Homogénéisation guidée par les données de l'équation de Langevin multi-échelle

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Motivation

Applications

- ▷ Multiscale diffusions suitable model for
  - oceanography
  - finance
  - ...
- ▷ Infer from data effective simple models for complex phenomena

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- ▷ Pre-processing needed to obtain effective dynamics
- ▷ Subsampling widely employed, but
  - requires knowledge of scale-separation
  - strongly dependent on rate
  - throws a lot of data

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Necessity for new method

- ▷ Consistence w.r.t. homogenization theory
- ▷ Robustness w.r.t. subsampling techniques
- ▷ Ease of applicability

Outline

Setting – The model

Multiscale Langevin SDE

$$\mathrm{d}X_t^\varepsilon = -\alpha \cdot V'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon} p'\left(\frac{X_t^\varepsilon}{\varepsilon}\right)\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}W_t$$

Parameters	Is it known?
- drift coefficient $\alpha \in \mathbb{R}^M$	✗
- slow potential $V\colon \mathbb{R} \to \mathbb{R}^M$	✓
- L-periodic fast potential $p\colon \mathbb{R} \to \mathbb{R}$	✗
- diffusion coefficient $\sigma > 0$	✗
- multiscale parameter $\varepsilon > 0$	✗

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$$\downarrow$$

<u>Remark:</u> we consider a semi-parametric framework

$$\alpha \cdot V(x) = \sum_{m=1}^{M} \alpha_m V_m(x)$$

indeed $\{V_m\}_{m=1}^M$ can be chosen as basis of appropriate function space

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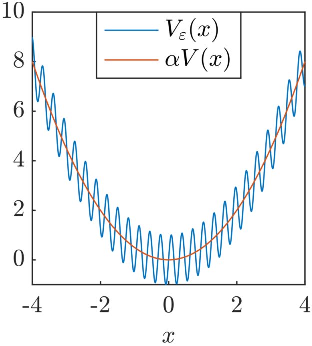

Motion in multiscale potential

$$V_\varepsilon(x) = \alpha V(x) + p(x/\varepsilon)$$

where:

- $V(x) = x^2/2$
- $\alpha = 1$
- $p(y) = \sin(y)$
- $\varepsilon = 0.05$

Setting – Homogenization

$X_t^\varepsilon \to X_t^0$ as $\varepsilon \to 0$ as random variables in $\mathcal{C}^0([0,T])$[4]

Homogenized Langevin SDE

$$\mathrm{d}X_t^0 = -A \cdot V'(X_t^0)\,\mathrm{d}t + \sqrt{2\Sigma}\,\mathrm{d}W_t$$

New parameters	Is it known?
- effective drift coefficient $A = K\alpha \in \mathbb{R}^M$	✗
- effective diffusion coefficient $\Sigma = K\sigma > 0$	✗

where $0 < K < 1$ dependent on p and σ and $K \to 1$ as $\sigma \to \infty$

[4] Bensoussan et al. (1978)

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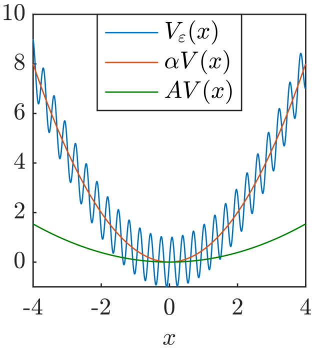

Effect of σ on the homogenized potential $AV(x)$ with

- $\sigma = 0.5 \implies K = 0.19$
- $V(x) = x^2/2$
- $\alpha = 1$
- $p(y) = \sin(y)$
- $\varepsilon = 0.05$
- $V_\varepsilon(x) = \alpha V(x) + p(x/\varepsilon)$

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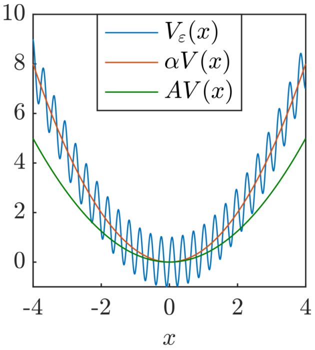

Effect of σ on the homogenized potential $AV(x)$ with

- $\sigma = 1 \quad \Longrightarrow \quad K = 0.62$
- $V(x) = x^2/2$
- $\alpha = 1$
- $p(y) = \sin(y)$
- $\varepsilon = 0.05$
- $V_\varepsilon(x) = \alpha V(x) + p(x/\varepsilon)$

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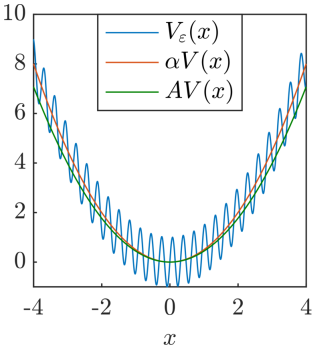

Effect of σ on the homogenized potential $AV(x)$ with

- $\sigma = 2 \quad \Longrightarrow \quad K = 0.88$
- $V(x) = x^2/2$
- $\alpha = 1$
- $p(y) = \sin(y)$
- $\varepsilon = 0.05$
- $V_\varepsilon(x) = \alpha V(x) + p(x/\varepsilon)$

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Setting – Parameter inference

$$\mathrm{d}X_t^\varepsilon = -\alpha \cdot V'(X_t^\varepsilon)\,\mathrm{d}t - \frac{1}{\varepsilon} p'\left(\frac{X_t^\varepsilon}{\varepsilon}\right)\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}W_t \qquad \longrightarrow \qquad \text{data}$$

$$\mathrm{d}X_t^0 = -A \cdot V'(X_t^0)\,\mathrm{d}t + \sqrt{2\Sigma}\,\mathrm{d}W_t \qquad \longrightarrow \qquad \text{model}$$

Goal: estimate $A \in \mathbb{R}^M$ and $\Sigma > 0$ given:

- continuous observations $X^\varepsilon = (X_t^\varepsilon, 0 \leq t \leq T)$
  - ▷ maximum likelihood estimator
  - ▷ quadratic variation
  - ▷ subsampling
  - ▷ filtering

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  - ▷ quadratic variation
  - ▷ subsampling
  - ▷ filtering
- discrete observations $\widetilde{X}^\varepsilon = (\widetilde{X}_n^\varepsilon = X_{n\Delta}^\varepsilon, n = 0, \dots, N, \Delta = T/N)$
  - ▷ martingale estimating functions
  - ▷ eigenvalues and eigenfunctions of the generator

Outline

Classic estimators ($M = 1$)

Assume to know continuous observations $X^\varepsilon = (X_t^\varepsilon, 0 \le t \le T)$

Drift: maximum likelihood estimator (MLE)

$$\widehat{A}_{\mathrm{MLE}}(X^\varepsilon, T) = -\frac{\int_0^T V'(X_t^\varepsilon)\,\mathrm{d}X_t^\varepsilon}{\int_0^T V'(X_t^\varepsilon)^2\,\mathrm{d}t} \qquad \text{(Girsanov formula)}$$

Diffusion: quadratic variation (partition $P = \{0 = t_0 < \cdots < t_{K_P} = T\}$)

$$\widehat{\Sigma}_{\mathrm{QV}}(X^\varepsilon, T) = \frac{\langle X^\varepsilon \rangle_T}{2T}, \qquad \langle X^\varepsilon \rangle_T = \lim_{\|P\| \to 0} \sum_{k=0}^{K_P - 1} (X_{t_{k+1}}^\varepsilon - X_{t_k}^\varepsilon)^2$$

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Issue: estimators are asymptotically biased[5]

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \widehat{A}_{\mathrm{MLE}}(X^\varepsilon, T) = \alpha \qquad a.s.$$

$$\widehat{\Sigma}_{\mathrm{QV}}(X^\varepsilon, T) = \sigma$$

[5] Pavliotis and Stuart (2007)

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Solution in literature: subsample the data with rate $\delta = \varepsilon^\zeta$, $\zeta \in (0,1)$[5]

$$\widehat{A}_{\mathrm{sub}}^\delta(X^\varepsilon, T) = -\frac{\sum_{n=0}^{N-1} V'(\widetilde{X}_n^\varepsilon)(\widetilde{X}_{n+1}^\varepsilon - \widetilde{X}_n^\varepsilon)}{\delta \sum_{n=0}^{N-1} V'(\widetilde{X}_n^\varepsilon)^2} \xrightarrow[\varepsilon \to 0]{T = \varepsilon^{-\gamma}} A \text{ in probability}$$

$$\widehat{\Sigma}_{\mathrm{sub}}^\delta(X^\varepsilon, T) = \frac{1}{2T} \sum_{n=0}^{N-1} (\widetilde{X}_{n+1}^\varepsilon - \widetilde{X}_n^\varepsilon)^2 \xrightarrow[\varepsilon \to 0]{T \text{ fixed}} \Sigma \text{ in probability}$$

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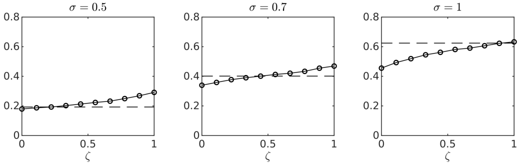
Warning: subsampling has several disadvantages:

- **not robust** with respect to subsampling rate δ
- knowledge of ε is required
- majority of data is wasted



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The filtered data approach

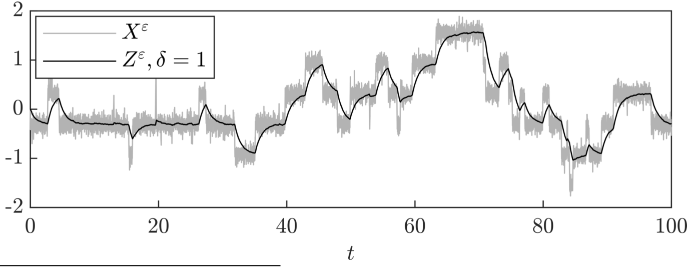
Idea: filter the data to cancel fast scale components[6]

$$Z_t^\varepsilon = \int_0^t k_{\exp}^{\delta,\beta}(t-s) X_s^\varepsilon \, \mathrm{d}s$$

where $\delta, \beta > 0$ and

$$k_{\exp}^{\delta,\beta}(r) = \frac{\beta}{\Gamma(1/\beta)\delta^{1/\beta}} e^{-\frac{r^\beta}{\delta}}$$

In the plot $\beta = 1$:



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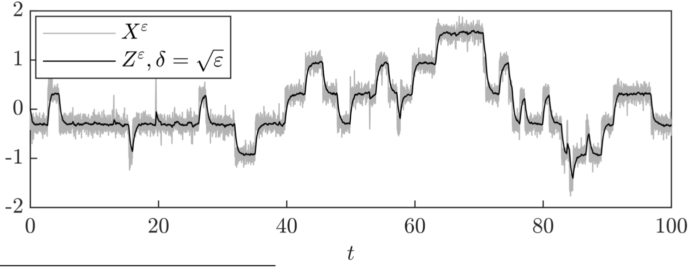

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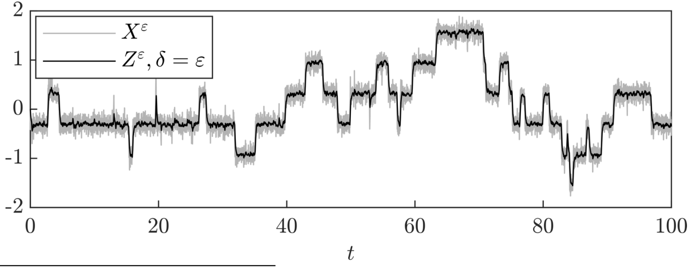

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The filtered data approach – The estimators

Drift: modification of the MLE

$$\widehat{A}^{\delta,\beta}_{\exp}(X^\varepsilon, T) = -\frac{\int_0^T V'(Z_t^\varepsilon)\,\mathrm{d}X_t^\varepsilon}{\int_0^T V'(Z_t^\varepsilon)V'(X_t^\varepsilon)\,\mathrm{d}t} \qquad (M=1)$$

Diffusion:

$$(i) \quad \widehat{\Sigma}^{\delta}_{\exp}(X^\varepsilon, T) = \frac{1}{\delta T}\int_0^T (X_t^\varepsilon - Z_t^\varepsilon)^2\,\mathrm{d}t \qquad (\beta = 1)$$

$$(ii) \quad \widetilde{\Sigma}^{\delta,\beta}_{\exp}(X^\varepsilon, T) = \frac{\widehat{A}^{\delta,\beta}_{\exp}(X^\varepsilon, T)}{\widehat{A}_{\mathrm{MLE}}(X^\varepsilon, T)}\widehat{\Sigma}_{\mathrm{QV}}(X^\varepsilon, T)$$

Why (ii)? $\quad \Sigma = K\sigma = \dfrac{A}{\alpha}\sigma$

The filtered data approach – Convergence analysis

Drift:

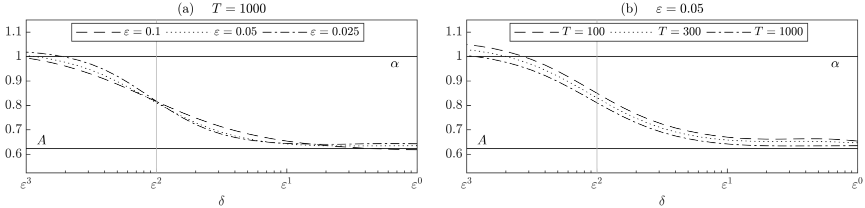
$$\widehat{A}^{\delta,\beta}_{\exp}(X^\varepsilon, T) = -\frac{\int_0^T V'(Z_t^\varepsilon)\,\mathrm{d}X_t^\varepsilon}{\int_0^T V'(Z_t^\varepsilon)V'(X_t^\varepsilon)\,\mathrm{d}t} \qquad (M=1)$$

Theorem ($\beta = 1$)

$$(i)\ \lim_{\varepsilon\to 0}\lim_{T\to\infty} \widehat{A}^{\delta,1}_{\exp}(X^\varepsilon, T) = A, \quad a.s., \quad \delta \textit{ independent of } \varepsilon$$

$$(ii)\ \lim_{\varepsilon\to 0}\lim_{T\to\infty} \widehat{A}^{\delta,1}_{\exp}(X^\varepsilon, T) = A, \quad a.s., \quad \delta = \varepsilon^\zeta,\ 0<\zeta<2$$

$$(iii)\ \lim_{\varepsilon\to 0}\lim_{T\to\infty} \widehat{A}^{\delta,1}_{\exp}(X^\varepsilon, T) = \alpha, \quad a.s., \quad \delta = \varepsilon^\zeta,\ \zeta>2$$



The filtered data approach – Convergence analysis

Diffusion:

$$\widehat{\Sigma}^{\delta}_{\text{exp}}(X^{\varepsilon}, T) = \frac{1}{\delta T} \int_0^T (X_t^{\varepsilon} - Z_t^{\varepsilon})^2 \, \mathrm{d}t$$

Theorem

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \widehat{\Sigma}^{\delta}_{\text{exp}}(X^{\varepsilon}, T) = \Sigma, \quad a.s., \quad \delta = \varepsilon^{\zeta}, \; 0 < \zeta < 2$$

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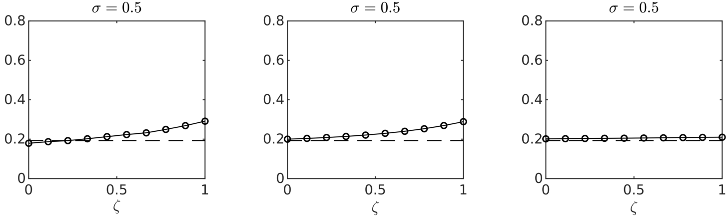
Numerical experiments – Filtering vs Subsampling

Setting: drift estimation for Ornstein–Uhlenbeck process

$$V(x) = \frac{x^2}{2},\ p(y) = \cos(y),\ \alpha = 1,\ \varepsilon = 0.1,\ T = 1000,\ \delta = \varepsilon^\zeta$$

Subsampling | Filtering, $\beta = 1$ | Filtering, $\beta = 5$



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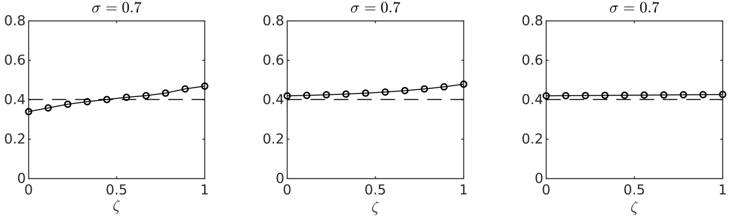

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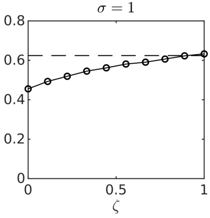
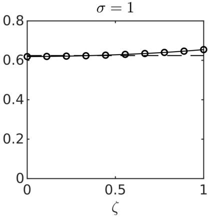
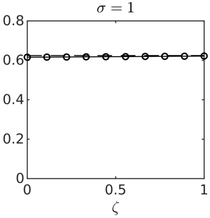


Filtering, $\beta = 1$



Filtering, $\beta = 5$



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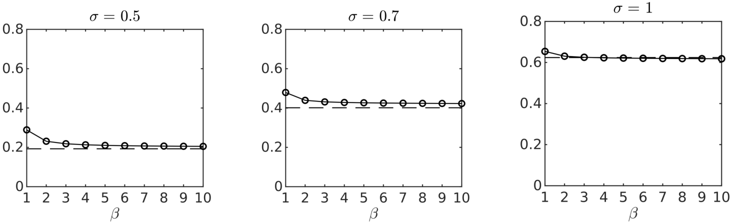
Sensitivity analysis w.r.t. β ($\delta = \varepsilon$)

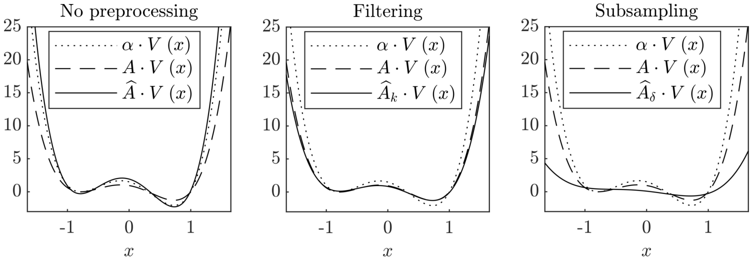


Numerical experiments – Multidimensional drift

Setting: let $T_m(x)$ be Chebyshev polynomials

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x)$$

- Drift estimation for $V_i(x) = T_i(x)$, $i = 1, \dots, 4$ and $p(y) = \cos(y)$, $\varepsilon = 0.05$, $T = 1000$
- Filtering with $\delta = 1, \beta = 1$
- Subsampling with $\delta = \varepsilon^{2/3}$

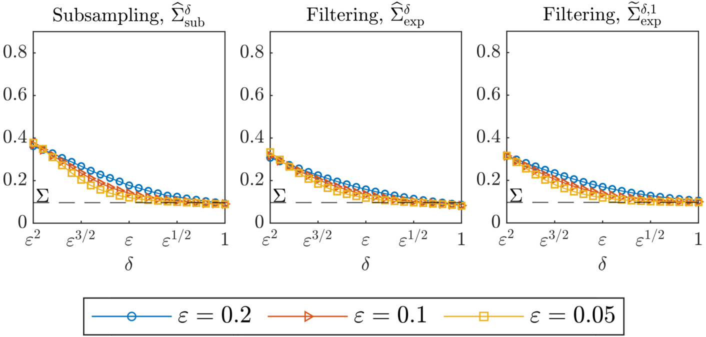


Numerical experiments – Diffusion coefficient

Setting: diffusion estimation for Ornstein–Uhlenbeck process

$$V(x) = \frac{x^2}{2},\ p(y) = \cos(y),\ \alpha = 1,\ T = 1000$$

$$\sigma = 0.5$$

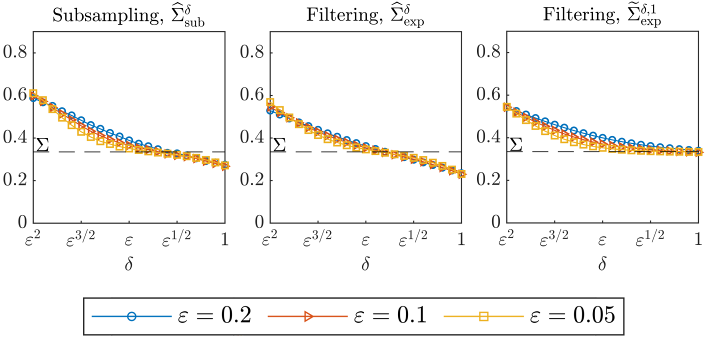


Numerical experiments – Diffusion coefficient

Setting: diffusion estimation for Ornstein–Uhlenbeck process

$$V(x) = \frac{x^2}{2},\ p(y) = \cos(y),\ \alpha = 1,\ T = 1000$$

$$\sigma = 0.75$$



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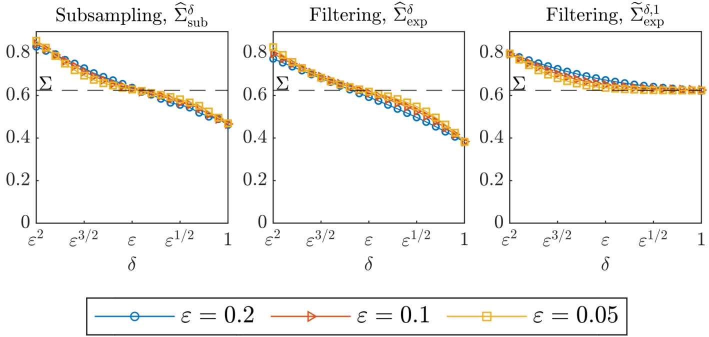

Outline

Martingale estimating functions

Assume to know discrete observations $\{\widetilde{X}_n^\varepsilon\}_{n=0}^N, \widetilde{X}_n^\varepsilon = X_{n\Delta}^\varepsilon, \Delta = T/N$

Issue: **discretization** of MLE is **biased** if Δ independent of ε

Martingale estimating functions

Assume to know discrete observations $\{\widetilde{X}_n^\varepsilon\}_{n=0}^N, \widetilde{X}_n^\varepsilon = X_{n\Delta}^\varepsilon, \Delta = T/N$

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Idea: employ martingale estimating functions based on eigenvalues and eigenfunctions of the generator of the homogenized dynamics[7]

Homogenized dynamics:	$\mathrm{d}X_t^0 = -a \cdot V'(X_t^0)\,\mathrm{d}t + \sqrt{2s}\mathrm{d}W_t$
Generator:	$\mathcal{L}_{(a,s)}u(x) = -a \cdot V'(x)u'(x) + su''(x)$
Eigenvalue problem:	$-\mathcal{L}_{(a,s)}\phi_j(x;a,s) = \lambda_j(a,s)\phi_j(x;a,s)$
Smooth functions:	$\beta_j(\cdot;a,s) \colon \mathbb{R} \to \mathbb{R}^{M+1}$

[7] Kessler and Sørensen (1999); Abdulle et al. (2022)

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If Δ small: combine previous filtering methodology

$$\widetilde{Z}_n^\varepsilon = \Delta \sum_{k=0}^{n-1} e^{-\Delta(n-k)} \widetilde{X}_k^\varepsilon \qquad (\delta = 1,\ \beta = 1)$$

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Martingale estimating functions – The estimators

Estimator implicitly defined from nonlinear system of dimension $M+1$

Without filtered data: $\widehat{G}^{\varepsilon}_{N,J}(a,s) = 0 \;\Rightarrow\; \widehat{A}^{J}_{\text{eigen}}(\widetilde{X}^{\varepsilon}, N), \widehat{\Sigma}^{J}_{\text{eigen}}(\widetilde{X}^{\varepsilon}, N)$

$$\widehat{G}^{\varepsilon}_{N,J}(a,s) = \frac{1}{\Delta}\sum_{n=0}^{N-1}\sum_{j=1}^{J}\left\{\beta_j(\widetilde{X}^{\varepsilon}_n; a, s)\left(\phi_j(\widetilde{X}^{\varepsilon}_{n+1}; a, s) - e^{-\lambda_j(a,s)\Delta}\phi_j(\widetilde{X}^{\varepsilon}_n; a, s)\right)\right\}$$

With filtered data: $\widetilde{G}^{\varepsilon}_{N,J}(a,s) = 0 \;\Rightarrow\; \widetilde{A}^{J}_{\text{eigen}}(\widetilde{X}^{\varepsilon}, N), \widehat{\Sigma}^{J}_{\text{eigen}}(\widetilde{X}^{\varepsilon}, N)$

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Why this definition? $\quad \mathbb{E}\left[\phi_j(\widetilde{X}^{0}_{n+1}; A,\Sigma)\middle|\widetilde{X}^{0}_n\right] = e^{-\lambda_j(A,\Sigma)\Delta}\phi_j(\widetilde{X}^{0}_n; A,\Sigma)$

$$\Longrightarrow \mathbb{E}\left[\widehat{G}^{0}_{N,J}(A,\Sigma)\right] = \mathbb{E}\left[\widetilde{G}^{0}_{N,J}(A,\Sigma)\right] = 0$$

Martingale estimating functions – Convergence analysis

Theorem (Without filtered data)

If Δ is independent of ε or $\Delta = \varepsilon^{\zeta}$ with $0 < \zeta < 1$ then

$$(i)\ \lim_{\varepsilon \to 0} \lim_{N \to \infty} \widehat{A}^{J}_{\text{eigen}}(X^{\varepsilon}, N) = A, \quad \textit{in probability}$$

$$(ii)\ \lim_{\varepsilon \to 0} \lim_{N \to \infty} \widehat{\Sigma}^{J}_{\text{eigen}}(X^{\varepsilon}, N) = \Sigma, \quad \textit{in probability}$$

Martingale estimating functions – Convergence analysis

Theorem (Without filtered data)

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Theorem (With filtered data)

If Δ *is independent of* ε *or* $\Delta = \varepsilon^\zeta$ *with* $\zeta > 0, \zeta \neq 1, \zeta \neq 2$ *then*

$$(i) \lim_{\varepsilon \to 0} \lim_{N \to \infty} \widetilde{A}^J_{\text{eigen}}(X^\varepsilon, N) = A, \quad \textit{in probability}$$

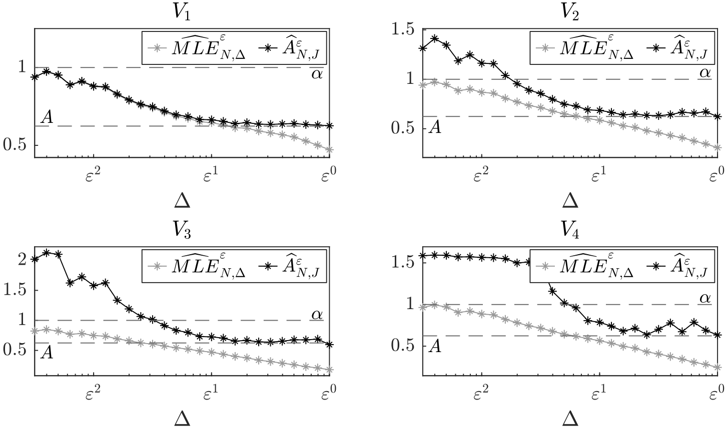
$$(ii) \lim_{\varepsilon \to 0} \lim_{N \to \infty} \widetilde{\Sigma}^J_{\text{eigen}}(X^\varepsilon, N) = \Sigma, \quad \textit{in probability}$$

Numerical experiments – Comparison estimators

Setting: drift estimation for polynomial potentials

$$V_1(x) = \frac{x^2}{2}, \quad V_2(x) = \frac{x^4}{4}, \quad V_3(x) = \frac{x^6}{6}, \quad V_4(x) = \frac{x^4}{4} - \frac{x^2}{2}$$

$$p(y) = \cos(y),\ \alpha = 1,\ \sigma = 1,\ \varepsilon = 0.1,\ T = 1000,\ J = 1$$



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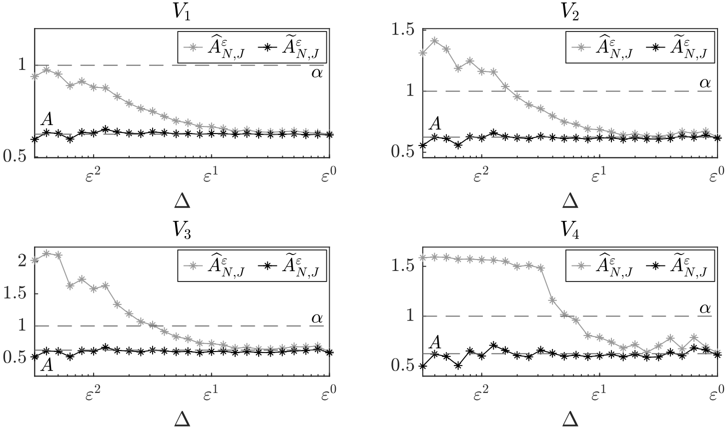

Numerical experiments – Sensitivity analysis w.r.t. J

Setting: drift estimation for polynomial potentials

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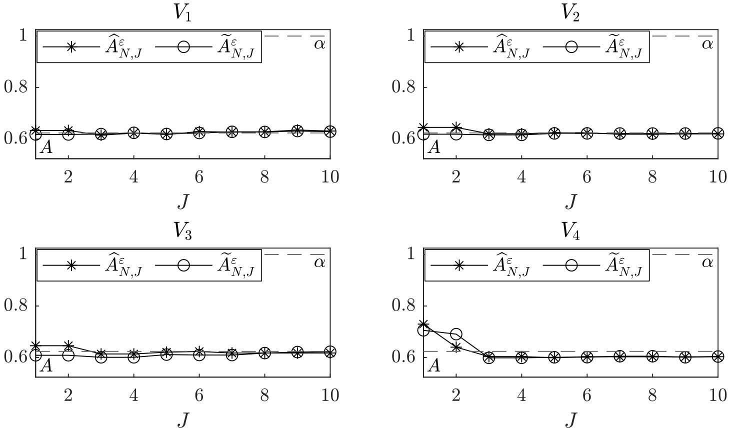


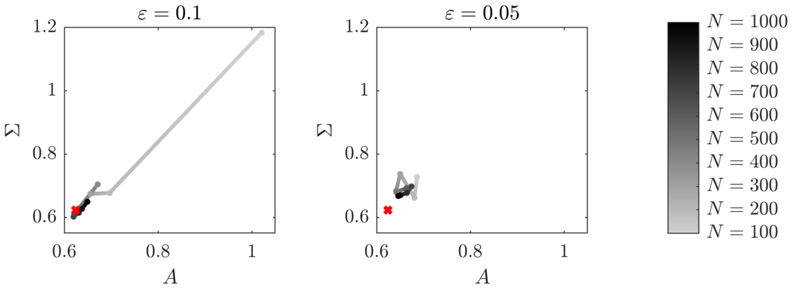
Numerical experiments – Multidimensional parameter

Setting: drift and diffusion estimation for Ornstein–Uhlenbeck process

$$V(x) = \frac{x^2}{2},\ p(y) = \cos(y),\ \alpha = 1,\ \sigma = 1$$

$$\varepsilon = 0.1,\ T = 1000,\ J = 2,\ \Delta = 1$$

Estimator $\left(\widehat{A}^J_{\text{eigen}}(X^\varepsilon, N), \widehat{\Sigma}^J_{\text{eigen}}(X^\varepsilon, N)\right)$ without filtered data



Conclusions

- We considered the problem of fitting effective dynamics from continuous and discrete multiscale data (Langevin SDE)

- We proposed novel estimators for the effective drift and diffusion coefficients based on filtered data and eigenpairs of the generator of the homogenized dynamics

- We proved the asymptotic unbiasedness of our estimators

- We showed numerically that our estimators outperform traditional techniques like subsampling both in terms of accuracy and robustness

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