A posteriori error estimator for 1D-2D Coupled Stokes Model

Hussein ALBAZZAL[1,2]

Supervisors: Alexei LOZINSKI[1], Roberta TITTARELLI[2]

[1] Laboratoire de Mathématiques-Université de Franche-Comté ,Besancon , France

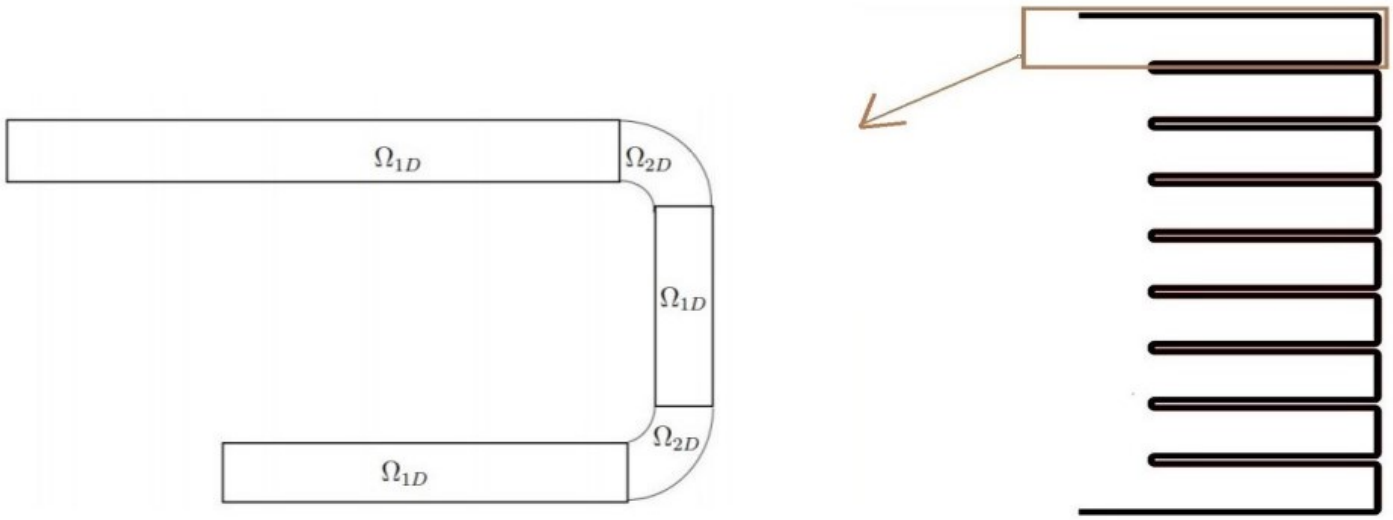
[2] FEMTO-st (Dep.Energie) ,Besancon ,France

14 June 2022

Goal

Our goals :

- We have non coupled 2D Stokes model.
- We have long channels and we want to find where the velocity are poiseuille in some 1D sections.
- Reduce the model to 1D-2D model.
- Determine the 1D-2D interface according to some tolerance
- Study a posteriori error.



Fuel Cell

Whole Domain Model

Stokes Equations

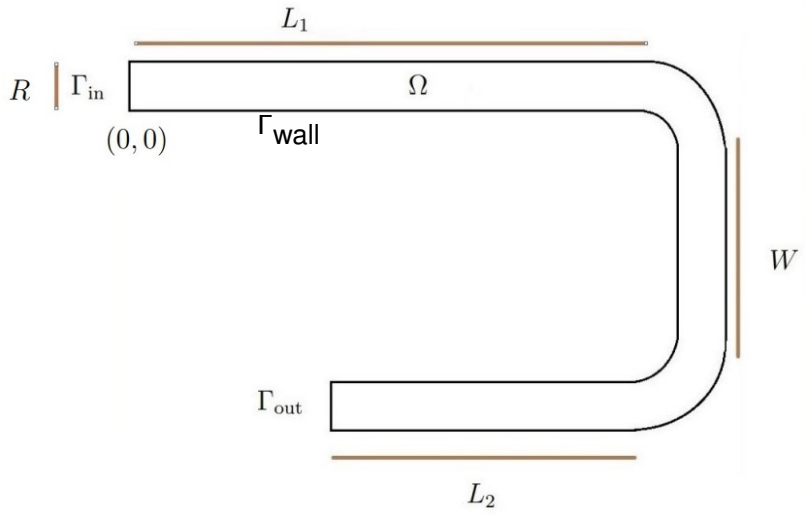
$$\begin{cases} -\Delta u + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot u = 0 & \text{in } \Omega \\ u = u_g & \text{on } \partial\Omega \end{cases}$$

Where, $u_g = \begin{cases} u_{in} & \text{on } \Gamma_{in}, \quad u_{in} \text{ is poiseuille}, \\ u_{out} & \text{on } \Gamma_{out}, \quad u_{out} \text{ is poiseuille}, \\ 0 & \text{on } \Gamma_{wall} \end{cases}$

- $u_{in} = (6u_{av}\frac{(R-y)y}{R^2}, 0)$
- $u_{out} = (-6u_{av}\frac{(-W-y)(y+W+R)}{R^2}, 0)$

constraint condition

$p + c$ is also a solution, we fix this constant by:
$\int_\Omega p = 0$

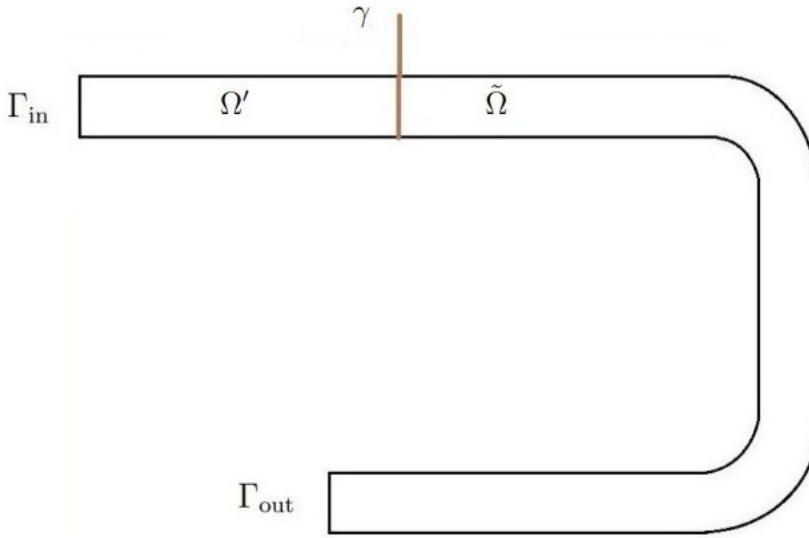


Derivation of 1D Model

Rely on [Gerbeau & Perthame,2000] and [Tayachi, 2014] to derive 1D simplified model.

1D Model

We put on Ω': $u_1' = 6u_{av}\frac{(R-y)y}{R^2}$, $u_2' = 0$ and $p' = -\frac{12u_{av}}{R^2}x + c_{\Omega'}$, with $c_{\Omega'}$ to be fixed after.



[Gerbeau & Perthame,2000] Jean-Frédéric Gerbeau—Benoît Perthame.Derivation of viscous saint-venant system for laminar shallow water; numerical validation.2000
[Tayachi,2014] M Tayachi, Antoine Rousseau, Eric Blayo, Nicole Goutal, and Véronique Martin.Design and analysis of a schwarz coupling method for a dimensionally heterogeneous problem.International Journal for Numerical Methods n Fluids, 75(6):446–465, 2014.

Simplified 1D-2D Model

2D Model

$$\begin{cases} -\Delta\tilde{u} + \nabla\tilde{p} = 0 & \text{in } \tilde{\Omega}, \\ \nabla\cdot\tilde{u} = 0 & \text{in } \tilde{\Omega}, \\ \tilde{u} = \tilde{u}_g & \text{on } \partial\tilde{\Omega}\ , \end{cases}$$

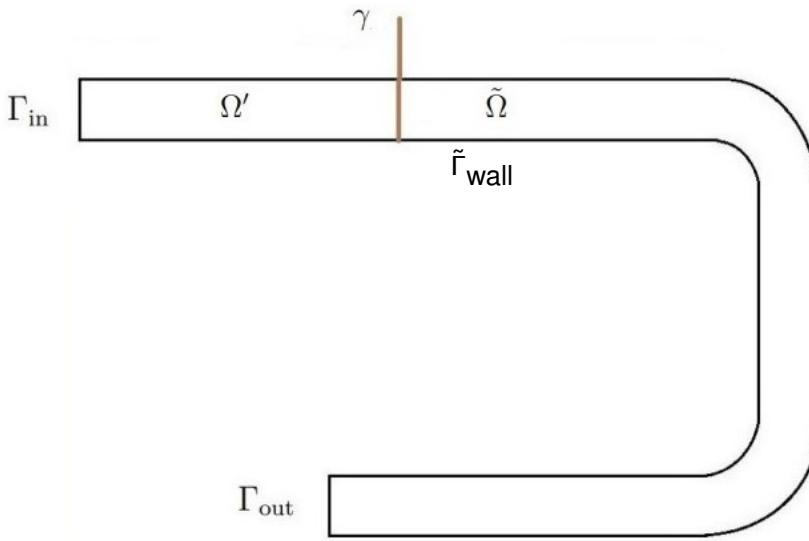
where, $\tilde{u}_g = \begin{cases} \tilde{u} = u_{in} & \text{on } \gamma \\ \tilde{u} = u_{out} & \text{on } \Gamma_{out} \\ \tilde{u} = 0 & \text{on } \tilde{\Gamma}_{wall} \end{cases}$

1D Model

We put on Ω': $u_1' = 6u_{av}\frac{(R-y)y}{R^2}$, $u_2' = 0$ and $p' = -\frac{12u_{av}}{R^2}x + c_{\Omega'}$

Coupled Conditions

$\tilde{p} + c_{\tilde{\Omega}}$ is also sol of 2D Model. Fix $c_{\tilde{\Omega}}$ & $c_{\Omega'}$ by :
$\tilde{u} = u'$ on γ & $\int_\gamma \tilde{p} = \int_\gamma p'$ & $\int_{\Omega'} p' + \int_{\tilde{\Omega}} \tilde{p} = 0$.



Approximated 1D-2D Solution on Ω

Definition (Approximated 1D-2D Solution)

find $(\tilde{u}_h, \tilde{p}_h) \in \tilde{V}_h^g \times \tilde{M}_h$ such that

$$\begin{cases} (\nabla \tilde{u}_h, \nabla \tilde{v}_h)_{\tilde{\Omega}} - (\nabla \cdot \tilde{v}_h, \tilde{p}_h)_{\tilde{\Omega}} = 0 & \forall \tilde{v}_h \in \tilde{V}_h^0, \\ -(\nabla \cdot \tilde{u}_h, \tilde{q}_h)_{\tilde{\Omega}} = 0 & \forall \tilde{q}_h \in \tilde{M}_h, \end{cases}$$

- $\tilde{\mathcal{T}}_h$ be a regular triangular mesh on $\tilde{\Omega}$
- $\tilde{V}_h := \{v_h \in [C^0(\bar{\tilde{\Omega}})]^2 \quad \text{s.t.} \quad v_h \in [\mathbb{P}_2(\tilde{\mathcal{T}}_h)]^2\}$
- $\tilde{M}_h := \{q_h \in C^0(\bar{\tilde{\Omega}}) \quad \text{s.t.} \quad q_h \in \mathbb{P}_1(\tilde{\mathcal{T}}_h) \quad \text{and} \int_{\tilde{\Omega}} q_h = 0\}$
- $\tilde{V}_h^g := \{v_h \in \tilde{V}_h \quad \text{s.t.} \quad v_h|_{\partial\tilde{\Omega}} = \tilde{u}_g\}$
- $\tilde{V}_h^0 := \{v_h \in \tilde{V}_h \quad \text{s.t.} \quad v_h|_{\partial\tilde{\Omega}} = 0\}$

The approximate solution on the whole Ω :

$$u_h^s = \begin{cases} u' = (u_1', u_2') = (u_{\text{in}}, 0) \text{ on } \Omega' \\ \tilde{u}_h \text{ on } \tilde{\Omega} \end{cases}$$

$$p_h^s = \begin{cases} p' = -\frac{12 u_{av}}{R^2} x + c_{\Omega'} \text{ on } \Omega' \\ \tilde{p}_h = \tilde{p}_h + c_{\tilde{\Omega}} \text{ on } \tilde{\Omega} \end{cases}$$

where $c_{\Omega'}$ and $c_{\tilde{\Omega}}$ are determined from coupled conditions $\int_\gamma [p_h^s] = 0$ and $\int_\Omega p_h^s = 0$

Properties of the approximated 1D-2D solution

We will construct some tools such as flux reconstruction to be used to derive a posteriori error estimator.

Let (u, p) be the weak solution of 2D Stokes then

- $u \in [H^1_g(\Omega)]^2 := \{u \in [H^1(\Omega)]^2; \quad u = u_g \quad \text{on} \quad \partial\Omega\}$
- $\sigma := \nabla u - pI \in [H(\text{div}, \Omega)]^2 := \{(\sigma_{ij})_{1 \leq i,j \leq 4}; \sigma_{ij} \in L^2(\Omega); \nabla \cdot \sigma \in [L^2(\Omega)]^2\}$
- $\nabla \cdot \sigma = 0$.

Definition (Approximate Flux)

Let (u^s_h, p^s_h) be the of approximated 1D-2D solution then, we call approximate flux: $\nabla u^s_h - p^S_h I$

Let (u^s_h, p^s_h) be the approximated 1D-2D solution then, $u^s_h \in H^1_g(\Omega)$ but

- $\nabla u^s_h - p^s_h I \notin [H(\text{div}, \Omega)]^2$ in general.
- $\nabla \cdot (\nabla u^s_h - p^s_h I) \neq 0$ in general.

Flux Reconstruction

- Modify Vohralik approach [Ern & Vohralik, 2015] to get suitable reconstructed flux to our simplified 1D-2D model.
- Let (u_h^s, p_h^s) be the approximated 1D-2D solution.
- Ideally we would look $\sigma_h \in \Sigma_h \subset [H(\mathrm{div}, \Omega)]^2$ such that:

$$\sigma_h := \underset{\substack{v_h \in \Sigma_h, \\ \mathrm{div}\, v_h = 0 \quad \mathrm{on} \quad \Omega}}{\arg\min} \; ||\nabla u_h^s - p_h^s I - v_h||_{L^2(\Omega)}.$$

- In practice, $\Sigma_h := \left(RTN_1 \mathbb{1}_{\tilde{\Omega}} + H(\mathrm{div}, \Omega')\mathbb{1}_{\Omega'}\right) \cap H(\mathrm{div}, \Omega)$.
  Let $RTN_1(K)$ be the Raviart-Thomas mixed finite element space on $K \in \tilde{\mathcal{T}}_h$ defined by:

$$RTN_1(K) := [\mathbb{P}_1(K)]^2 + x\mathbb{P}_1(K)$$

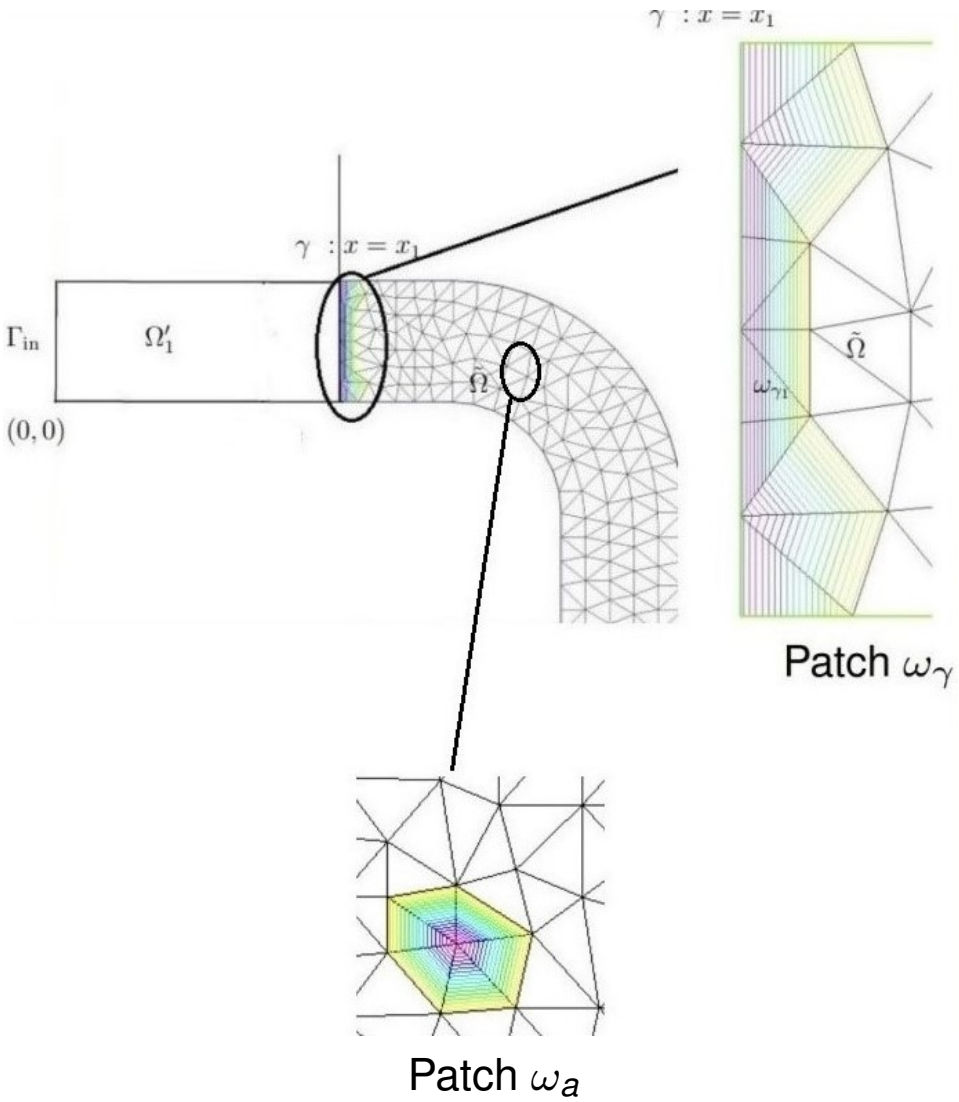
$$RTN_1 := \{v_h \in H(\mathrm{div}, \tilde{\Omega}); v_h|_K \in RTN_1(K), \forall K \in \tilde{\mathcal{T}}_h\}$$

- This σ_h is too expensive so localize this minimization.

[Ern & Vohralik, 2015]Alexandre Ern and Martin Vohralík.Polynomial-degree-robust a posteriori estimates in a unified setting for conforming,nonconforming, discontinuous galerkin, and mixed discretizations.SIAMJournalonNumericalAnalysis, 53(2):1058–1081, 2015

Flux Reconstruction

- ω_a is a patch of triangles sharing a vertex a & $\omega_\gamma := \cup_{a\in\gamma}\omega_a$

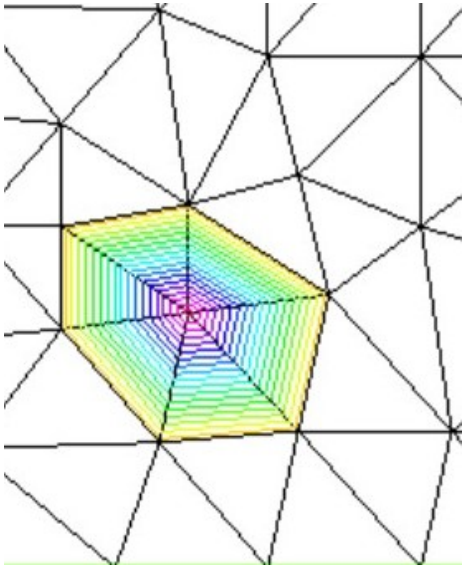


Flux Reconstruction

- $\mathbb{1}_\Omega = \mathbb{1}_{\Omega'} + \mathbb{1}_{\tilde{\Omega}} = \mathbb{1}_{\Omega'} + \sum_{a \in \gamma} \psi_a + \sum_{a \in \tilde{\Omega}\setminus\gamma} \psi_a = \mathbb{1}_{\Omega'} + \psi_\gamma + \sum_{a \in \tilde{\Omega}\setminus\gamma} \psi_a$
- Where, ψ_a hat fn & $\psi_\gamma = \sum_{a\in\gamma} \psi_a$
- $\sigma_h = (\sigma_h^\gamma + \sum_{a\in\tilde{\Omega}\setminus\gamma} \sigma_h^a)\mathbb{1}_{\tilde{\Omega}} + (\nabla u' - p' I)\mathbb{1}_{\Omega'}$
- <u>Case1: a in an internal node of $\tilde{\Omega}$</u>

$$\sigma_h^a := \underset{\substack{v_h^a \in \Sigma_h^a, \\ \operatorname{div} v_h^a = (\nabla \tilde{u}_h - \tilde{p}_h I)\cdot \nabla \psi_a}}{\arg\min} ||v_h^a - \psi_a(\nabla \tilde{u}_h - \tilde{p}_h I)||_{L^2(\omega_a)}$$

where, $\Sigma_h^a := \{\sigma_h \in RTN_1(\omega_a), \sigma_h \cdot n = 0 \quad \text{on} \quad \partial\omega_a\}$

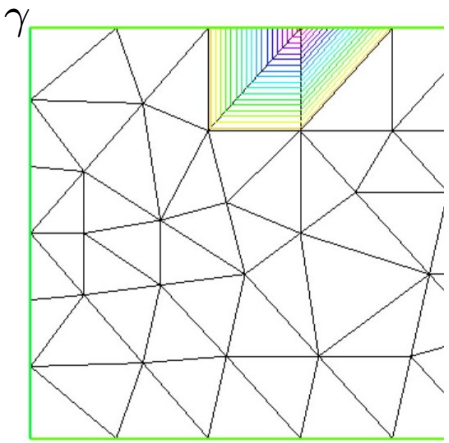


Flux Reconstruction

- $\sigma_h = (\sigma_h^\gamma + \sum_{a\in\tilde{\Omega}\backslash\gamma} \sigma_h^a)\mathbb{1}_{\tilde{\Omega}} + (\nabla u' - p' I)\mathbb{1}_{\Omega'}$
- Case2: a on the wall of $\tilde{\Omega}\backslash\gamma$

$$\sigma_h^a := \underset{\substack{v_h^a\in\Sigma_h^a,\\ \operatorname{div} v_h^a=(\nabla\tilde{u}_h-\tilde{p}_h I)\cdot\nabla\psi_a}}{\arg\min} ||v_h^a - \psi_a(\nabla\tilde{u}_h - \tilde{p}_h I)||_{L^2(\omega_a)}$$

where, $\Sigma_h^a := \{\sigma_h \in RTN_1(\omega_a), \sigma_h \cdot n = 0 \quad \text{on} \quad \partial\omega_a\backslash\partial\tilde{\Omega}\}$

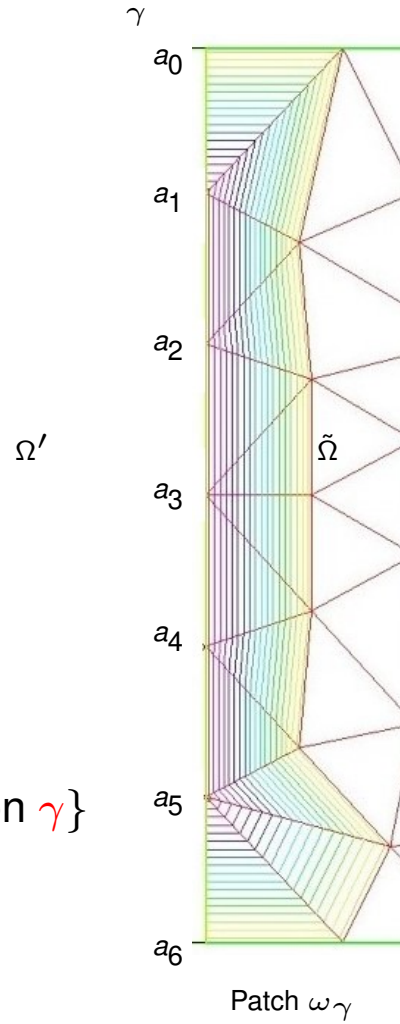


Flux Reconstruction

- $\sigma_h = (\sigma_h^\gamma + \sum_{a\in\tilde{\Omega}\setminus\gamma} \sigma_h^a)\mathbb{1}_{\tilde{\Omega}} + (\nabla u' - p' I)\mathbb{1}_{\Omega'}$
- Case3: a on the wall γ
- $\sigma_h^\gamma := \underset{\substack{v_h^\gamma \in \Sigma_h^\gamma, \\ \operatorname{div} v_h^\gamma = (\nabla \tilde{u}_h - \tilde{p}_h I)\cdot\nabla\psi_\gamma}}{\arg\min} ||v_h^\gamma - \psi_\gamma(\nabla \tilde{u}_h - \tilde{p}_h I)||_{L^2(\gamma)}$

where,

$$\Sigma_h^\gamma := \{\sigma_h \in RTN_1(\omega_\gamma),\ \sigma_h \cdot n = 0 \ \text{ on } \partial\omega_\gamma \setminus \partial\tilde{\Omega},\ \ \sigma_h \cdot n = (\nabla u' - p' I)n \quad \text{on } \gamma\}$$



Proposition

$\sigma_h = \tilde{\sigma}_h \mathbb{1}_{\tilde{\Omega}} + \sigma' \mathbb{1}_{\Omega'}$ where, $\tilde{\sigma}_h = \sigma_h^{\gamma} + \sum_{a \in \tilde{\Omega} \setminus \gamma} \sigma_h^a$, then $\nabla \cdot \tilde{\sigma}_h = 0$ on $\tilde{\Omega}$ and consequently $\nabla \cdot \sigma_h = 0$ on Ω.

Theorem (A general a posterior error estimate)

- *Let (u, p) be the weak 2D solution of Stokes on Ω.*
- *Let $u_h^s \in [H_g^1(\Omega)]^2$ and $p_h^s \in L_0^2(\Omega)$ defined as approximated 1D-2D solution.*
- *Let $\tilde{\mathcal{T}}_h$ be the mesh of $\tilde{\Omega}$, then $\forall K \in \tilde{\mathcal{T}}_h$ define:*
  - *Flux estimator: $\eta_{F,K} := ||\nabla \tilde{u}_h - \tilde{p}_h I - \tilde{\sigma}_h||_K$*
  - *Divergence estimator: $\eta_{D,K} := \frac{||\nabla \cdot \tilde{u}_h||_K}{\beta}$.*

Then,

$$e_U := ||\nabla(u - u_h)||_\Omega \leq \left(\sum_{K \in \tilde{\mathcal{T}}_h} \eta_{F,K}^2 + \sum_{K \in \tilde{\mathcal{T}}_h} \eta_{D,K}^2 \right)^{\frac{1}{2}} := \eta_U$$

$$e_P := \beta ||p - p_h||_\Omega \leq \left(\sum_{K \in \tilde{\mathcal{T}}_h} \eta_{F,K}^2 \right)^{\frac{1}{2}} + \left(\sum_{K \in \tilde{\mathcal{T}}_h} \eta_{D,K}^2 \right)^{\frac{1}{2}} := \eta_P$$

where, $\inf_{q \in L_0^2(\Omega)} \sup_{v \in [H_0^1(\Omega)]^2} \frac{(q, \nabla \cdot v)_\Omega}{||q||_\Omega ||\nabla v||_\Omega} = \beta > 0$

Results

- Plot errors & estimators for different positions of the interface γ and for mesh sizes $h = 0.07$ and $h = 0.02$

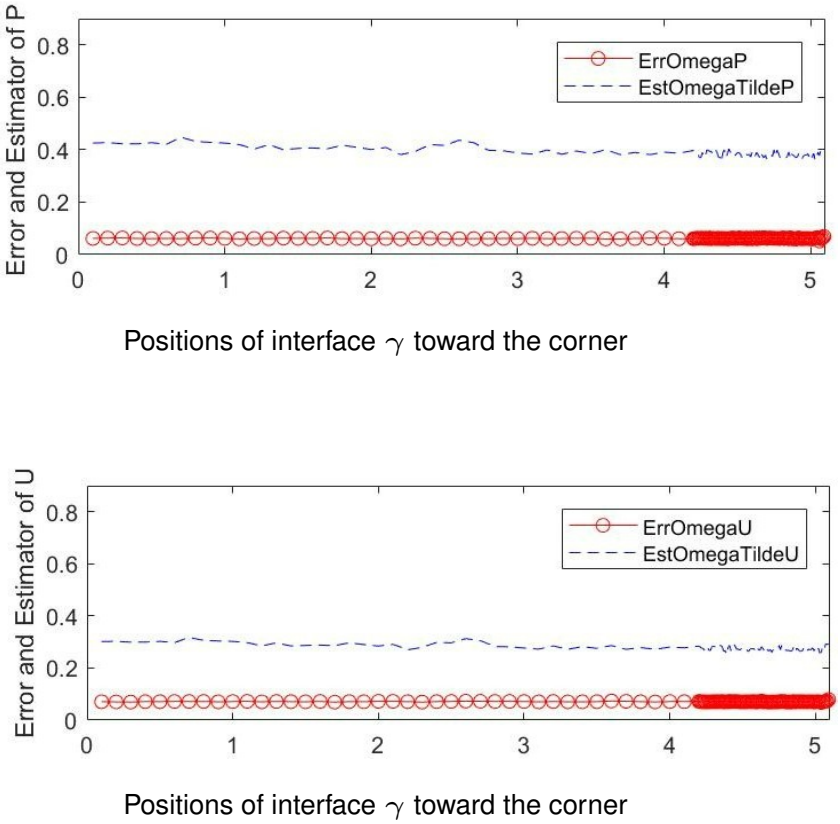


Figure: Errors and Estimators U and P for $h = 0.07$ for different positions of interface γ

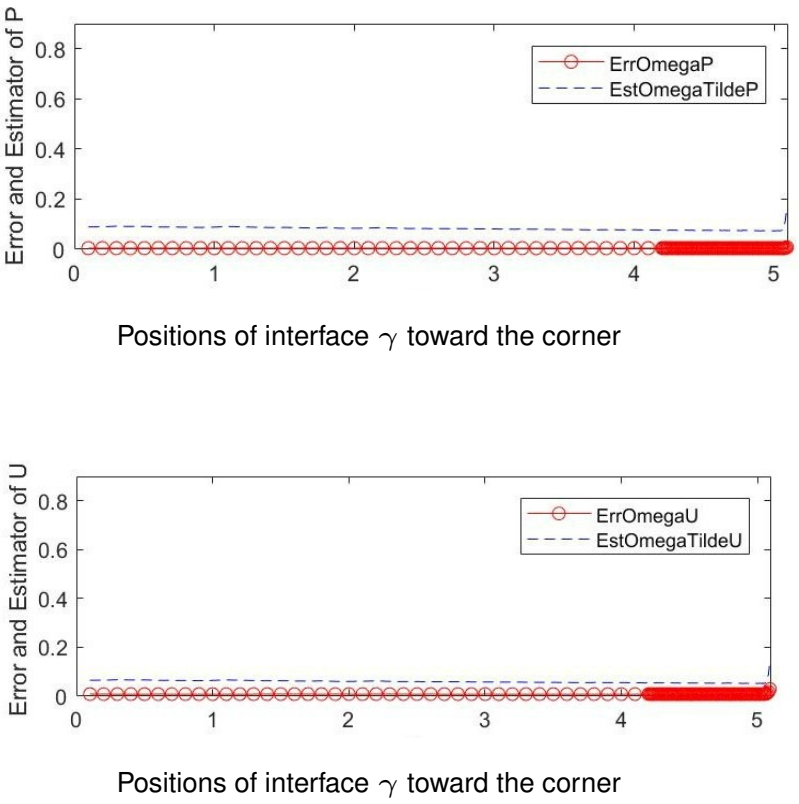


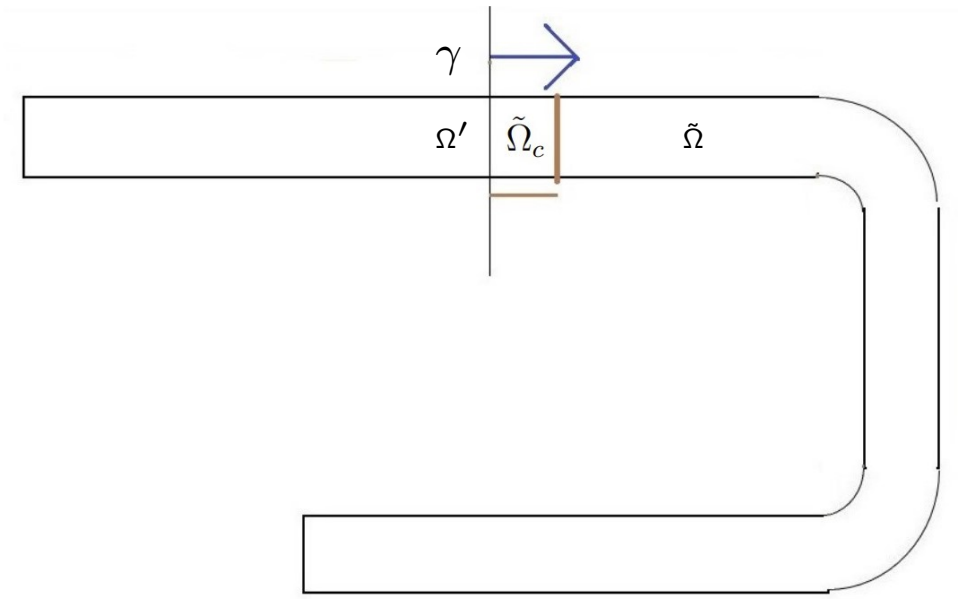
Figure: Errors and Estimators U and P for $h = 0.02$ for different positions of interface γ

Results

- Using Theorem we can't detect the suitable position for the interface.
- We introduce a "Detection of interface position".

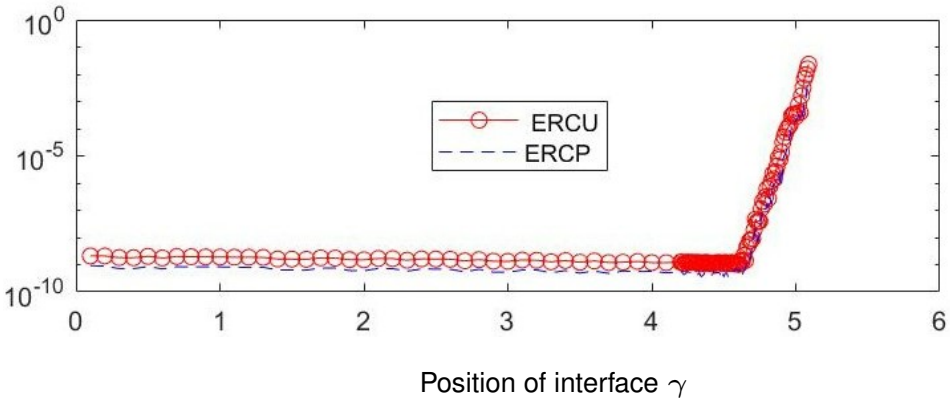
$$ERCU := ||\nabla(\tilde{u}_h - u')||_{\tilde{\Omega}_c}$$

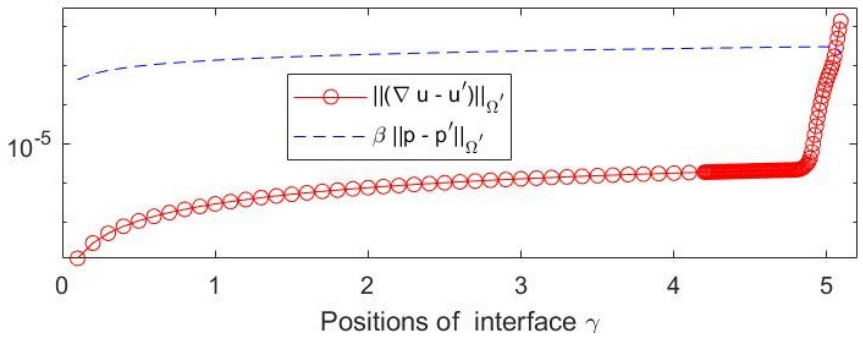
$$ERCP := \beta||\tilde{p}_h - p'||_{\tilde{\Omega}_c}$$



Results

- Plot Estimated Region Cut *ERCU* and *ERCP* on a region $\tilde{\Omega}_c$ for mesh sizes $h = 0.07$ and for different positions of the interface.
- Compare *ERCU* and *ERCP* on a region $\tilde{\Omega}_c$ with errors $||\nabla(u - u')||_{\Omega'}$ and $\beta||p - p'||_{\Omega'}$ on Ω'.





- the benefit of "ERCU" or "ERCP" is to determine the position of the interface without any knowledge about the exact solution u and p

Conclusion and Perispectives

We conclude that:

- The errors between non-coupled model and 2D-1D coupled model depends on the position of interfaces and mesh size.

- As we are near the bend channels the 2D effects are dominants and we can not reduce model in this region .

- We studied the posterior error estimator to get idea about the errors without any knowledge about the exact solution.

- We validate numerically the upper bound of a posteriori error estimator (for a chosen β to be calclated after) .

Perispective:

- We can make an approximation of inf-sup condition to determine β.
- We will study the efficiency of the estimators.