Contrôlabilité pour le problème de Stefan

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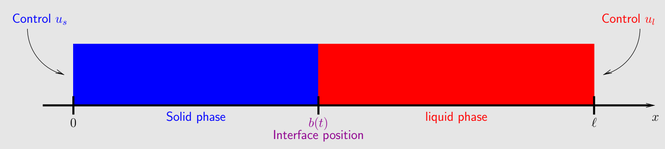
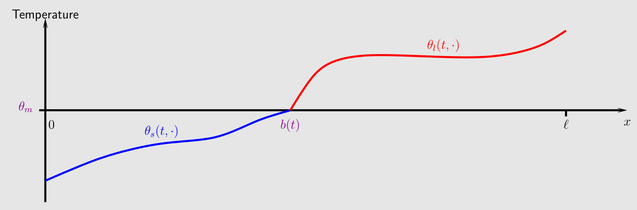
Introduction



In this poster, we are interested in the controllability of the Stefan problem, which is a mathematical model for solid-liquid phase transition, well-posedness of the problem has already been studied intensively, see *Fasano and Primicerio 1977* for example. Here, we present a steady state to steady state controllability result based on the flatness method which has been applied to the heat equation in *Martin et al. 2014* and to the Stefan problem in *Dunbar et al. 2003*.



Problem statement

If we denote by θ_s and θ_l the temperature in the two slabs (solid and liquid) and by b the position of the moving interface between the two, using dimensionless variables, the Stefan problem can be written as:

$$\dot\theta_s(t,x) = \partial_x^2\theta_s(t,x) \qquad (t>0,\ x\in(0,b(t))), \tag{1a}$$
$$\theta_s(t,b(t)) = 0 \qquad (t>0), \tag{1b}$$
$$\theta_s(t,0) = u_s(t) \qquad (t>0), \tag{1c}$$
$$\dot\theta_l(t,x) = c\,\partial_x^2\theta_l(t,x) \qquad (t>0,\ x\in(b(t),1)), \tag{2a}$$
$$\theta_l(t,b(t)) = 0 \qquad (t>0), \tag{2b}$$
$$\theta_l(t,1) = u_l(t) \qquad (t>0), \tag{2c}$$
$$\dot b(t) = \partial_x\theta_s(t,b(t)) - \partial_x\theta_l(t,b(t)) \qquad (t>0), \tag{3}$$

with some initial conditions,

$$b(0) = b^0, \tag{4a}$$
$$\theta_s(0,x) = \theta_s^0(x) \qquad (x\in(0,b^0)), \tag{4b}$$
$$\theta_l(0,x) = \theta_l^0(x) \qquad (x\in(b^0,1)), \tag{4c}$$

with the state constraints,

$$b(t)\in[0,1] \qquad (t>0), \tag{5a}$$
$$\theta_s(t,x)\leqslant 0 \qquad (t>0,\ x\in(0,b(t))), \tag{5b}$$
$$\theta_l(t,x)\geqslant 0 \qquad (t>0,\ x\in(b(t),1)), \tag{5c}$$

and with the control constraints,

$$u_s(t)\leqslant 0 \quad\text{and}\quad u_l(t)\geqslant 0 \qquad (t>0). \tag{6}$$

Let us mention that the steady states of this system are described by $\bar b\in(0,1)$ and a parameter $\bar v\in\mathbb{R}_+$, and given $\bar b\in(0,1)$ and $\bar v\in\mathbb{R}_+$, the associated steady state is given by

$$\bar u_s = -\bar v\bar b, \qquad \bar u_l = \bar v(1-\bar b),$$
$$\bar\theta_s(x) = \bar v(x-\bar b)\quad (x\in[0,\bar b]) \quad\text{and}\quad \bar\theta_l(x) = \bar v(x-\bar b)\quad (x\in[\bar b,1]). \tag{7}$$

Let us then define the set of steady states:

$$\mathcal{S}_+^*(\bar b) = \Big\{(\bar\theta_s,\bar\theta_l)\in\Big(H^2(0,\bar b)\cap H_R^1(0,\bar b)\Big)\times\Big(H^2(\bar b,1)\cap H_L^1(\bar b,1)\Big)\ \Big|$$
$$\exists\bar v\in\mathbb{R}_+^*,\ \forall x\in(0,\bar b),\ \theta_s(x)=\bar v(x-\bar b)\ \text{ and }\ \forall x\in(\bar b,1),\ \theta_l(x)=\bar v(x-\bar b)\Big\}. \tag{8a}$$

Main result: Steady state to steady state controllability theorem

Let $\bar b_0,\bar b_1\in(0,1)$, $(\bar\theta_s^0,\bar\theta_l^0)$ and $(\bar\theta_s^1,\bar\theta_l^1)$ be steady states, and let $\bar v_0,\bar v_1\in\mathbb{R}_+$ be such that

$$\bar\theta_s^0(x) = \bar v_0(x-\bar b_0)\quad (x\in[0,\bar b_0]) \quad\text{and}\quad \bar\theta_l^0(x) = \bar v_0(x-\bar b_0)\quad (x\in[\bar b_0,1])$$

and

$$\bar\theta_s^1(x) = \bar v_1(x-\bar b_1)\quad (x\in[0,\bar b_1]) \quad\text{and}\quad \bar\theta_l^1(x) = \bar v_1(x-\bar b_1)\quad (x\in[\bar b_1,1]).$$

Then for all $T>0$, there exists $u_s,u_l\in C^\infty(0,T)$, Gevrey, such that the solution of (1)–(3) is steered from $(\bar\theta_s^0,\bar\theta_l^0,\bar b^0)$ to $(\bar\theta_s^1,\bar\theta_l^1,\bar b^1)$ in time T.
In addition, if $(\bar\theta_s^0,\bar\theta_l^0)\in\mathcal{S}_+^*(\bar b^0)$ and $(\bar\theta_s^1,\bar\theta_l^1)\in\mathcal{S}_+^*(\bar b^1)$ (i.e., if $\bar v_0,\bar v_1\in\mathbb{R}_+^*$), then there exists $T>0$ and $u_s,u_l\in C^\infty(0,T)$, satisfying (5)-(6) such that the solution of (1)–(3) is steered from $(\bar\theta_s^0,\bar\theta_l^0,\bar b^0)$ to $(\bar\theta_s^1,\bar\theta_l^1,\bar b^1)$ in time T.

Gevrey functions

Let $n\geq 1$, $U\subset\mathbb{R}^n$ and $f\in\mathcal{C}^\infty(U,\mathbb{R})$, we say that f is Gevrey, if there exist $M\geq 0$, $R_1,\dots,R_n>0$ and $\sigma_1,\dots,\sigma_n\geq 0$ such that:

$$\forall x\in U,\ \forall p_1,\dots,p_n\in\mathbb{N},\qquad \left|\partial_{x_1}^{p_1}\dots\partial_{x_n}^{p_n}f(x)\right|\leq M\prod_{i=1}^n\frac{(p_i)!^{\sigma_i}}{R_i^{p_i}}$$

And if $p\in\mathbb{N}$ and $f\in\mathcal{C}^\infty(U,\mathbb{R}^p)$, we say that f is Gevrey if its coordinates are Gevrey in the above sense. If we are working on an interval I of $\mathbb{R}$, we denote $\mathcal{G}(M,R,\sigma)$ the set of Gevrey functions on I of order σ and constants $M,R\geq 0$

Estimate Result

Let $\chi\geq 0$, $I\subset\mathbb{R}$ an interval, $\sigma\in[1,2]$, $M_\alpha>0, M_\beta,M_f\geq 0$ and $R>0$. Let $f\in\mathcal{G}(M_f,R,\sigma)$, $\alpha_0\in\mathcal{G}(M_\alpha,R,\sigma)$ and $\beta_0\in\mathcal{G}(M_\beta,R,\sigma)$ be given Gevrey functions defined on I. Consider the sequence defined by

$$\begin{cases}\beta_{i+1} = \chi\dot\beta_i - \chi f\alpha_i,\\ \alpha_{i+1} = \chi\dot\alpha_i - \chi f\beta_{i+1},\end{cases}$$

where β_i and α_i are real functions defined on I (we initialize the sequences with α_0,β_0).
Then, for every $i\in\mathbb{N}$, α_i and β_i are Gevrey functions of order σ defined on I. In addition, for every $\rho\in(0,\rho^*(M,R,\chi)]$, we have, for every $l\in\mathbb{N}$ and every $i\in\mathbb{N}$,

$$\|\alpha_i^{(l)}\|_{L^\infty(I)}\leqslant\frac{M_\alpha\chi^i}{R^l\rho^i}\frac{(l+2i)!^\sigma}{i!^p(2i)!^{\sigma-1}} \quad\text{and}\quad \|\beta_{i+1}^{(l)}\|_{L^\infty(I)}\leqslant\frac{\mu\chi^{i+1}}{R^l\rho^i}\frac{(l+2i+1)!^\sigma}{i!^p(2i+1)!^{\sigma-1}}, \tag{9}$$

where we have set $\mu := \frac{M_\beta}{R}+M_\alpha M_f$, $p=2-\sigma\in[0,1]$ and

$$\rho^*(M,R,\chi) = \min\left\{\left(\frac{4+\chi M_\beta M_f/(2M_\alpha)}{R}+\frac{\chi M_f^2}{2}\right)^{-1},\frac{2R}{3}\right\}.$$

Step and bump functions

If $\sigma>1$ and $k:=(\sigma-1)^{-1}$ the function defined by:

$$\phi_\sigma(t) := \begin{cases}1 & \text{if } t\leq 0,\\ 0 & \text{if } t\geq 1,\\ \dfrac{e^{-(1-t)^{-k}}}{e^{-(1-t)^{-k}}+e^{-t^{-k}}} & \text{if } t\in(0,1),\end{cases}\qquad (t\in\mathbb{R}), \tag{10}$$

is Gevrey of order σ on $\mathbb{R}$ and verifies:

1. $\forall i\in\mathbb{N}^*, \phi_\sigma^{(i)}(0)=\phi_\sigma^{(i)}(1)=0$
2. $\forall t\in\mathbb{R}, 0\leq\phi_\sigma(t)\leq 1.$
3. $\forall t\in\mathbb{R}, \phi_\sigma(t)+\phi_\sigma(1-t)=1$

If $\sigma>1$ then, it exists η a Gevrey function of order σ on $\mathbb{R}$ such that:

1. η vanishes on $\mathbb{R}\setminus[0,1]$
2. $\eta\geq 0$
3. $\int_{\mathbb{R}}\eta(t)dt=1$

It is constructed using the step function.

Sketch of the proof of the main result

We are looking for solutions under the form:

$$\theta_s(t,x) = \sum_{i=0}^\infty\alpha_i^s(t)\frac{(x-b(t))^{2i+1}}{(2i+1)!}+\sum_{i=0}^\infty\beta_i^s(t)\partial_x\frac{(x-b(t))^{2i}}{(2i)!}, \tag{11a}$$
$$\theta_l(t,x) = \sum_{i=1}^\infty\alpha_i^l(t)\frac{(x-b(t))^{2i+1}}{(2i+1)!}+\sum_{i=0}^\infty\beta_i^l(t)\partial_x\frac{(x-b(t))^{2i}}{(2i)!}. \tag{11b}$$

The problem statement gives us the following conditions:

$$\begin{cases}\beta_{i+1}^s = \dot\beta_i^s - \dot b\alpha_i^s,\\ \alpha_{i+1}^s = \dot\alpha_i^s - \dot b\beta_{i+1}^s\end{cases}\quad\text{and}\quad\begin{cases}c\beta_{i+1}^l = \dot\beta_i^l - \dot b\alpha_i^l,\\ c\alpha_{i+1}^l = \dot\alpha_i^l - \dot b\beta_{i+1}^l\end{cases} \tag{12}$$

and

$$\dot b = \alpha_0^s - \alpha_0^l. \tag{13}$$

The previous estimate result tells us that if α_0^s and α_0^l are chosen in some $\mathcal{G}(M,R,\sigma)$ with $\sigma\in(1,2)$ then the previous series converge and are $\mathcal{C}^\infty$ and are in fact Gevrey (we necessarily have $\beta_0^s=\beta_0^l=0$).
If $b_1\geq b_0$ (else we switch the two), we set:

$$\alpha_0^l(t) := \alpha\left(\tfrac{t}{T}\right)\quad\text{and}\quad\alpha_0^s(t) := \alpha\left(\tfrac{t}{T}\right)+\frac{1}{T}\varphi\left(\tfrac{t}{T}\right)\qquad (t\in[0,T]). \tag{14}$$

with

$$\alpha(t) = \bar v_0\phi_\sigma(t)+\bar v_1\phi_\sigma(1-t)\quad\text{and}\quad\varphi(t) = (\bar b_1-\bar b_0)\eta(t)\qquad (t\in[0,1]),$$

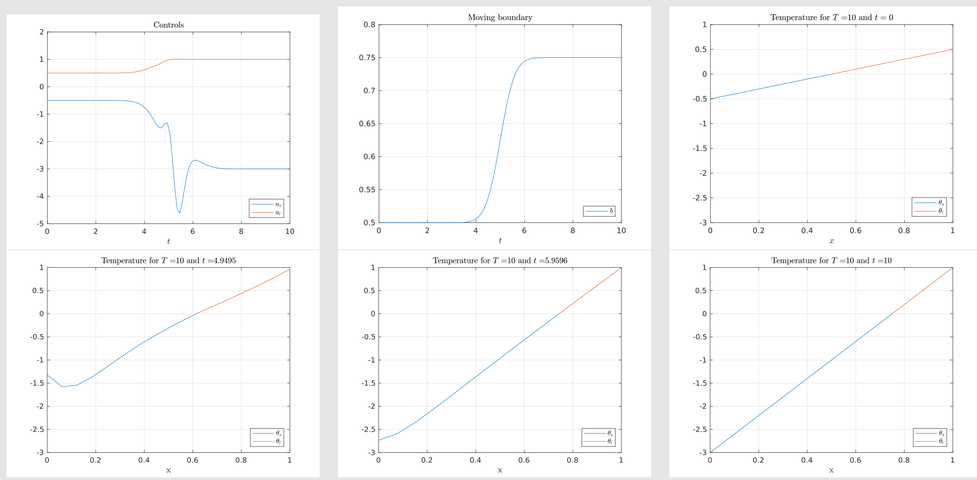
to have

- $\alpha_0^s(0)=\alpha_0^l(0)=\bar v_0$;
- $\alpha_0^s(T)=\alpha_0^l(T)=\bar v_1$;
- $\alpha_0^{s(i)}(0)=\alpha_0^{l(i)}(0)=\alpha_0^{s(i)}(T)=\alpha_0^{l(i)}(T)=0$ for every $i\in\mathbb{N}$;
- $\int_0^T\left(\alpha_0^s(t)-\alpha_0^l(t)\right)dt=\bar b_1-\bar b_0$.

And thus we designed a solution to our problem. We get u_s and u_l as the traces of our solutions.
To show the preservation of the sign constraints for large T we derive finer estimates on our series depending on T, using the construction of α_0^s and α_0^l and our estimation result.

Numerical aspect

The series defining the trajectories we designed in our result converge at a rate: $o\left(\frac{e^{-\frac{p}{2}N(\ln(N)-1)}}{(N+1)^{\frac{p}{2}}\ln(N)}\right)$.



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