Bias in the representative volume element method: periodize the ensemble instead of its realizations

Nicolas CLOZEAU

Institute of Science and Technology Austria

Joint work with M. Josien, F. Otto and Q. Xu

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Goal: Compute ($d = 3$)

$$a_{\text{hom}} \quad \text{and} \quad (\phi_{e_i})_{i\in\{1,\cdots,d\}}$$

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RVE approach: $L \gg 1$

$a \rightsquigarrow a_L$ suitable periodization $\quad \begin{cases} -\nabla \cdot a_L(e_i + \nabla \phi_{e_i,L}) = 0 & \text{in } [0,L)^d, \\ \phi_{e_i,L} \text{ is } L\text{-periodic.} \end{cases}$

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Two sources of error:

$$\mathbb{E}[|\bar{a}_L - a_{\text{hom}}|^2] = \underbrace{\text{var}(\bar{a}_L)}_{\text{random error}} + \underbrace{|\mathbb{E}[\bar{a}_L] - a_{\text{hom}}|^2}_{\text{bias}}.$$

Previous works:

- A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. Ann. Probab., 39(3):779–856, 2011
- A. Gloria, S. Neukamm, and F. Otto. An optimal quantitative two-scale expansion in stochastic homogenization of discrete elliptic equations. ESAIM Math. Model. Numer. Anal., 48(2):325–346, 2014
- A. Gloria, S. Neukamm, and F. Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. Invent. Math., 199(2):455–515, 2015
- J.-C. Mourrat and F. Otto. Correlation structure of the corrector in stochastic homogenization. Ann. Probab., 44(5):3207–3233, 2016.
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Novelty: Characterization of the asymptotic behaviour

$$\lim_{L\uparrow\infty} L^{\alpha}(\mathbb{E}[\bar{a}_L] - a_{\mathrm{hom}}) \quad \text{for} \quad \alpha > 0$$

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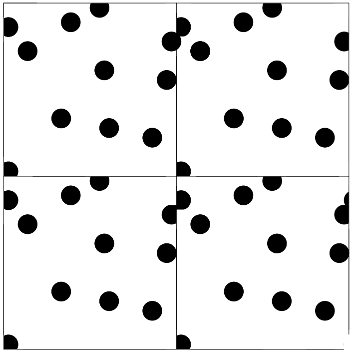

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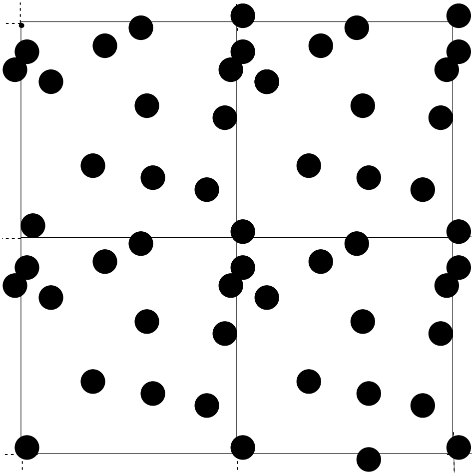

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$$\sup_x (1 + |x|)^{d+\alpha} (|c(x)| + (1 + |x|)^2 |\nabla^2 c(x)|) < \infty.$$

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$$\mathbb{P}_L = \text{Gaussian with covariance } c_L(x) := \sum_{k \in \mathbb{Z}^d} c(x + kL)$$

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$$\frac{\mathrm{d}}{\mathrm{d}L} \mathbb{E}_L[F] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{d}x \, \mathrm{d}y \, \mathbb{E}_L\left[\frac{\partial^2 F}{\partial g(x) \partial g(y)} \right] \frac{\mathrm{d}c_L}{\mathrm{d}L}(x - y) \quad \text{(Otto-Josien-C.)}$$

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- We let $L \uparrow \infty$
- Need a regularization : $-\nabla\cdot a\nabla \rightsquigarrow \frac{1}{T} - \nabla\cdot a\nabla$ and a re-summation

Thank you for your attention