + P\right) = \rho g, \\ \partial_t(\rho E) + \partial_x(u\rho E + Pu) = \rho u g, \end{cases}$$

Desired properties of a numerical method:

-> **Stability**, namely positive preserving and entropy inequality:

$$\rho > 0, \; P > 0, \; \text{ and } \partial_t \rho s + \partial_x u \rho s \geq 0,$$

-> **All-regime**: The Discretization error has to be independent of the mach number Ma.
-> **Well-balanced**: Preserves the hydrostatic equilibrium $\partial_x P = -\rho \partial_x \phi, \quad u_0 = 0$ exactly

—> The splitted acoustic/transport solver of Chalons et al. 2016 satisfies all properties !

The splitted acoustic/transport solver of Chalons et al. 2016

Acoustic system $\mathbf{Ac}$

$$\begin{cases} \partial_t \rho + \rho \partial_x u = 0 \\ \partial_t(\rho u) + \rho u \partial_x u + \partial_x p = \rho g \\ \partial_t(\rho E) + \rho E \partial_x u + \partial_x(pu) = \rho u g \end{cases}$$

Transport system $\mathbf{T}$

$$\begin{cases} \partial_t \rho + u \partial_x \rho = 0 \\ \partial_t(\rho u) + u \partial_x(\rho u) = 0 \\ \partial_t(\rho E) + u \partial_x(\rho E) = 0 \end{cases}$$

The idea is to resolve both system successively: $U^{n+1} = \mathbf{T}(\mathbf{Ac}(U^n)) = \mathbf{T}(U^{Ac})$:

1) Starting from $(\rho, \rho u, \rho E)^n$, compute $(\rho, \rho u, \rho E)^{Ac}$ by approximating $\mathbf{Ac}$
2) Starting from $(\rho, \rho u, \rho E)^{Ac}$, compute $(\rho, \rho u, \rho E)^{n+1}$ by approximating $\mathbf{T}$

The splitted acoustic/transport solver of Chalons et al. 2016

reads:

$$
\begin{cases}
\rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}\rho^{Ac}\right]_{j\pm 1/2} \\
(\rho u)_j^{n+1} = (\rho u)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho u)^{Ac} + \Pi^{*,n}\right]_{j\pm 1/2} + \Delta t\{\rho g\}_j^n \\
(\rho E)_j^{n+1} = (\rho E)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho E)^{Ac} + \Pi u^{*,n}\right]_{j\pm 1/2} + \Delta t\{\rho u g\}_j^n
\end{cases}
$$

Our new method:

reads:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}\rho^{n}\right]_{j\pm1/2} \\ (\rho u)_j^{n+1} = (\rho u)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho u)^{n} + \Pi^{*,n}\right]_{j\pm1/2} + \Delta t\{\rho g\}_j^n \\ (\rho E)_j^{n+1} = (\rho E)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho E)^{n} + \Pi u^{*,n}\right]_{j\pm1/2} + \Delta t\{\rho u g\}_j^n \end{cases}$$

Our new method, advantages

reads:

$$\begin{cases} \rho_j^{n+1} = \rho_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}\rho^n\right]_{j\pm 1/2} \\ (\rho u)_j^{n+1} = (\rho u)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho u)^n + \Pi^{*,n}\right]_{j\pm 1/2} + \Delta t\{\rho g\}_j^n \\ (\rho E)_j^{n+1} = (\rho E)_j^n - \dfrac{\Delta t}{\Delta x}\left[u^{*,n}(\rho E)^n + \Pi u^{*,n}\right]_{j\pm 1/2} + \Delta t\{\rho u g\}_j^n \end{cases}$$

- Easy to implement (Easy to plug in a flux based finite volume code)
- Easy to combine with a high order algorithm (MUSCL, WENO, MOOD etc…)
- Memory print is dramatically reduced (no need for an intermediate array) -> HPC
- Same good properties as the splitted method
- Stencil reduction 2->1
- Half CFL condition (no miracle…)
- Entirely new kind of mathematical background: Flux splitting method* instead of operator splitting.

*Intensively studied by Steger & Warming, Zha & Bilgen, Liou & Steffen, Jameson, Bouchut, Toro

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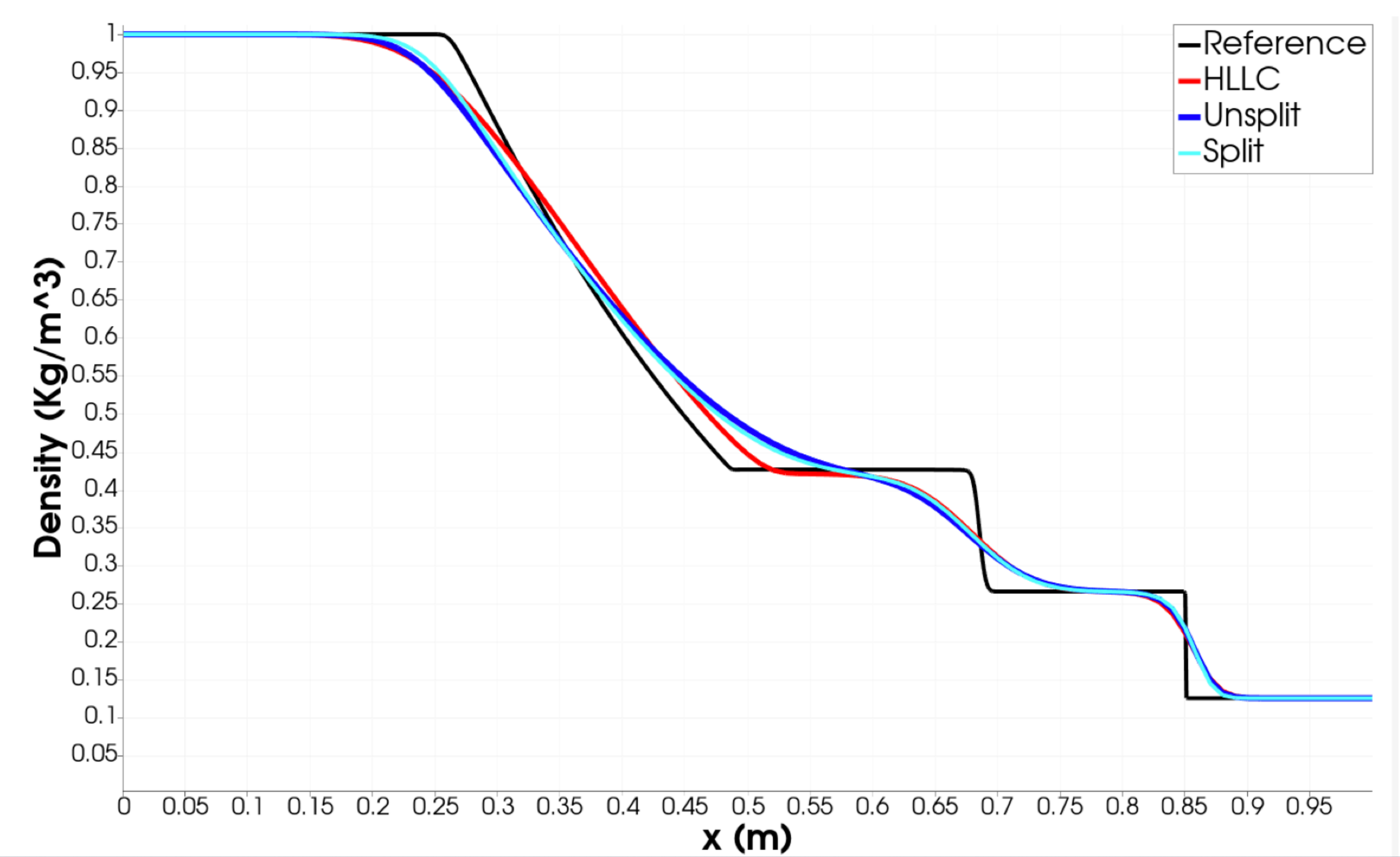
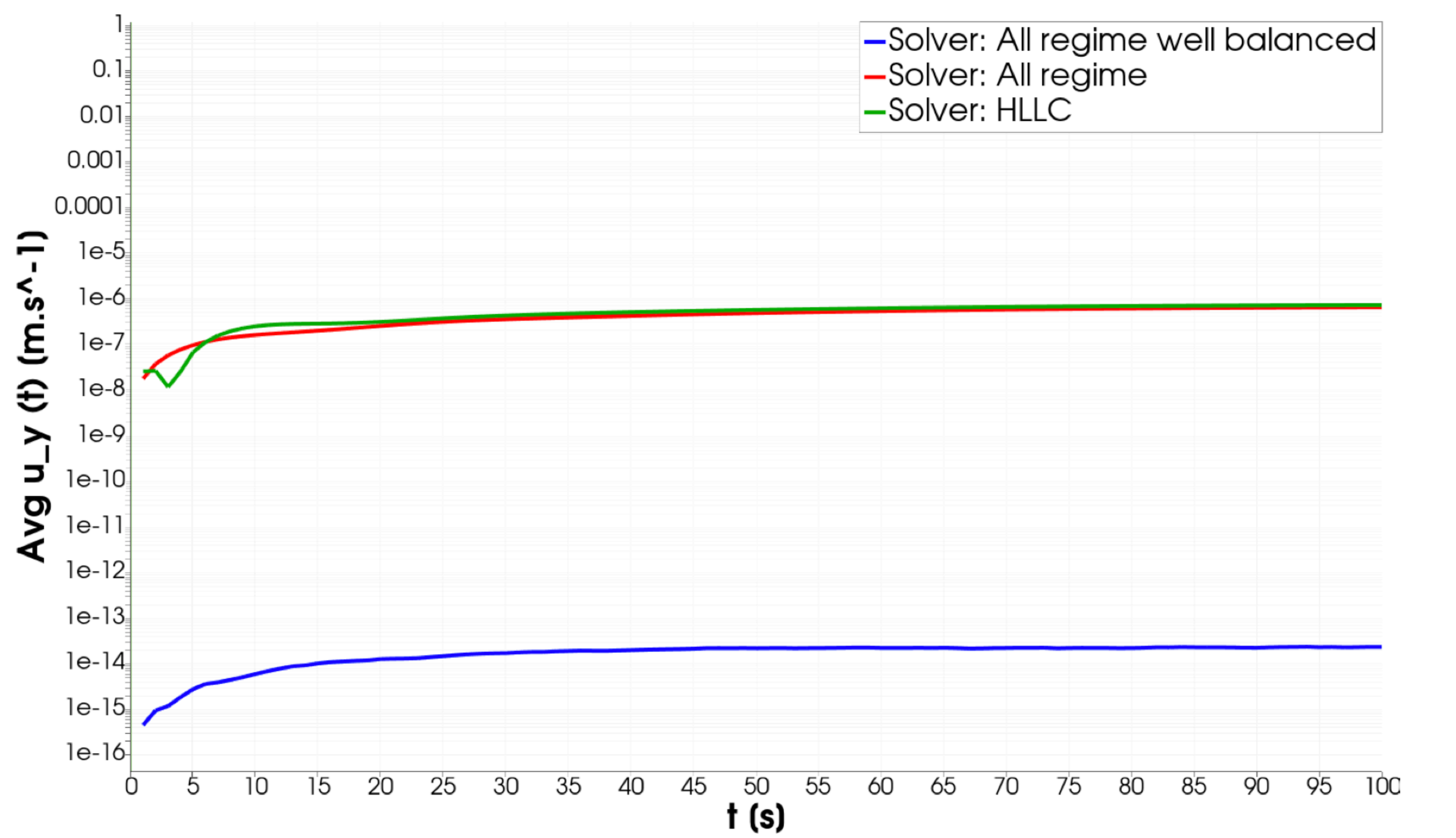
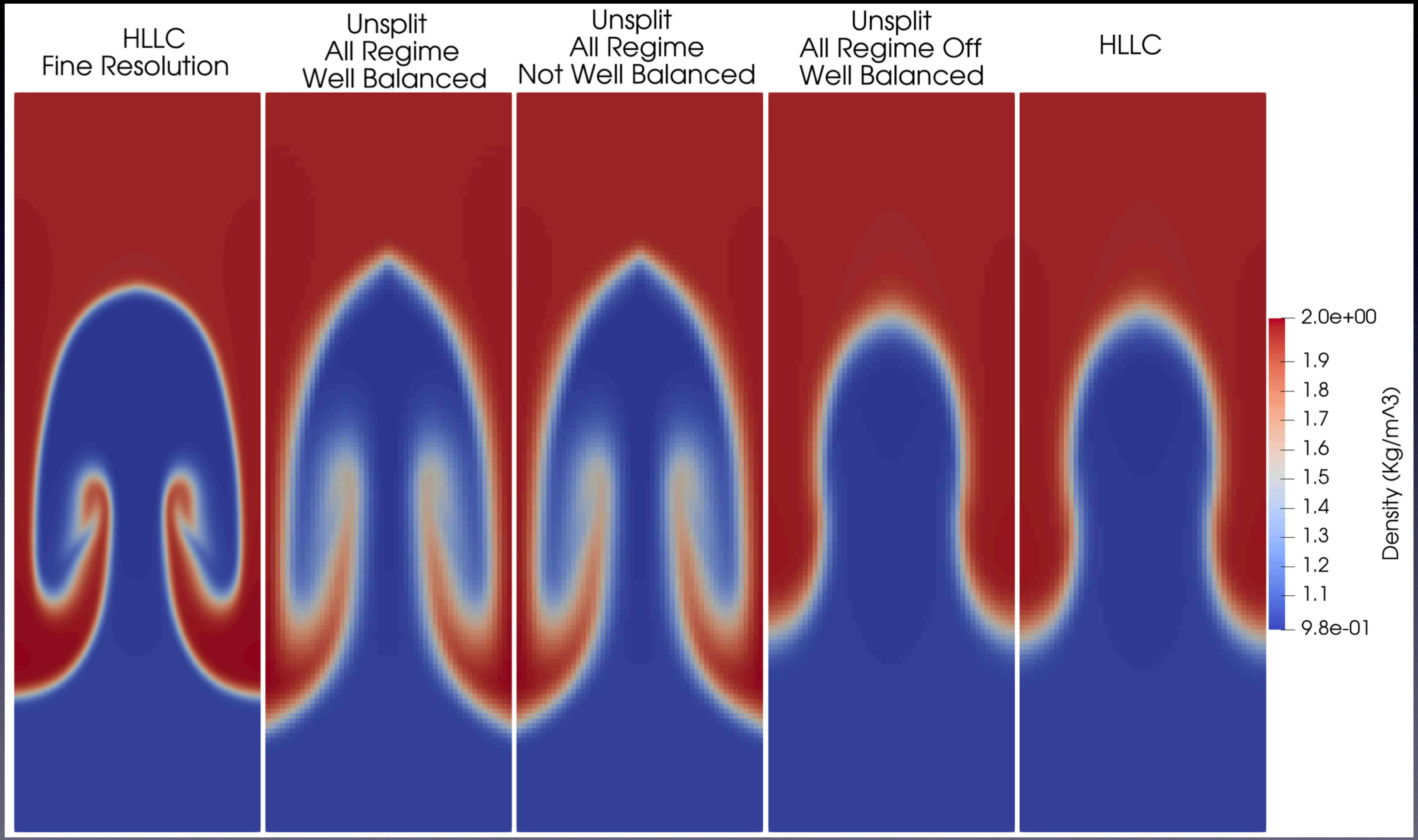
Numerical results, Sod's shock tube (stable)



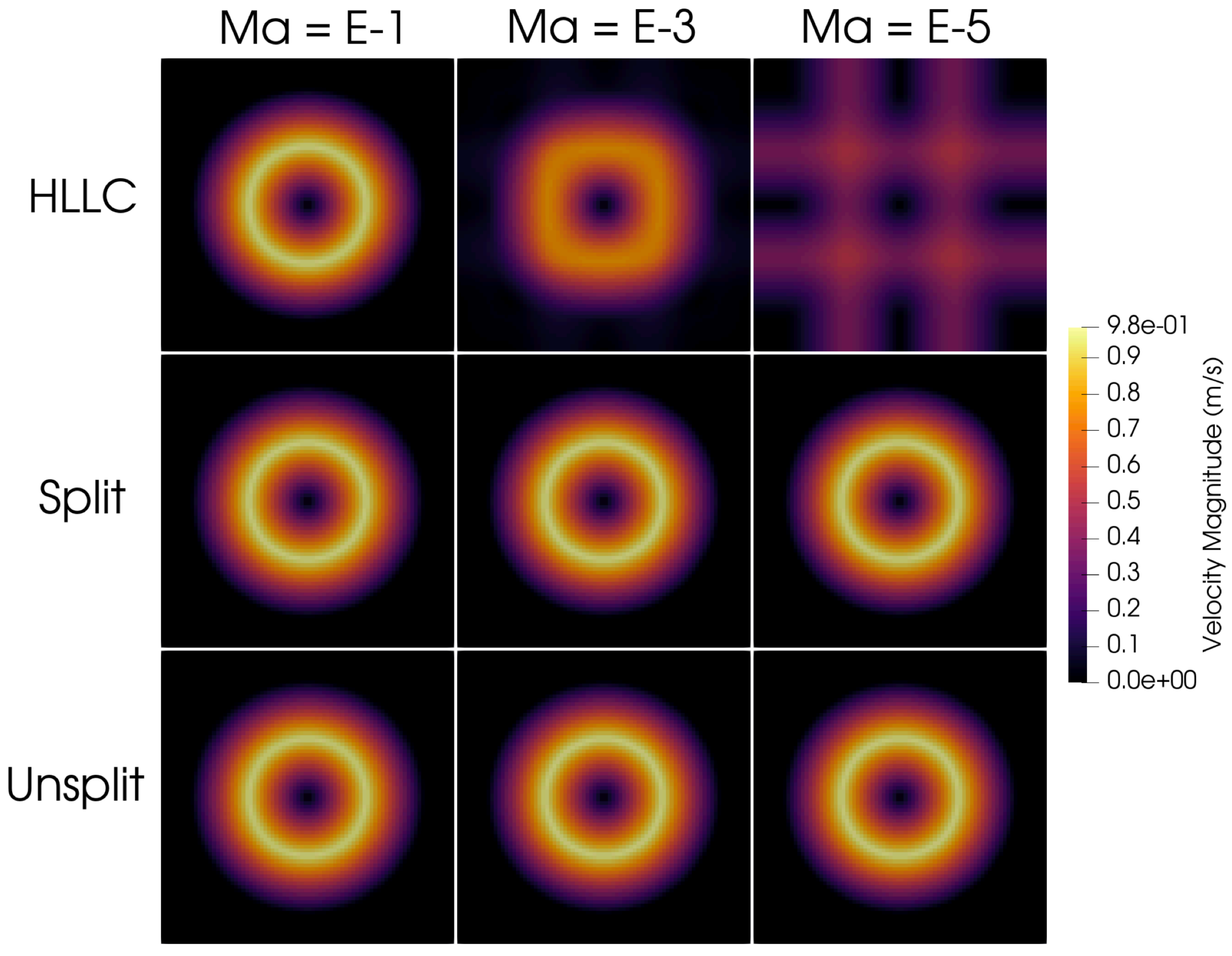
Numerical results, Atmosphere at rest (Well-Balanced)



Numerical results, Rayleigh-Taylor inst. (All-Regime)



Numerical results, Gresho Vortex (All-Regime)



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Why does it works? -Relaxation and Flux splitting-

$$
\begin{cases}
\partial_t \rho + \partial_x(\rho u) = 0 \\
\partial_t(\rho u) + \partial_x(u\rho u + \Pi) = -\rho \partial_x \phi \\
\partial_t(\rho E) + \partial_x(u\rho E + \Pi u) = -\rho u \partial_x \phi \\
\partial_t(\rho \Pi) + \partial_x\left(u\rho\Pi + a^2 u\right) = \rho\lambda(p - \Pi) \\
\partial_t(\rho \mathscr{T}) = 0
\end{cases}
$$

Advection system

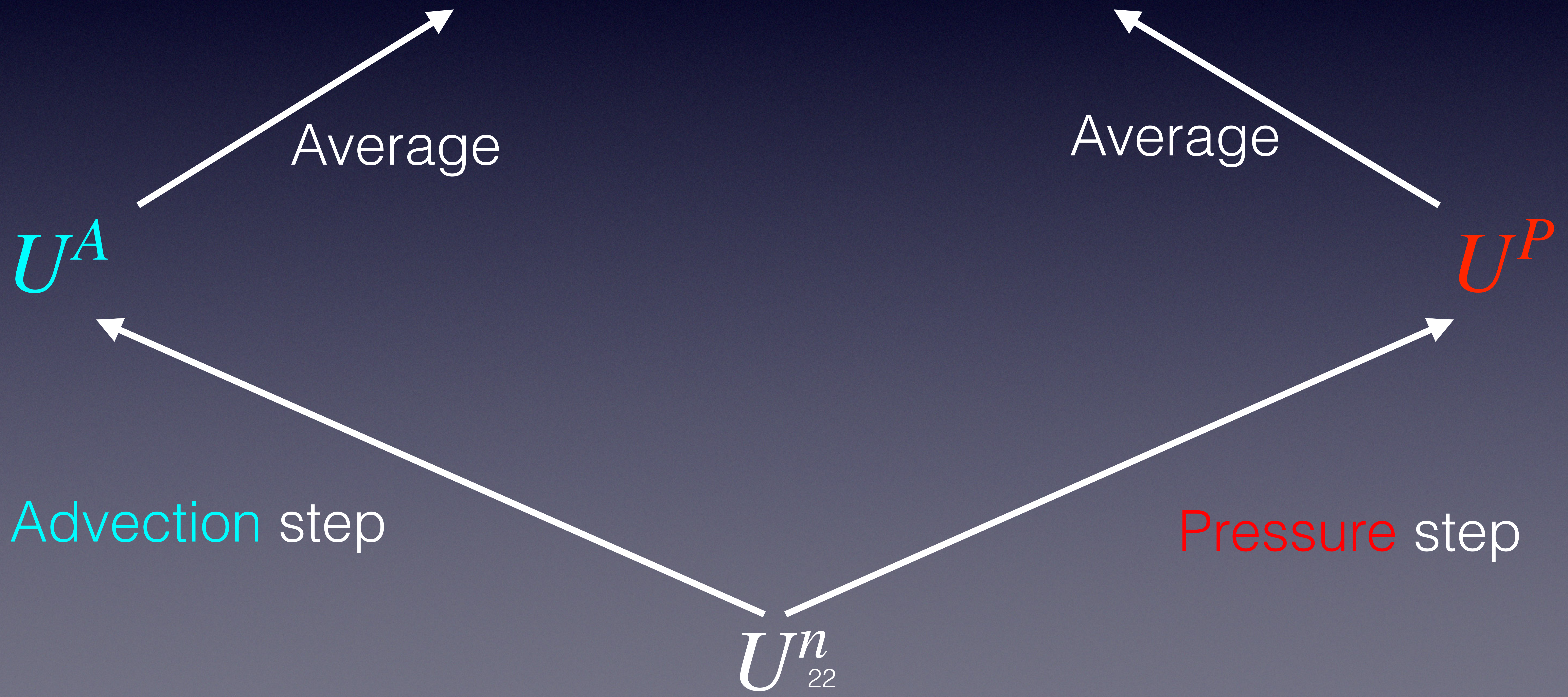
$$
\begin{cases}
\partial_t \rho + 2\partial_x(\rho u) = 0 \\
\partial_t(\rho u) + 2\partial_x(u\rho u) = 0 \\
\partial_t(\rho E) + 2\partial_x(u\rho E) = 0 \\
\partial_t(\rho \Pi) + 2\partial_x(u\rho\Pi) = 0 \\
\partial_t(\rho \mathscr{T}) + 2\partial_x(u\rho\mathscr{T}) = 0
\end{cases}
$$

Pressure system

$$
\begin{cases}
\partial_t \rho = 0 \\
\partial_t(\rho u) + 2\partial_x(\Pi) = -2\rho \partial_x \phi \\
\partial_t(\rho E) + 2\partial_x(\Pi u) = -2\rho u \partial_x \phi \\
\partial_t(\rho \Pi) + 2\partial_x\left(a^2 u\right) = 0 \\
\partial_t(\rho \mathscr{T}) - 2\partial_x u = 0,
\end{cases}
$$

Why does it works? -The averaging interpretation-

$$U^{n+1} = \frac{U^P + U^A}{2}$$



Why does it works ? -time consistency-

$$U_j^{n+1} = \frac{U_j^A + U_j^P}{2} = \begin{cases} \rho_j^n - \frac{\Delta t}{\Delta x}\left[u^{*,n}\rho^n\right]_{j\pm1/2} \\ (\rho u)_j^n - \frac{\Delta t}{\Delta x}\left[u^{*,n}(\rho u)^n + \Pi^{*,n}\right]_{j\pm1/2} + \Delta t\{\rho g\}_j^n \\ (\rho E)_j^n - \frac{\Delta t}{\Delta x}\left[u^{*,n}(\rho E)^n + \Pi u^{*,n}\right]_{j\pm1/2} + \Delta t\{\rho u g\}_j^n \end{cases}$$

The update is consistent with the Euler system because of the factor 2 in fluxes of the sub-systems.
—>The associated waves are twice as fast.
—>In practice, it means that we have to use a CFL number < 0.5

Why does it works ? -stability of the Advection flux udpate-

$$\forall b \in \{u, E, \Pi, \mathscr{T}\} \quad (\rho b)_i^A = (\rho b)_i^n - \frac{\Delta t}{\Delta x}\left(u^*_{i+1/2}(\rho b^n)_{i+1/2} - u^*_{i-1/2}(\rho b)^n_{i-1/2}\right),$$

$$\leftrightarrow b_j^A = a_1 \; b^n_{j+1} + a_2 \; b^n_j + a_3 \; b^n_{j-1}$$

With $a_1 + a_2 + a_3 = 1$ and all $a_j > 0$

Convex combination under CFL condition $\frac{\Delta t}{\Delta x}\left(u^{*,+}_{j+1/2} - u^{*,-}_{j-1/2}\right) < \frac{1}{2}$!

Why does it works ? -stability of the Advection flux update-

$\forall b \in \{u, E, \Pi, \mathcal{T}\} \quad b_j^A = a_1\ b_{j+1}^n + a_2\ b_j^n + a_3\ b_{j-1}^n$ convex combination

Since $(\mathcal{T}, E, u) \to s(\mathcal{T}, e(E, u))$ is concave:

$$(\rho s)_j^A - (\rho s)_j^n + 2\frac{\Delta t}{\Delta x}\left(u^*_{j+1/2}(\rho s)^n_{j+1/2} - u^*_{j-1/2}(\rho s)^n_{j-1/2}\right) \geq 0\,*$$

Entropy inequality for the advection fluxes!

*Subtlety : $(\rho s)_j^A = \rho_j^A s\left(\mathcal{T}_j^A, e_j^A\right)$

Why does it works ? -stability of the Pressure flux update-

Exact same steps as in Chalons et al. 2016 provide:

$$(\rho s)_j^P - (\rho s)_j^n + 2\frac{\Delta t}{\Delta x}\left(q_{j+1/2}^n - q_{j-1/2}^n\right) \geq 0 \; *$$

Where the flux function is consistent with 0, under the pressure CFL condition.

-Result comes from the resolution of the LD Riemann problem for the pressure system
-The entropy inequality can be violated for low values of the Low-Mach correction -> Checkerboard modes, not AP

*Subtlety : $(\rho s)_j^P = \rho_j^A s\left(\mathcal{T}_j^P, e_j^A\right)$

Why does it works ? -stability of the full scheme-

$$(\rho s)_j^A - (\rho s)_j^n + 2\frac{\Delta t}{\Delta x}\left(u^*_{j+1/2}(\rho s)^n_{j+1/2} - u^*_{j-1/2}(\rho s)^n_{j-1/2}\right) \geq 0$$

$$(\rho s)_j^P - (\rho s)_j^n + 2\frac{\Delta t}{\Delta x}\left(q^n_{j+1/2} - q^n_{j-1/2}\right) \geq 0$$

$$\rightarrow \frac{(\rho s)_j^P + (\rho s)_j^A}{2} - (\rho s)_j^n + \frac{\Delta t}{\Delta x}\left(u^*_{j+1/2}(\rho s)^n_{j+1/2} + q^n_{j+1/2} - u^*_{j-1/2}(\rho s)^n_{j-1/2} - q^n_{j-1/2}\right) \geq 0$$

We have $(\rho s)_j^{n+1} > \dfrac{(\rho s)_j^P + (\rho s)_j^A}{2}$

By concavity of $\eta : (\rho, \rho\mathcal{T}, \rho u, \rho E) \mapsto \rho s\left(\dfrac{\rho\mathcal{T}}{\rho}, \dfrac{(\rho E)}{\rho} - \dfrac{(\rho u)^2}{2\rho^2}\right)$

Why does it works ? -stability of the full scheme-

Full entropy inequality:

$$(\rho s)_j^{n+1} - (\rho s)_j^n + \frac{\Delta t}{\Delta x}\left(u^*_{j+1/2}(\rho s)^n_{j+1/2} + q^n_{j+1/2} - u^*_{j-1/2}(\rho s)^n_{j-1/2} - q^n_{j-1/2}\right) \geq 0$$

Is indeed a discrete equivalent of:

$$\partial_t \rho s + \partial_x u \rho s \geq 0.$$

Conclusion-perspectives

- The procedure was also applied successfully to splitted Lagrange-projection methods for other systems; -Ideal MHD, M1 model for radiative transfer. Should also work for shallow-water equations and the 5 equations two phases flow model.

- The flux-based update we obtained was successfully plugged in existing MUSCL and MOOD based codes,

- Implicit/explicit approach for the low Mach CFL issue should be straightforward,

- One important issue remains: The checkerboard modes.

- Main takeaway: If you use a splitted method, give the un-splitting a shot !

Thanks for your attention

Why does it works ? -stability of the advection step-

$$\forall b \in \{u, E, \Pi, \mathcal{T}\} \quad (\rho b)_i^A = (\rho b)_i^n - \frac{\Delta t}{\Delta x}\left(u^*_{i+1/2}(\rho b^n)_{i+1/2} - u^*_{i-1/2}(\rho b)^n_{i-1/2}\right),$$

$$\leftrightarrow \left(\frac{\rho b}{\rho}\right)_j^A = \lambda_j^{(+1)}\left(\frac{\rho b}{\rho}\right)_{j+1}^n + \lambda_j^{(0)}\left(\frac{\rho b}{\rho}\right)_j^n + \lambda_j^{(-1)}\left(\frac{\rho b}{\rho}\right)_{j-1}^n .$$

Where $\lambda_j^{(+1)} = -\frac{\Delta t}{\Delta x}u^{*,-}_{j+1/2}\left(\frac{\rho^n_{j+1}}{\rho^A_j}\right)$, $\lambda_j^{(0)} = \left[1 - \frac{\Delta t}{\Delta x}\left(u^{*,+}_{j+1/2} - u^{*,-}_{j-1/2}\right)\right]\left(\frac{\rho^n_j}{\rho^A_j}\right)$, $\lambda_j^{(-1)} = \frac{\Delta t}{\Delta x}u^{*,+}_{j-1/2}\left(\frac{\rho^n_{j-1}}{\rho^A_j}\right)$.

Since $\rho_j^A = -\frac{\Delta t}{\Delta x}u^{*,-}_{j+1/2}\rho^n_{j+1} + \frac{\Delta t}{\Delta x}u^{*,+}_{j-1/2}\rho^n_{j-1} + \left[1 - \frac{\Delta t}{\Delta x}\left(u^{*,+}_{j+1/2} - u^{*,-}_{j-1/2}\right)\right]\rho^n_j$, we have

$\lambda_j^{(+1)} + \lambda_j^{(0)} + \lambda_j^{(-1)} = 1$: convex combination under CFL condition $\frac{\Delta t}{\Delta x}\left(u^{*,+}_{j+1/2} - u^{*,-}_{j-1/2}\right) < 1$!

Why does it works ? -stability of the advection step- simplifier

$$\forall b \in \{u, E, \Pi, \tau\}, \left(\frac{\rho b}{\rho}\right)_j^A = \lambda_j^{(+1)} \left(\frac{\rho b}{\rho}\right)_{j+1}^n + \lambda_j^{(0)} \left(\frac{\rho b}{\rho}\right)_j^n + \lambda_j^{(-1)} \left(\frac{\rho b}{\rho}\right)_{j-1}^n, \ \lambda_j^{(+1)} + \lambda_j^{(0)} + \lambda_j^{(-1)} = 1$$

Noting $e(u, E) = E - \frac{1}{2}u^2$, we have that: $e_j^A > e_{j+1}^n \lambda_j^{(+1)} + e_j^n \lambda_j^{(0)} + e_{j-1}^n \lambda_j^{(-1)}$ by concavity

Since $(\mathcal{T}, E, u) \to s(\mathcal{T}, e(E, u))$ is concave too:

$$\rho_j^A s\left(\mathcal{T}_j^A, e_j^A\right) - \rho_j^n s_j^n + \frac{\Delta t}{\Delta x}\left(u^*_{j+1/2}\rho^n_{j+1/2}s^n_{j+1/2} - u^*_{j-1/2}\rho^n_{j-1/2}s^n_{j-1/2}\right) \geq 0$$

Why does it works ? -stability of the Pressure step-

Exact same steps as in Chalons et al. 2016 provide:

$$\rho_j^P s\left(\mathcal{T}_j^P, e_j^P\right) - \rho_j^n s\left(1/\rho_j^n, e_j^n\right) + \frac{\Delta t}{\Delta x}\left(q_{j+1/2}^n - q_{j-1/2}^n\right) \geq 0$$

Where the flux function is consistent with 0, under the pressure CFL condition.

-Result comes from the resolution of the LD Riemann problem for the pressure system
-The entropy inequality can be violated for low values of the Low-Mach correction -> Checkerboard modes, not AP

Why does it works ? -stability of the full scheme- simplifier

$$\rho_j^A s\left(\mathcal{T}_j^A, e_j^A\right) - \rho_j^n s_j^n + \frac{\Delta t}{\Delta x_j}\left(u^*_{j+1/2}\rho^n_{j+1/2}s^n_{j+1/2} - u^*_{j-1/2}\rho^n_{j-1/2}s^n_{j-1/2}\right) \geq 0$$

$$\rho_j^P s\left(\mathcal{T}_j^P, e_j^P\right) - \rho_j^n s\left(1/\rho_j^n, e_j^n\right) + \frac{\Delta t}{\Delta x}\left(q^n_{j+1/2} - q^n_{j-1/2}\right) \geq 0$$

$$\rightarrow \frac{\rho_j^P s\left(\mathcal{T}_j^P, e_j^P\right) + \rho_j^A s\left(\mathcal{T}^A, e_j^A\right)}{2} - \rho_j^n s\left(1/\rho_j^n, e_j^n\right) + \frac{\Delta t}{2\Delta x}\left(u^*_{j+1/2}\rho^n_{j+1/2}s^n_{j+1/2} + q^n_{j+1/2} - u^*_{j-1/2}\rho^n_{j-1/2}s^n_{j-1/2} - q^n_{j-1/2}\right) \geq 0$$

We need $\rho_j^{n+1} s(1/\rho_j^{n+1}, e_j^{n+1}) > \dfrac{\rho_j^P s\left(\mathcal{T}_j^P, e_j^P\right) + \rho_j^A s\left(\mathcal{T}_j^A, e_j^A\right)}{2}$

Why does it works ? -stability of the full scheme- simplifier

We need $\rho_j^{n+1} s(1/\rho_j^{n+1}, e_j^{n+1}) > \dfrac{\rho_j^P s\left(\mathcal{T}_j^P, e_j^P\right) + \rho_j^A s\left(\mathcal{T}_j^A, e_j^A\right)}{2}$

We define $\eta : (\rho, \rho\mathcal{T}, \rho u, \rho E) \mapsto \rho s\left(\dfrac{\rho\mathcal{T}}{\rho}, \dfrac{(\rho E)}{\rho} - \dfrac{(\rho u)^2}{2\rho^2}\right)$

$$\rho_j^{n+1} s\left(1/\rho_j^{n+1}, e_j^{n+1}\right) = \rho_j^{n+1} s\left(\frac{1}{\rho_j^{n+1}}, \frac{(\rho E)_j^{n+1}}{\rho_j^{n+1}} - \frac{1}{2}\left(\frac{(\rho u)_j^{n+1}}{\rho_j^{n+1}}\right)^2\right) = \eta\left(\rho_j^{n+1}, 1, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}\right)$$

$$= \eta\left(\rho_j^{n+1}, (\rho\mathcal{T})_j^{n+1}, (\rho u)_j^{n+1}, (\rho E)_j^{n+1}\right) = \eta\left(\sum_{k=A,P} \frac{\rho_j^k}{2}, \sum_{k=A,P} \frac{(\rho\mathcal{T})_j^k}{2}, \sum_{k=A,P} \frac{(\rho u)_j^k}{2}, \sum_{k=A,P} \frac{(\rho E)_j^k}{2}\right).$$

Lemma: η is concave, thus:

$$\rho_j^{n+1} s\left(1/\rho_j^{n+1}, e_j^{n+1}\right) \geq \sum_{k=A,P} \frac{1}{2}\eta\left(\rho_j^k, (\rho\mathcal{T})_j^k, (\rho u)_j^k, (\rho E)_j^k\right) = \sum_{k=A,P} \frac{1}{2}\rho_j^k s\left(\mathcal{T}_j^k, e_j^k\right)$$