A new brittle-elastoviscoplastic fluid based on the Drucker-Prager plasticity

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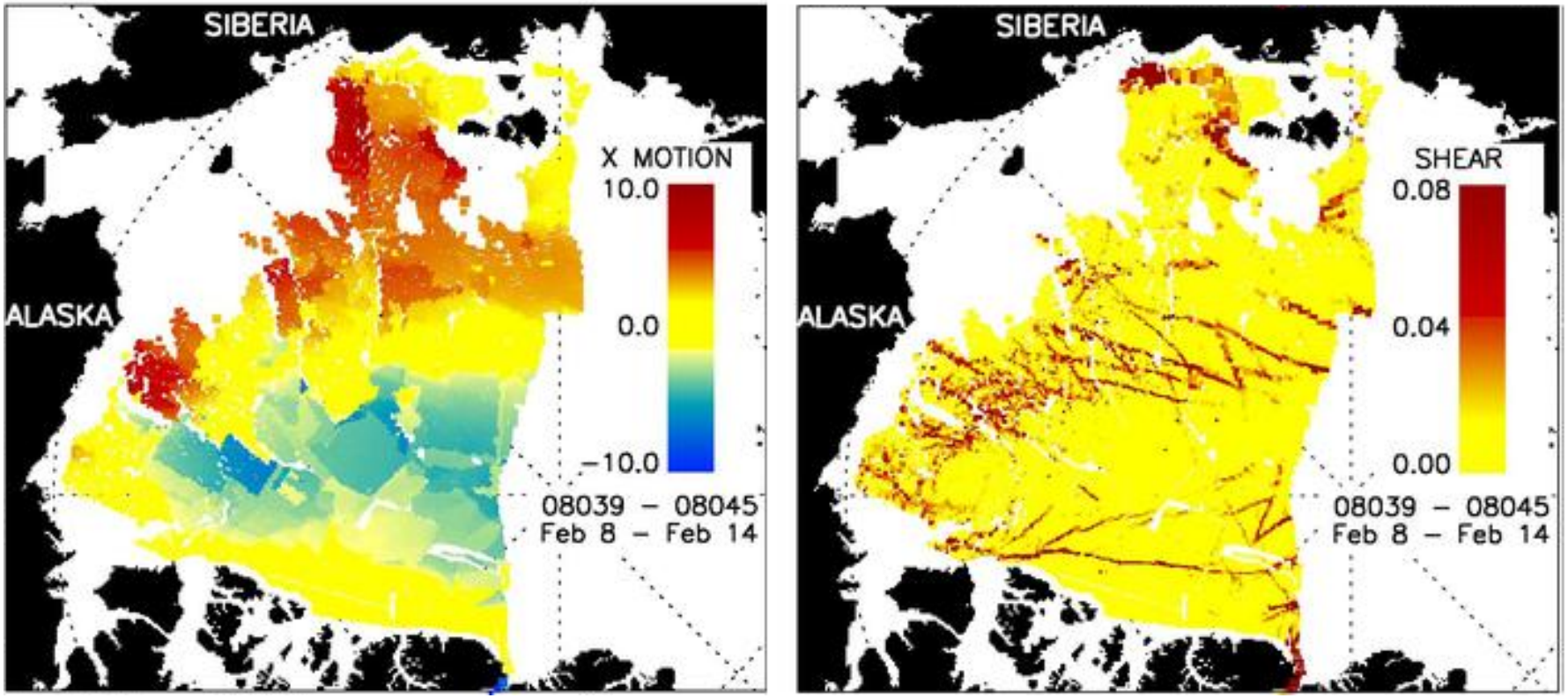
Motivation: sea ice

[Polar region atlas, CIA, 1978]

- sea ice drift:
  due to ocean currents
  and wind
- accelerated:
  due to global heating

Motivation: sea ice

velocity $|\boldsymbol{u}|$ deformation rate $|D(\boldsymbol{u})|$

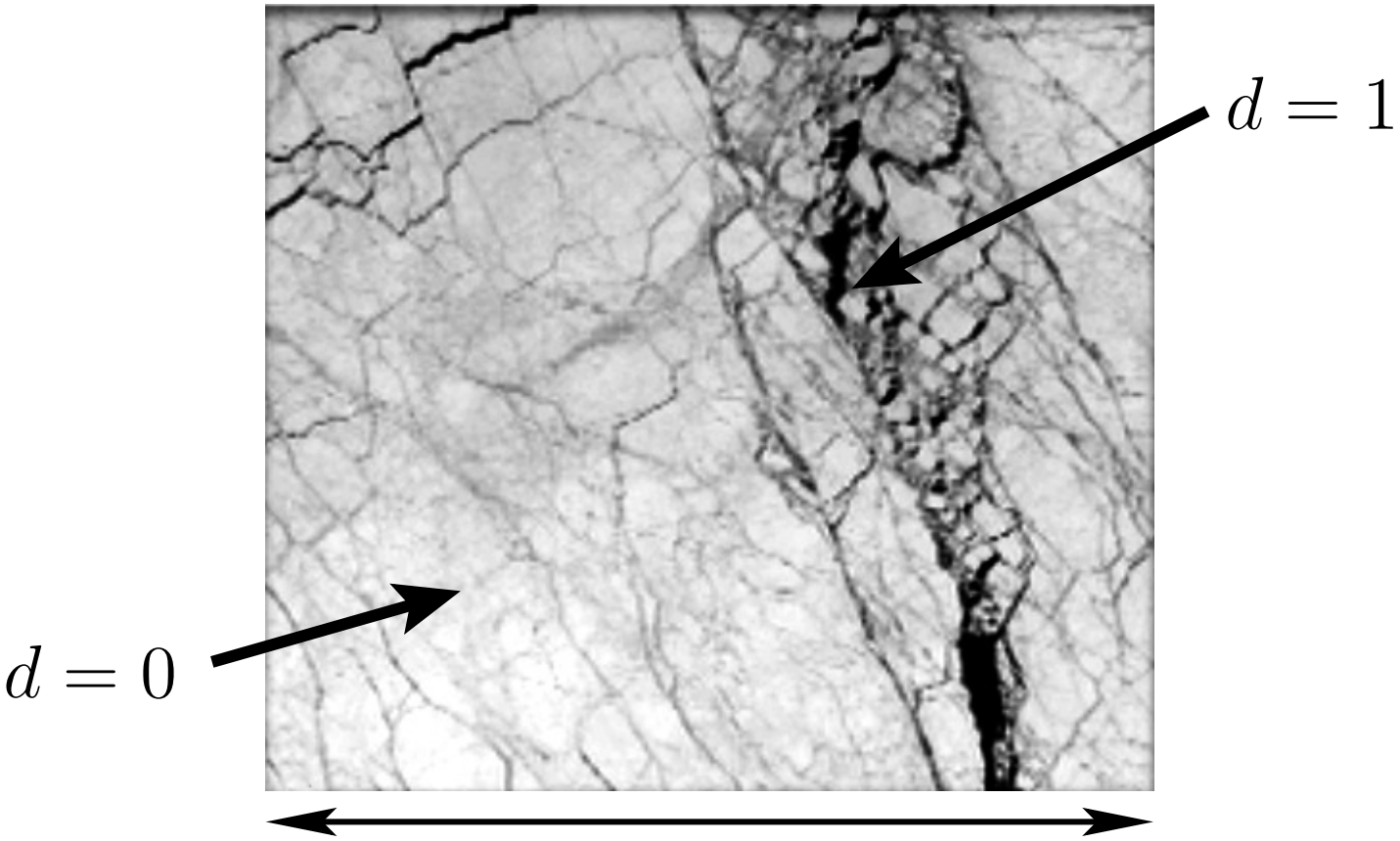


[Kwok 2010] satellite observation

$\Longrightarrow$ piecewise constant velocity
localization, fracture network

Progressive damage

$$0 \leqslant d(t, \boldsymbol{x}) \leqslant 1$$



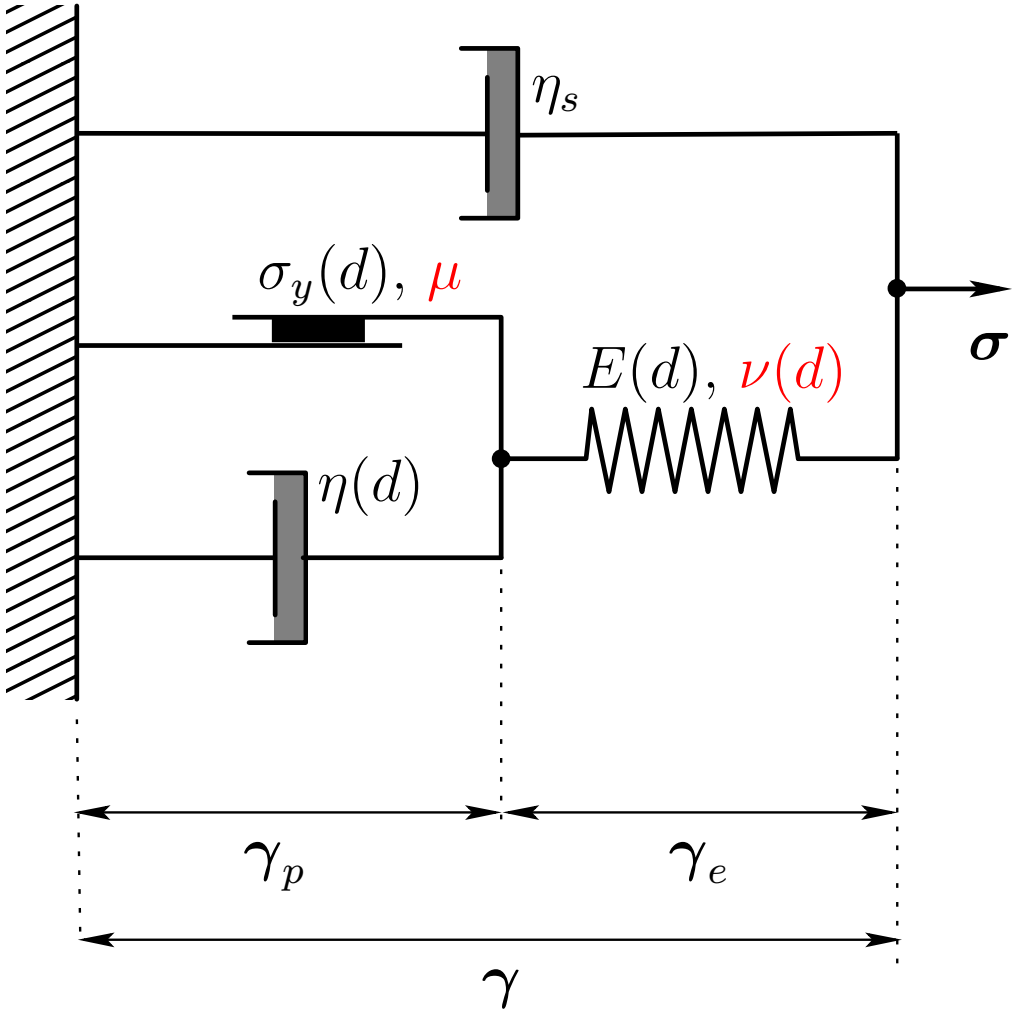
59 km : spot satelite

$$1 - d \;=\; \frac{E}{E_0} \;\rightsquigarrow\; \text{structural parameter}$$

[Kachanov 1958]

Outline

1. Rheology and thermodynamics
2. The new BEVP model
3. Numerical experiments



$\Longrightarrow$ elastoviscoplastic fluid

extension: μ, $\nu(d)$

some parameters depend upon damage d

[Saramito 2007]

Why thermodynamics ?

- it has a physical meaning
- it directly leads to constitutive equations
- it leads to well-posed math problems

  avoids fields to tend to infinity...
- it permits efficient numerical methods

Thermodynamics

- **variables**

γ	:	deformation tensor	$\dot{\gamma}, \dot{\gamma}_p, \dot{d}$	:	rate variables
γ_p	:	its irreversible part			
d	:	damage			

- **functions**

$\psi(\gamma, \gamma_p, d)$: Helmholtz free energy

$\phi(\dot{\gamma}, \dot{\gamma}_p, \dot{d})$: dissipation potential

$\Rightarrow$ **constitutive equations**

$$\begin{aligned} \boldsymbol{\sigma} &= \rho \frac{\partial \psi}{\partial \boldsymbol{\gamma}} + \frac{\partial \phi}{\partial \dot{\boldsymbol{\gamma}}} \\ 0 &= \rho \frac{\partial \psi}{\partial \boldsymbol{\gamma}_p} + \frac{\partial \phi}{\partial \dot{\boldsymbol{\gamma}}_p} \\ 0 &= \rho \frac{\partial \psi}{\partial d} + \frac{\partial \phi}{\partial \dot{d}} \end{aligned}$$

standard generalized materials

[Germain 1973 ; Moreau 1974 ; Halphen & Nguyen 1975 ; Saramito 2016]

Example 1: Maxwell viscoelatic fluid

- **variables** : γ, γ_p

- **functions**

$$\begin{aligned} \psi(\gamma, \gamma_p) &= \frac{G}{\rho}|\gamma - \gamma_p|^2 \\ \phi(\dot{\gamma}, \dot{\gamma}_p) &= \eta\,|\dot{\gamma}_p|^2 \end{aligned}$$

$\Rightarrow$ **constitutive equations**

$$\begin{aligned} \boldsymbol{\sigma} + p\boldsymbol{I} &= \boldsymbol{\sigma}_e \\ \frac{\dot{\boldsymbol{\sigma}}_e}{G} + \frac{\boldsymbol{\sigma}_e}{\eta} &= 2\dot{\gamma} \end{aligned}$$

where $\boldsymbol{\sigma}_e = 2G\gamma_e =$ elastic stress

From kinematics: $\dot{\gamma} \to D(\boldsymbol{u})$ and $\dot{\boldsymbol{\sigma}}_e \to \overset{\nabla}{\boldsymbol{\sigma}}_e$

Present choice 1: Hooke's elasticity

- variables : γ, γ_p, d

- Helmholtz free energy

$$\psi(\gamma, \gamma_p, d) = \frac{G(d)}{\rho}|\gamma - \gamma_p|^2 + \frac{\lambda(d)}{2\rho}\mathrm{tr}(\gamma - \gamma_p)^2$$

$$\Longrightarrow \quad \boldsymbol{\sigma}_e = \rho\frac{\partial\psi}{\partial\gamma}$$

$$= \; 2G(d)\gamma_e + \lambda(d)\,\mathrm{tr}(\gamma_e)\boldsymbol{I} \qquad \text{elastic stress}$$

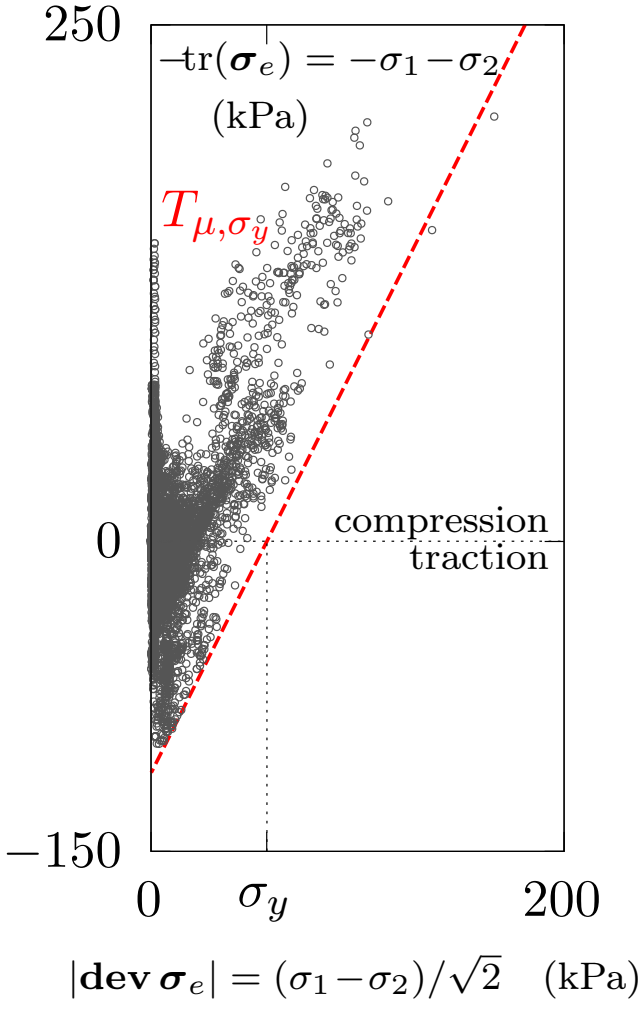
with $\quad G(d) = \dfrac{E(d)}{2(1+\nu(d))}$

$$\lambda(d) = \frac{E(d)\;\nu(d)}{(1+\nu(d))(1-2\nu(d))} \qquad \text{Lamé coefs}$$

and $\quad E(d) = (1-d)E_0 \qquad$ Kachanov's elastic modulus

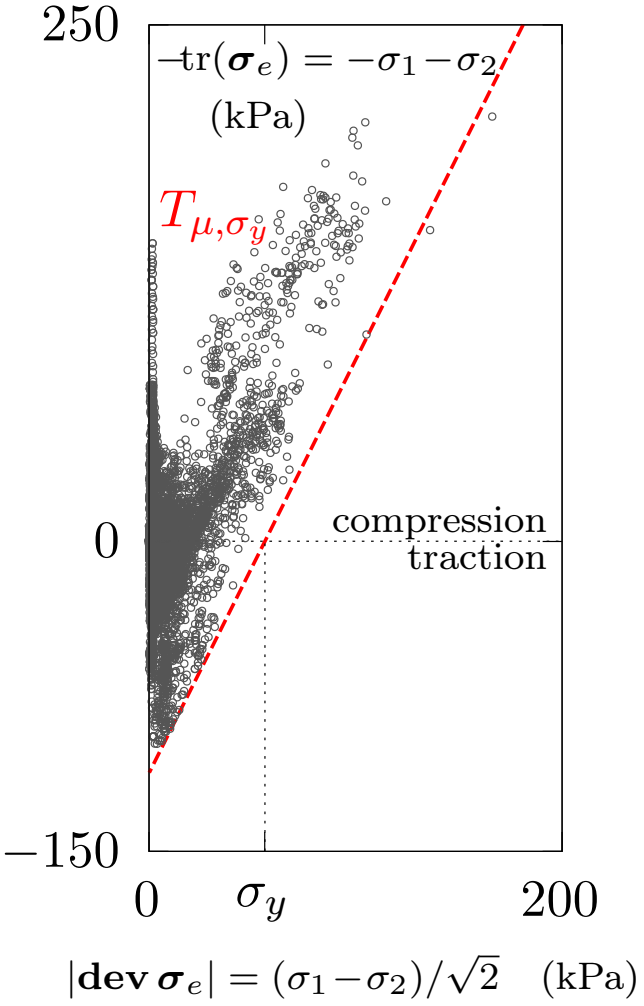
$\nu(d) = \nu_0 + (\nu_1 - \nu_0)d \qquad$ increasing Poisson ratio, $\nu_1 \geqslant \nu_0$

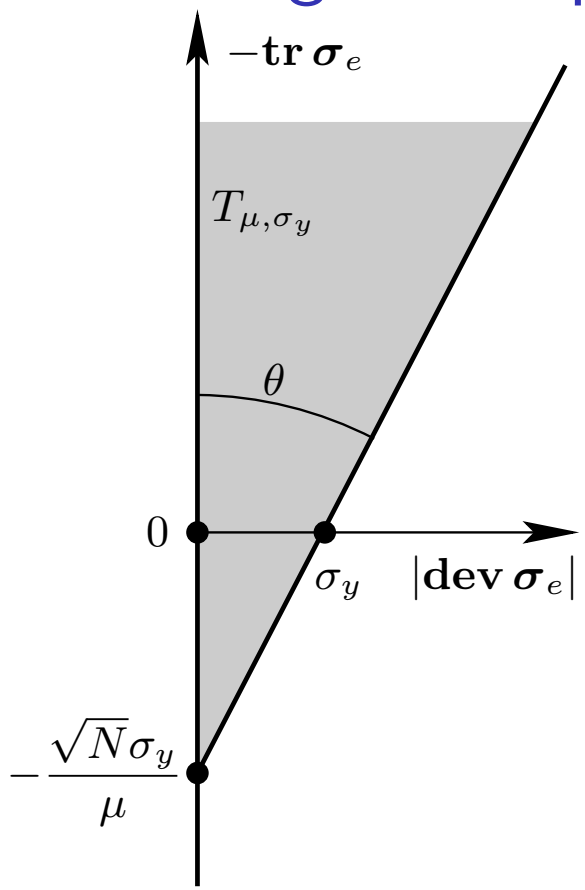
Present choice 2: Drucker-Prager viscoplasticity



data from: [Weiss Schulson, J Phys D, 2009]

Present choice 2: Drucker-Prager viscoplasticity





fit:

$\sigma_y \approx 58$ Pa

$\mu \approx 1/\sqrt{2}$

dependency:

$$\begin{aligned} \sigma_y(d) &= (1-d)\sigma_{y0} \\ \eta(d) &= (1-d)\eta_0 \end{aligned}$$

.

$$\begin{aligned} T_{\mu,\sigma_y} &= \text{translated Drucker-Prager cone} \\ &= \left\{ \boldsymbol{\sigma}_e \in \mathbb{R}^{N\times N}_s \ ; \ |\mathbf{dev}\,\boldsymbol{\sigma}_e| - \sigma_y \leqslant \frac{\mu}{\sqrt{N}} \,\mathrm{tr}\,\boldsymbol{\sigma}_e \right\} \end{aligned}$$

Present choice 3: brittle damage

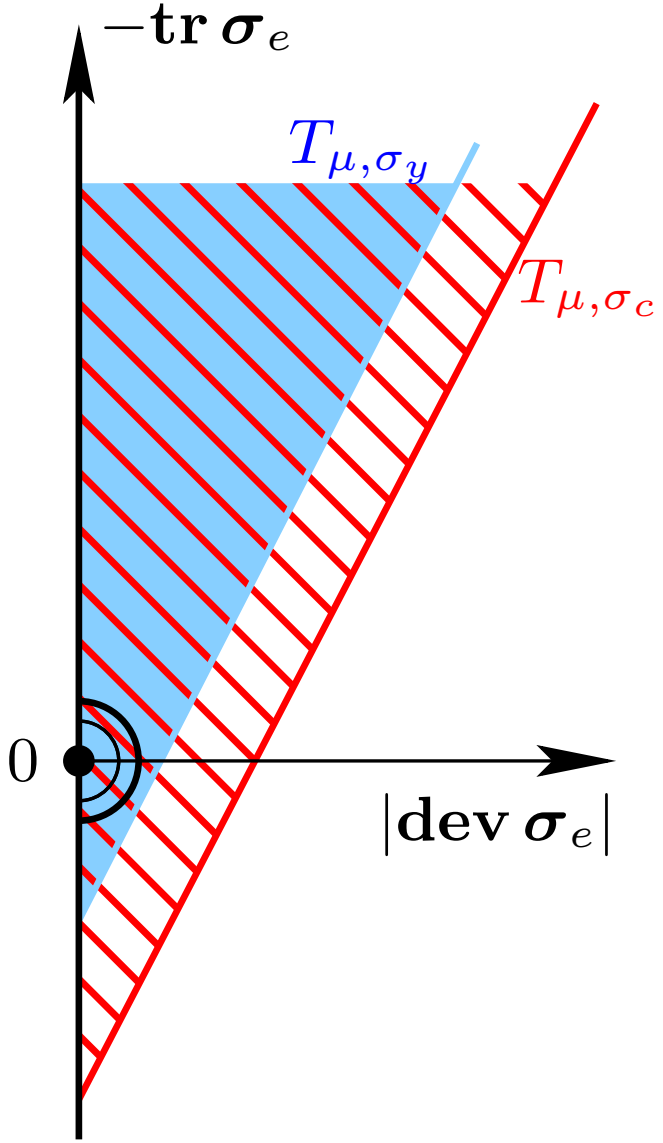
Damage evolution:

$$\dot{d} = 0 \qquad \text{when } \boldsymbol{\sigma}_e \in T_{\mu,\sigma_c}$$
$$\dot{d} > 0 \qquad \text{otherwise}$$

Two embedded DP cones

$$\sigma_y < \sigma_c \;\Rightarrow\; T_{\mu,\sigma_y} \subset T_{\mu,\sigma_c}$$

$\Rightarrow$ plasticity, then damage



Present BEVP model

Dissipation potential

$$\phi(\dot{\gamma}, \dot{\gamma}_p, \dot{d}) \quad = \quad \eta_s \, |\dot{\gamma}|^2 + \underbrace{\eta \, |\dot{\gamma}_p|^2 + \left(\mathscr{I}_{-T_{\mu,\sigma_y}}\right)^*(\dot{\gamma}_p)}_{\phi_p(\dot{\gamma}_p)} + \phi_d(\dot{d})$$

Main results

- satisfies the second principle of thermodynamics
- satisfies the Onsager symmetry principle

[Saramito, JNNFM, 2021]

Problem statement

(P): *find* d, γ_e, $\boldsymbol{u}$, p *such that*

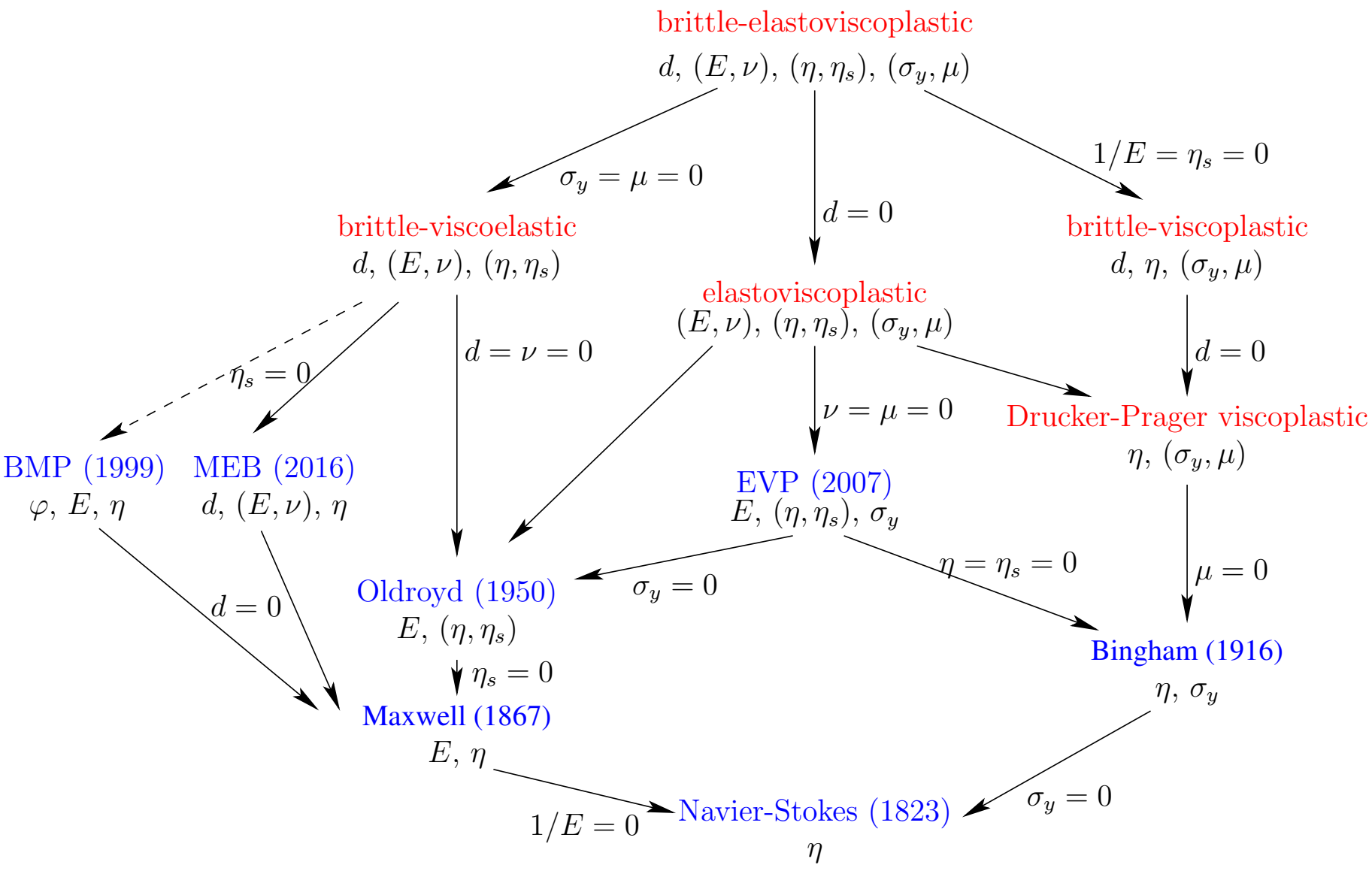
$$\begin{cases} \dot{d} - \nabla\phi^*_{\boldsymbol{d}}(d,\gamma_e) = 0 \\ \overset{\triangledown}{\gamma}_e + \nabla\phi^*_{\boldsymbol{p}}(d,\gamma_e) - D(\boldsymbol{u}) = 0 \\ \rho\dot{\boldsymbol{u}} - \mathbf{div}\,\boldsymbol{\sigma} = \boldsymbol{f} \\ \mathrm{div}\,\boldsymbol{u} = 0 \\ +B.C. + I.C. \end{cases}$$

with

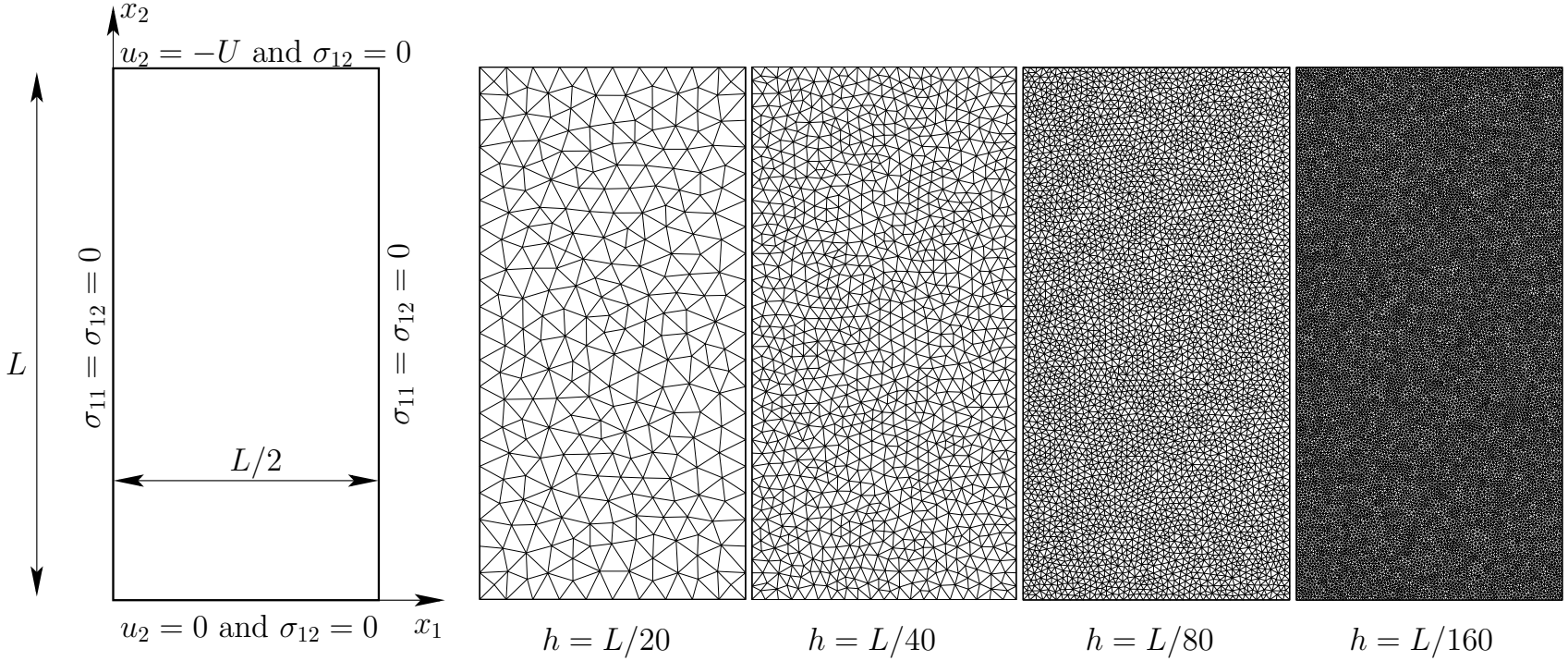
$$\begin{aligned} \boldsymbol{\sigma} &= -p\boldsymbol{I} + 2\eta_s D(\boldsymbol{u}) + \boldsymbol{\sigma}_e \\ D(\boldsymbol{u}) &= \frac{\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T}{2} \\ \overset{\triangledown}{\gamma}_e &= \partial_t\gamma_e + (\boldsymbol{u}.\nabla)\gamma_e - \nabla\boldsymbol{u}\gamma_e - \gamma_e\nabla\boldsymbol{u}^T \end{aligned}$$

Nonlinear Oldroyd-like + kinetic eqn for d
$\rightarrow$ classical structure

[Saramito, JNNFM, 2021]


brittle-elastoviscoplastic
d, (E,ν), (η,η_s), (σ_y,μ)
$\sigma_y = \mu = 0$
$d = 0$
$1/E = \eta_s = 0$
brittle-viscoelastic
d, (E,ν), (η,η_s)
elastoviscoplastic
(E,ν), (η,η_s), (σ_y,μ)
brittle-viscoplastic
d, η, (σ_y,μ)
$\eta_s = 0$
$d = \nu = 0$
$d = 0$
$\nu = \mu = 0$
Drucker-Prager viscoplastic
η, (σ_y,μ)
BMP (1999)
φ, E, η
MEB (2016)
d, (E,ν), η
EVP (2007)
E, (η,η_s), σ_y
$\eta = \eta_s = 0$
$\mu = 0$
Oldroyd (1950)
E, (η,η_s)
$\sigma_y = 0$
$d = 0$
Bingham (1916)
η, σ_y
$\eta_s = 0$
Maxwell (1867)
E, η
$1/E = 0$
Navier-Stokes (1823)
η
$\sigma_y = 0$

Uniaxial compression benchmark

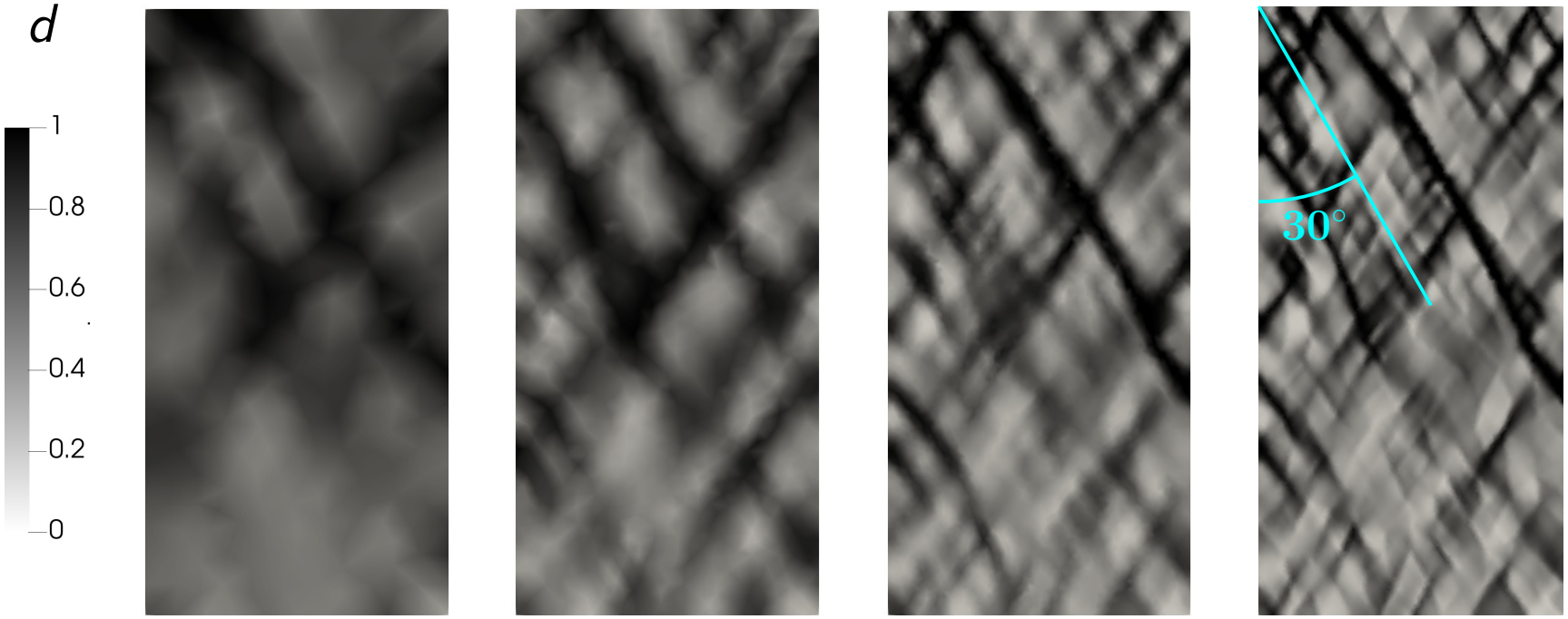


Uniform random heterogeneity

$$
\begin{aligned}
\sigma_{y0}(\boldsymbol{x}) &= \bar{\sigma}_{y0}\,(1 + 0.3\,\chi(\boldsymbol{x})) \\
\sigma_c(\boldsymbol{x}) &= \bar{\sigma}_c\,(1 + 0.3\,\chi(\boldsymbol{x})) \\
\chi(\boldsymbol{x}) &\in [-1, 1]
\end{aligned}
$$

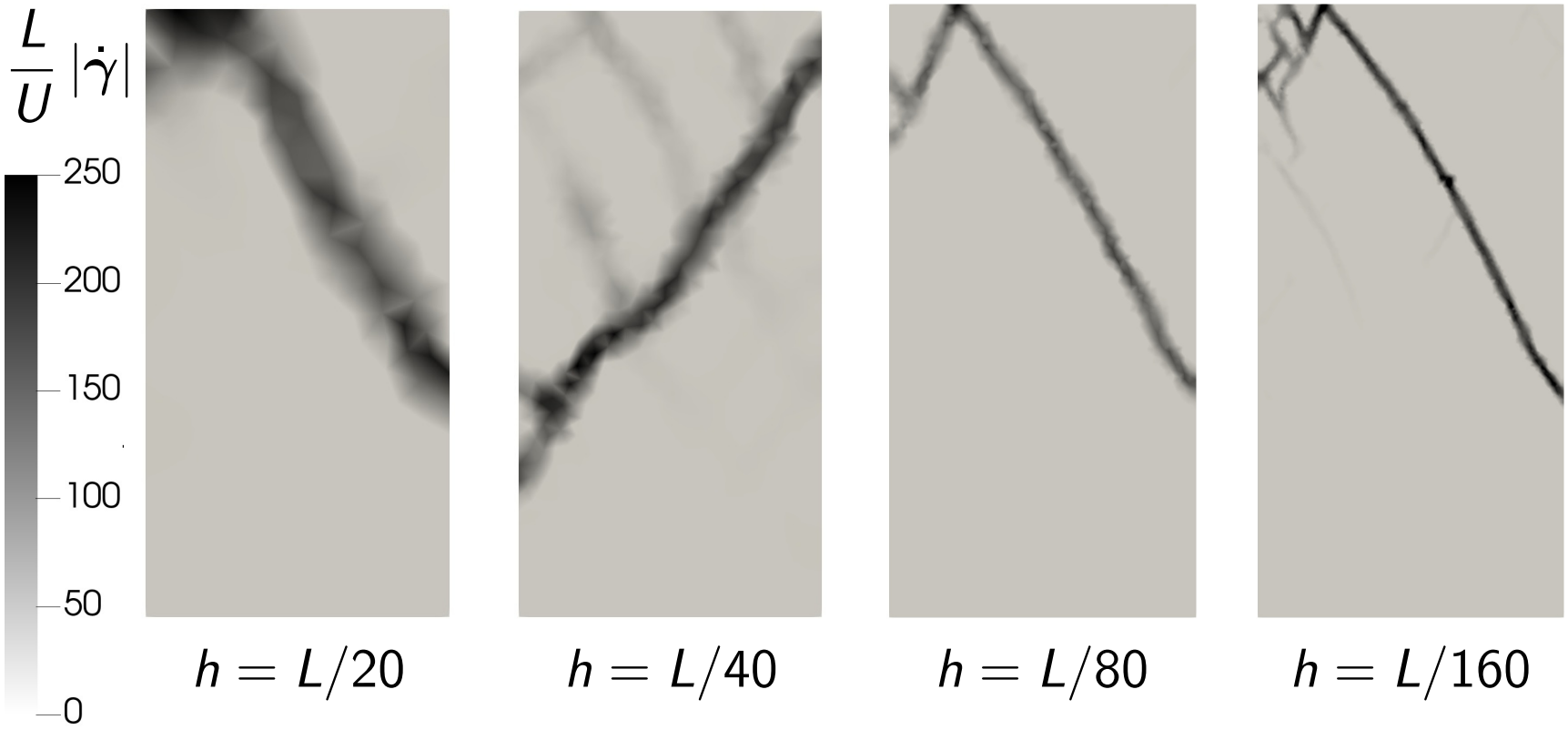
$\rightarrow$ breaks symmetry

Damage value

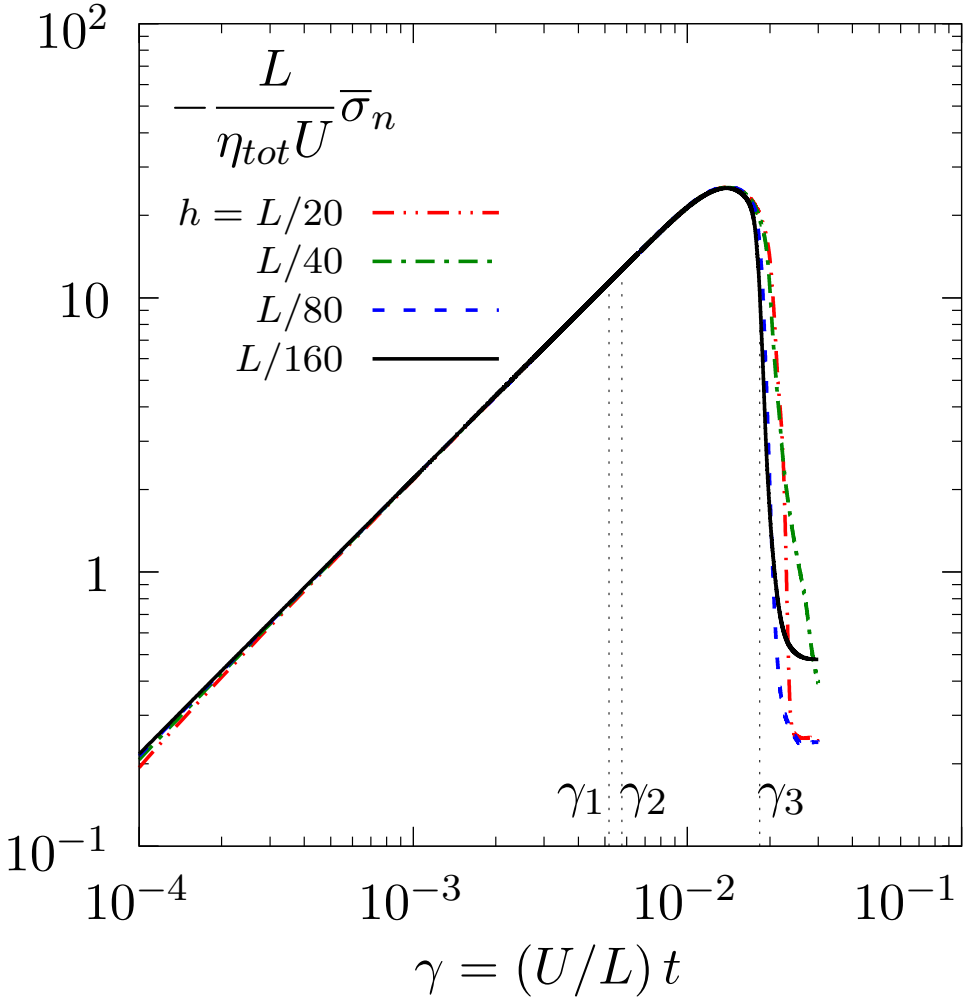


$h = L/20$ $h = L/40$ $h = L/80$ $h = L/160$

Deformation rate



Averaged normal stress on top boundary



$$\overline{\sigma}_n(t) = \frac{2}{L}\int_{top} \sigma_{yy}\,\mathrm{d}x$$

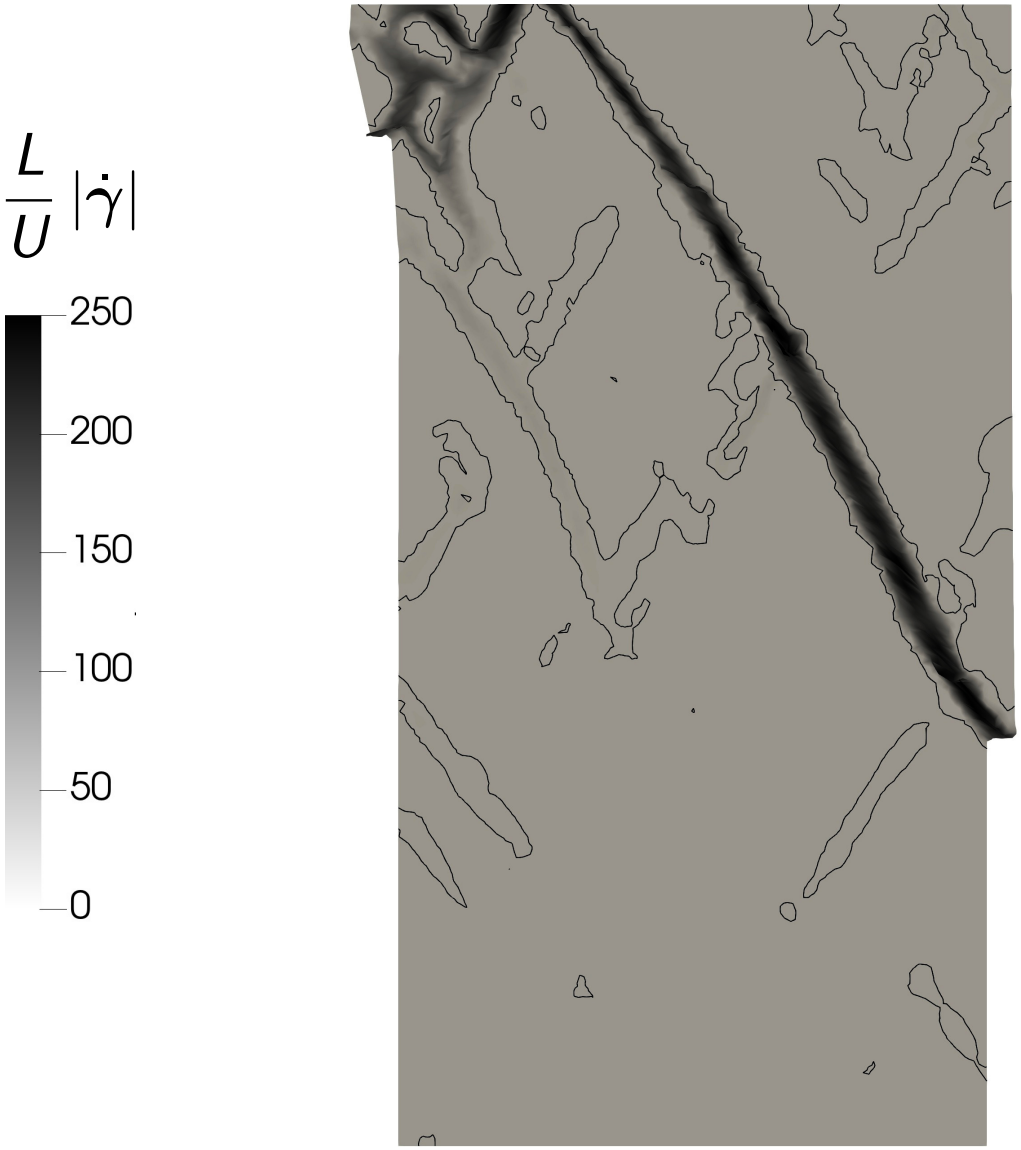
Four flow regimes

γ_1: first plastic even

γ_2: first damage

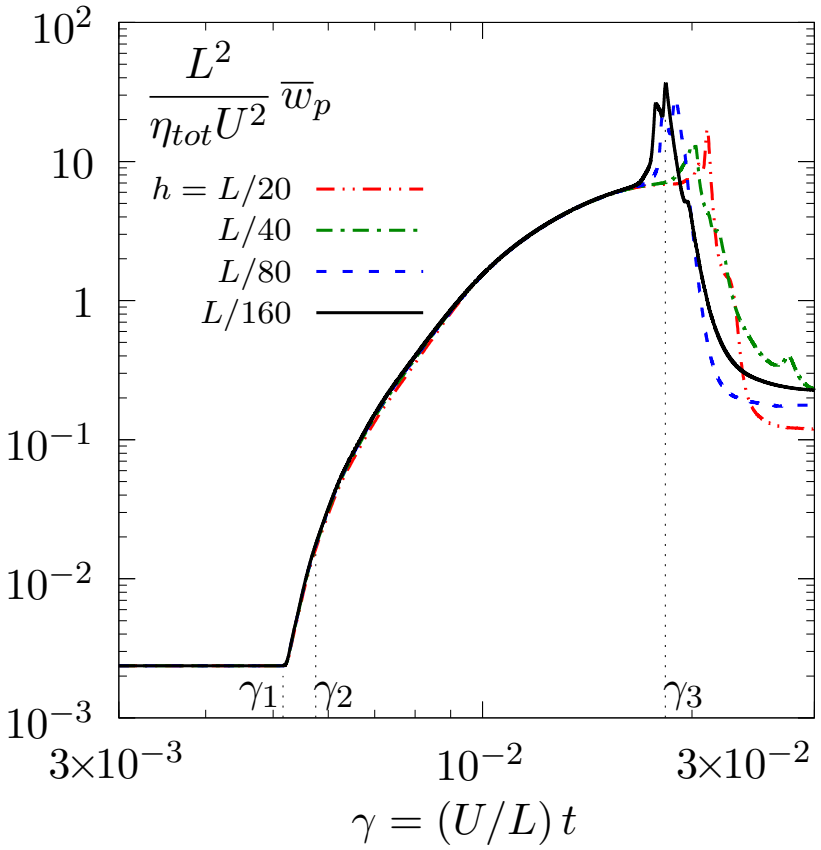
γ_3: post-failure

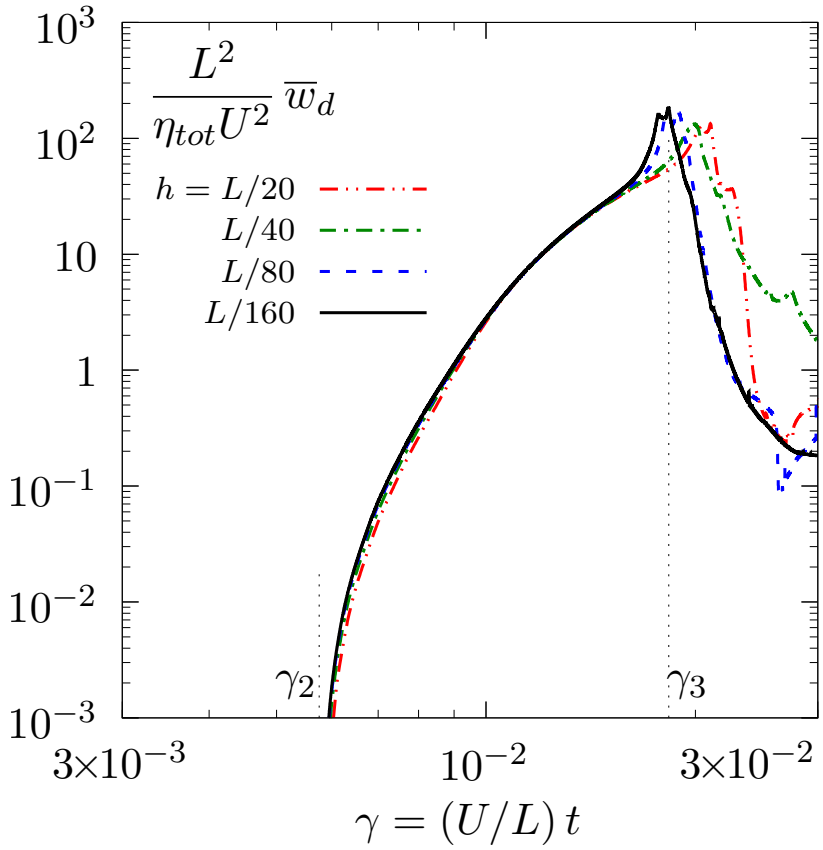
Post-failure: deformed geometry & yield surfaces



Dissipation: Clausius-Duhem

$$
\begin{aligned}
w &= -\rho\dot{\psi} + \boldsymbol{\sigma} : \dot{\boldsymbol{\gamma}} \\
&= w_p + w_d \geqslant 0
\end{aligned}
$$





Conclusion

New BEVP model

- second principle of thermodynamics
- link : soft solids & complex fluids
- new Drucker-Prager viscoplastic fluid model

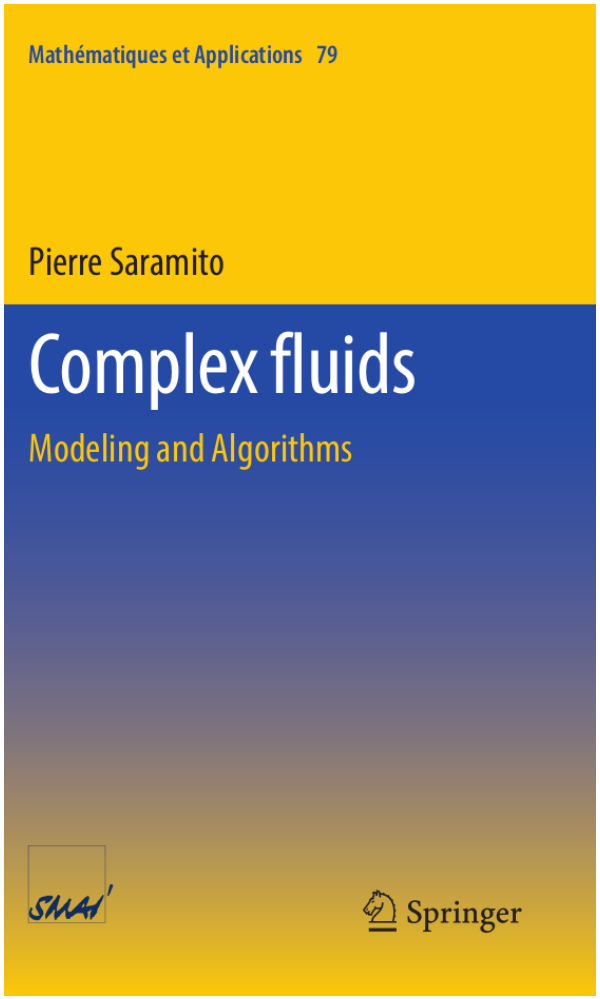
Perspectives

- sea-ice $\rightarrow$ climate changes
- earth cracks
- granular matter & suspensions

More reading

paper: Saramito, JNNFM, 2021

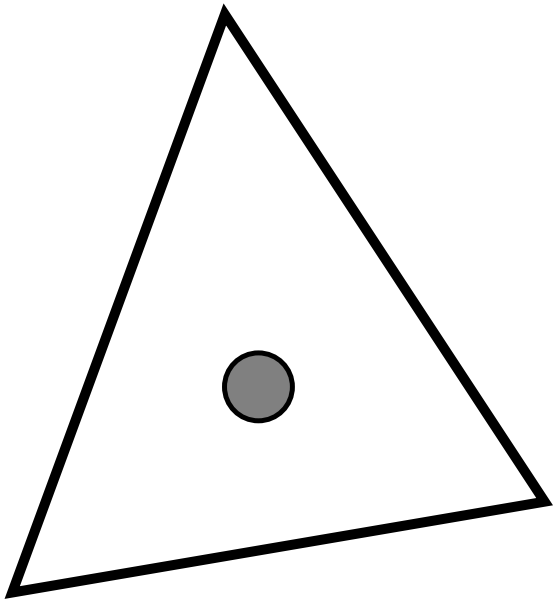
book: Saramito, *Complex fluids*
Springer, 2016



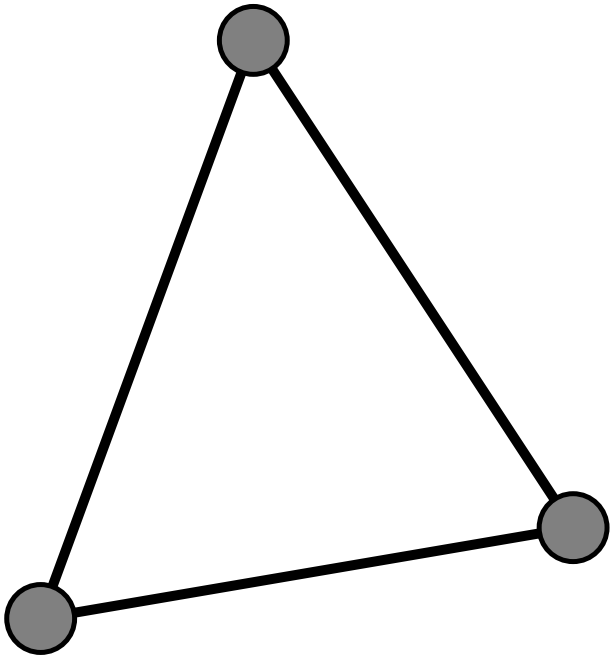
code: Saramito, 2018
Rheolef FEM C++ library
Free software: GPL licence
http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef

Discretization

$\mathrm{d}, \boldsymbol{\gamma}_e$: P_0
discontinuous



$\boldsymbol{u}$: P_1
continuous



Parameter set

param	value	dimension
L	200×10^{3}	m
U	2×10^{-3}	$\mathrm{m.s}^{-1}$
E_0	28×10^{6}	Pa
σ_{y0}	50×10^{3}	Pa
σ_c	56×10^{3}	Pa
η_0	1.4×10^{12}	Pa.s
η_s	1.4×10^{8}	Pa.s
η_d	2.8×10^{8}	Pa.s

number	value	expression
We	5×10^{-4}	$U(\eta_s+\eta_0)/(LE_0)$
We_d	10^{-7}	$U\eta_d/(LE_0)$
γ_y	1.8×10^{-3}	σ_{y0}/E_0
γ_c	2×10^{-3}	σ_c/E_0
ν_0	0.30	
ν_1	0.49	
μ	0.7	
$1-\alpha$	10^{-4}	$\eta_s/(\eta_s+\eta_0)$

Viscoelasticity: expansion

$$\overset{\nabla}{\gamma}_e \; + \nabla\phi_p^*(\boldsymbol{\sigma}_e) \; - \; D(\boldsymbol{u}) = 0$$

where

$$\boldsymbol{\sigma}_e = 2G(d)\gamma_e + \lambda(d)\,\mathrm{tr}(\gamma_e)\boldsymbol{I}$$

$$\nabla\phi_p^*(\boldsymbol{\sigma}_e) = \frac{\kappa_{\mu,\sigma_y}(\boldsymbol{\sigma}_e)}{2\eta\,(1+\mu^2)}\left(\boldsymbol{\sigma}_e - \frac{\xi_{\mu,\sigma_y}(\boldsymbol{\sigma}_e)}{\sqrt{N}\,\mu}\boldsymbol{I}\right)$$

and

$$\kappa_{\mu,\sigma_y}(\boldsymbol{\sigma}_e) = \begin{cases} 1+\mu^2 & \text{when } -\mu^2|\mathbf{dev}\,\boldsymbol{\sigma}_e| \;\geqslant\; \sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\sigma}_e \\ 1 - \dfrac{\sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\sigma}_e}{|\mathbf{dev}\,\boldsymbol{\sigma}_e|} & \text{when } -\mu^2|\mathbf{dev}\,\boldsymbol{\sigma}_e| \;<\; \sigma_y - \dfrac{\mu}{\sqrt{N}}\,\mathrm{tr}\,\boldsymbol{\sigma}_e \\ & \qquad\qquad\qquad\qquad < \; |\mathbf{dev}\,\boldsymbol{\sigma}_e| \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_{\mu,\sigma_y}(\boldsymbol{\sigma}_e) = \min\left(\sigma_y,\; \frac{\mu\,\mathrm{tr}\,\boldsymbol{\sigma}_e}{\sqrt{N}} \; - \; \mu^2|\mathbf{dev}\,\boldsymbol{\sigma}_e|\right)$$

Damage: expansion

$$\dot{d} = \nabla \phi_d^*(d)$$

$$\nabla \phi_d^*(d) = \frac{(1-d)\ \kappa_{\mu,\sigma_c}(\boldsymbol{\sigma}_e)}{2\eta_d(1+\mu^2)}\ Y$$

$$Y = -\left\{2G'(d)\boldsymbol{\gamma}_e + \lambda'(d)\,\mathrm{tr}(\boldsymbol{\gamma}_e)\boldsymbol{I}\right\} : \boldsymbol{\gamma}_e$$