Singular Perturbated Problems and Julia Package in Optimal Control

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in collaboration with :

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45eme Congrès National d'Analyse Numérique
13-17 June 2022, Evians-les-Bains.

Université Fédérale
Toulouse Midi-Pyrénées

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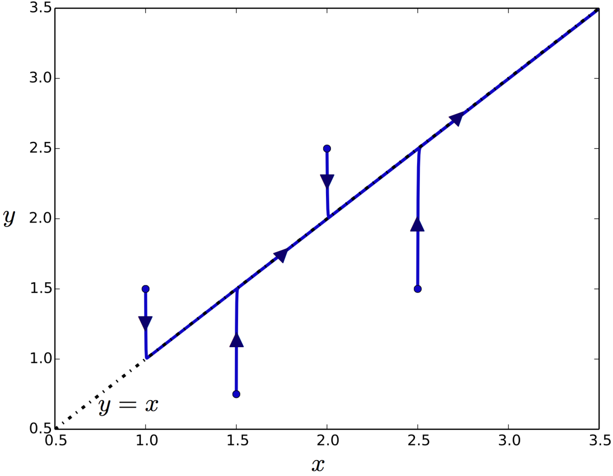
$$\begin{cases} \dot{x}(t) &= x(t), \qquad\qquad x(0) = x_0 \\ \varepsilon \dot{y}(t) &= x(t) - y(t), \quad y(0) = y_0 \end{cases}$$



- Problem of interest:

$$(P_\varepsilon)\begin{cases} \min \displaystyle\int_0^1 f^0(x(t),y(t),u(t))\,dt \\ \dot{x}(t) = f(x(t),y(t),u(t)), \quad x(t)\in\mathbb{R}^n, \quad x(0),\ x(1) \quad \textit{given} \\ \varepsilon\dot{y}(t) = g(x(t),y(t),u(t)), \quad y(t)\in\mathbb{R}^m, \quad y(0),\ y(1) \quad \textit{given} \end{cases}$$

where x, y are resp. **slow** and **fast** variables since $\varepsilon > 0$ is supposed to be **small** and where $u(t) \in \mathbb{R}^k$.

- Setting $\varepsilon = 0$, we define the *zero order reduced problem*:

$$(P_0)\begin{cases} \min \int_0^1 f^0(\overline{x}(t),\overline{y}(t),\overline{u}(t))\,dt \\ \dot{\overline{x}}(t) = f(\overline{x}(t),\overline{y}(t),\overline{u}(t)), \quad \overline{x}(0)=x(0), \quad \overline{x}(1)=x(1), \\ \quad 0 = g(\overline{x}(t),\overline{y}(t),\overline{u}(t)). \end{cases}$$

- Roughly speaking and under suitable assumptions the main result is:

$$x_\varepsilon(t) \to \overline{x}(t) \textit{ on } [0,1] \text{ and } y_\varepsilon(t) \to \overline{y}(t) \text{ on every } [a,b] \subset (0,1), \text{ when } \varepsilon \to 0.$$

- We'll first introduce the **turnpike framework** and show the link with **singularly perturbed optimal control problems**;

- Then we'll combine the ideas developed in both approaches (turnpike property: see Trélat and Zuazua [4] and singular perturbation theory: see Khalil [2]) **and propose a path following approach to provide a more efficient numerical resolution method**;

- Finally we'll present **the implementation in Julia** and some **numerical results**.

- Let's consider the optimal control problem

$$(OCP_{t_f})\begin{cases} \min \displaystyle\int_0^{t_f} f^0(y(t),u(t))\,dt, & t_f > 0 \textbf{ large enough} \\ \dot{y}(t) = f(y(t),u(t)), \quad y(t)\in\mathbb{R}^m, \quad u(t)\in\mathbb{R}^k, \\ y(0)=y_0, \quad y(t_f)=y_f. \end{cases}$$

- The associated reduced problem (or **static** optimal control problem) is

$$(SOCP_{t_f}) \qquad \min_{(y,u)\in\mathbb{R}^m\times\mathbb{R}^k} f^0(y,u) \quad \text{s.t.} \quad f(y,u)=0.$$

Turnpike property (Trélat and Zuazua [4]): under suitable assumptions, the optimal solution $(y_{t_f}(\cdot), u_{t_f}(\cdot))$ of $(OCP)_{t_f}$ remains most of the time **close to the static solution** $(\overline{y},\overline{u})$, i.e there exists positive constants C_1, C_2 such that

$$\|y_{t_f}(t)-\overline{y}\| + \|u_{t_f}(t)-\overline{u}\| \le C_1\left(e^{-C_2 t} + e^{-C_2(t_f-t)}\right) \tag{1}$$

for every $t\in[0,t_f]$.

$$\begin{cases} \min \dfrac{1}{2}\displaystyle\int_0^{t_f} \left[(y_1(t)-1)^2 + (y_2(t)-1)^2 + (u(t)-2)^2\right] dt, & t_f = 20, \\ \dot{y}_1(t) = y_2(t), & (y_1(0), y_1(t_f)) = (1,3) \\ \dot{y}_2(t) = 1 - y_1(t) + y_2^3(t) + u(t), & (y_2(0), y_2(t_f)) = (1,0) \end{cases}$$

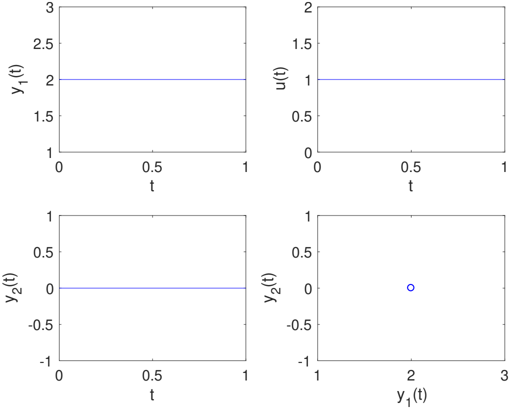


Figure 1: (Blue) Static solution: $(\overline{y}_1, \overline{y}_2, \overline{u}) = (2, 0, 1)$.

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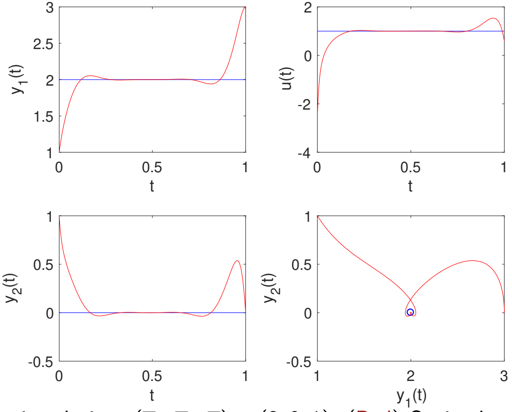

Figure 1: (Blue) Static solution: $(\overline{y}_1, \overline{y}_2, \overline{u}) = (2, 0, 1)$. (Red) Optimal solution computed by HamPath code.

- Setting $\tau = \varepsilon t$ with $\varepsilon = 1/t_f$, $(OCP)_{t_f}$ becomes

$$
(OCP_\varepsilon)\begin{cases}
\min\, t_f \displaystyle\int_0^1 f^0(y(\tau), u(\tau))\, d\tau, \\
\dot{y}(\tau) = f(y(\tau), u(\tau))t_f \iff \varepsilon\dot{y}(\tau) = f(y(\tau), u(\tau)) \\
y(0) = y_0, \quad y(1) = y_f.
\end{cases}
$$

- Thus: Turnpike control problems $\Leftrightarrow$ singular perturbation control problems with only fast variables.

Resolution of Optimal Control Problems by indirect method

Définition 1 – Pseudo-Hamiltonian

The pseudo-Hamiltonian is the function

$$\begin{array}{rccl} H: & \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k & \longrightarrow & \mathbb{R} \\ & (y, q, u) & \longmapsto & H(y, q, u) = -f^0(y, u) + \langle q, f(y, u) \rangle, \end{array}$$

where $\langle ., . \rangle$ is the dot product.

Theorem 2 – Pontryagin's Maximum Principle

Under classical assumptions, if (y, u) is a solution of (OCP_ε), then there exists an absolute continuous function called co-state q such that we have

- the co-state equation

$$\varepsilon \dot{q}(\tau) = -\frac{\partial H}{\partial y}(y(\tau), q(\tau), u(\tau)) \tag{2}$$

- The maximization of the pseudo-Hamiltonian

$$u(\tau) = \arg\max_{v \in \mathbb{R}^k} H(y(\tau), q(\tau), v)$$

- We suppose that the maximization of the pseudo-Hamiltonian can be analytically solved $u(y(\tau), q(\tau))$
- We call true Hamiltonian the function $\mathcal{H}(z) = \mathcal{H}(y, q) = H(y, q, u(y, q))$
- We note also

$$\vec{\mathcal{H}}(z) = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial q}(y, q) \\ -\frac{\partial \mathcal{H}}{\partial y}(y, q) \end{pmatrix}$$

-

$$\varepsilon \dot{z}(\tau) = \begin{pmatrix} \frac{\partial H}{\partial q}(y(\tau), q(\tau), u(\tau)) \\ -\frac{\partial H}{\partial y}(y(\tau), q(\tau), u(\tau)) \end{pmatrix} = \vec{\mathcal{H}}(z(\tau))$$

(OCP_ε) →

Boundary value Problem

$$(BVP_\varepsilon) \begin{cases} \varepsilon \dot{z}(\tau) = \vec{\mathcal{H}}(z(\tau)) \\ y(0) = y_0 \\ y(1) = y_f \end{cases}$$

→

Find a zero of
the shooting function
$S_\varepsilon(q_0) = y(1, q_0) - y_f$
with $y(1, q_0)$ solution of

$$(IVP_\varepsilon) \begin{cases} \varepsilon \dot{z}(\tau) = \vec{\mathcal{H}}(z(\tau)) \\ y(0) = y_0 \\ q(0) = q_0 \end{cases}$$

- **Goal:** Solve (OCP_ε) for ε **small**.

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- **Methodology:**
  - Step 1: **Resolution of the KKT** conditions of the static problem;
  - Step 2: **Continuation on the boundary conditions** for sufficiently large ε ;
  - Step 3: **Continuation on** ε.

- We define the shooting homotopic function by

$$
\begin{array}{rcl}
S:\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R} & \to & \mathbb{R}^{2m}\\
(q_0,q_1,\lambda) & \mapsto & S(q_0,q_1,\lambda)=z(0.5,0.,z_0)-z(0.5,1.,z_1)
\end{array}
$$

where $z(0.5,0.,z_0)$ and $z(0.5,1.,z_1)$ are the solutions at $\tau=0.5$ of

$$
(IVP_{\varepsilon,\lambda,0})\begin{cases}\varepsilon\dot z(\tau)=\vec{\mathcal{H}}(z(\tau))\\ z(0)=\begin{pmatrix}\lambda y_0+(1-\lambda)\overline{y}\\ q_0\end{pmatrix}\end{cases}\qquad (IVP_{\varepsilon,\lambda,1})\begin{cases}\varepsilon\dot z(\tau)=\vec{\mathcal{H}}(z(\tau))\\ z(1)=\begin{pmatrix}\lambda y_f+(1-\lambda)\overline{y}\\ q_1\end{pmatrix}\end{cases}.
$$

Remarque 1.

- For $\lambda=0$ the solution is the solution of the static problem $SOCP_{t_f}$: $(y(\tau),q(\tau))=(\bar y,\bar q)$ ($\bar q$ is the Lagrange multiplier).
- For $\lambda=1$ the solution is the solution of the (BVP_ε).

Now, we have to compute the path of zeros of $S(q_0,q_1,\lambda)=0$

How to compute the path of zeros of a homotopy function

$$\begin{array}{rccc} F\colon & \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ & (x, \lambda) & \longmapsto & F(x, \lambda) \end{array}$$

Theorem 3

Under the assumptions

i) For all $(x, \lambda) \in F^{-1}(0)$, $rank(F'(x, \lambda)) = n$

ii) For all $(x, 0) \in F^{-1}(0)$, $rank(\frac{\partial F}{\partial x}(x, 0)) = n$ and for all $(x, 1) \in F^{-1}(0)$, $rank(\frac{\partial F}{\partial x}(x, 1)) = n$

$F^{-1}(0)$ is a set of curves (a manifold of dimension 1)

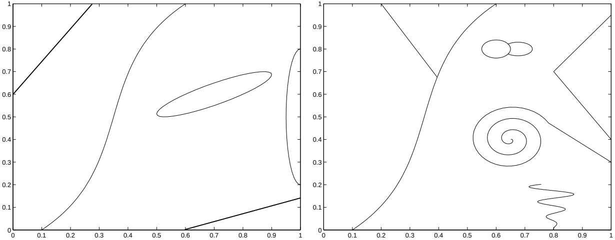


Figure 2: $F^{-1}(0)$, *possible path (left) and impossible (right) (z is in x-axis and λ in y-axis).*

Theorem 4

If $c(s) = (x(s), \lambda(s))$ is a smooth curve parametrized by the arc length such that

i) $c(0) = (x_0, 0)$

ii) $F(c(s)) = 0$

iii) $rank(F'(c(s))) = n$

iv) $\dot{c}(s) \neq 0$

then the tangent vector $\dot{c}(s) = T(c(s))$ is defined by

i) $F'(c(s))\dot{c}(s) = 0$

ii) $|\dot{c}(s)|=1$

iii) $\operatorname{sign}\det \begin{pmatrix} F'(c(s)) \\ \dot{c}(s)^T \end{pmatrix} > 0$

The smooth curve can be computed by integration of the Initial Value Problem

$$(IVP) \begin{cases} \dot{c}(s) = T(c(s)) \\ c(0) = (x_0, 0). \end{cases}$$

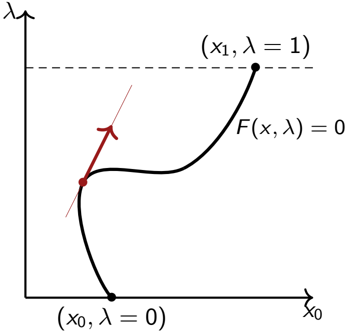
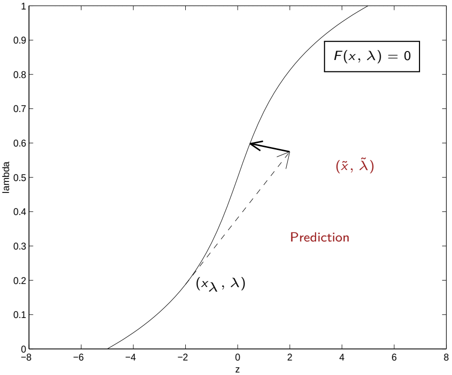


Figure 3: Illustration of the homotopy $F(x, \lambda) = 0$

- Prediction: Euler step

$$(\tilde{x}, \tilde{\lambda}) = (x_\lambda, \lambda) + \delta s T(F'(x_\lambda, \lambda))$$

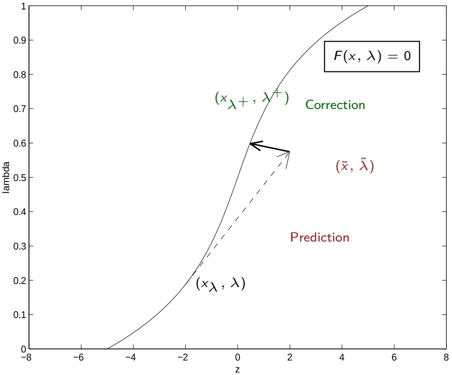


- Prediction: Euler step

$$(\tilde{x}, \tilde{\lambda}) = (x_\lambda, \lambda) + \delta s T(F'(x_\lambda, \lambda))$$

- Correction: $(x_{\lambda^+}, \lambda^+)$ solution of

$$\begin{cases} \min \{|(x,\lambda) - (\tilde{x}, \tilde{\lambda})|^2\} \\ s.t.\ F(x,\lambda) = 0 \end{cases}$$



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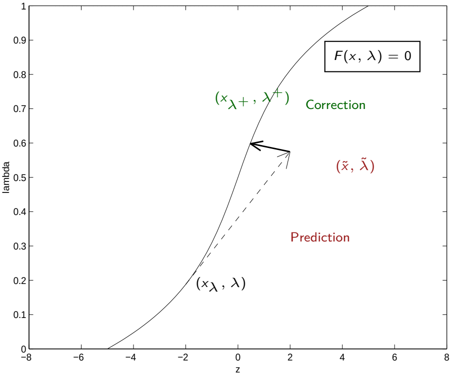
- until $\lambda = 1$.



True Hamiltonian

$\mathcal{H}(y, q)$

$\rightarrow$

Automatic differentiation:

$$\vec{\mathcal{H}}(y, q) = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial q}(y, q) \\ -\frac{\partial \mathcal{H}}{\partial y}(y, q) \end{pmatrix}$$

$\rightarrow$

Numerical integration:
Flow $z(\tau_f, \tau_0, z_0)$
solution at τ_f of

$$(IVP) \begin{cases} \varepsilon \dot{z}(\tau) = \vec{\mathcal{H}}(z(\tau)) \\ z(\tau_0) = z_0 \end{cases}$$

https://ct.gitlabpages.inria.fr/gallery/homotopy-julia/FGS.html ou
http://localhost:8888/lab

The following diagram

$$
\begin{tikzcd}[column sep=huge, row sep=large]
(IVP)\begin{cases} \varepsilon\dot{z}(\tau) = \vec{\mathcal{H}}(z(\tau)) \\ z(0) = z_0 \end{cases} \arrow[r, "\text{Numerical integration}"] \arrow[d, "\text{Automatic}"', "\text{Differentiation}"] & \begin{array}{c}\text{Flow of the } (IVP) \\ z(t_f, z_0)\end{array} \arrow[d, "\text{Automatic}"', "\text{differentiation}"] \\
(VAR)\begin{cases} \varepsilon\dot{z}(\tau) = \vec{\mathcal{H}}(z(\tau)) \\ \dot{\widehat{\delta z}}(t) = \frac{\partial \vec{\mathcal{H}}}{\partial z}(z(t))\delta z(t) \\ z(0) = z_0 \\ \delta z(0) = I \end{cases} \arrow[r, "\text{Numerical integration}"] & \delta z(t_f) = \frac{\partial z}{\partial z_0}(t_f, z_0)
\end{tikzcd}
$$

commutes if we use Runge-Kutta algorithm with variable steps in the case where the step control is only on the z variable (not on the δz for the (VAR) equations)

- What does exactly the `ForwardDiff` package on the flow in Julia?
- How can we implement in Julia the control on the z variable only for the numerical integration of the variationnal equation?

- Thanks to Homotopy method to obtain the numerical solutions.
- Generalization to singularly perturbed optimal control problems

$$(P_\varepsilon)\begin{cases} \min \int_0^1 f^0(x(t),y(t),u(t))\,dt \\ \dot{x}(t) = f(x(t),y(t),u(t)), \quad x(t)\in\mathbb{R}^n, \quad x(0),\ x(1) \quad \textit{given} \\ \varepsilon\dot{y}(t) = g(x(t),y(t),u(t)), \quad y(t)\in\mathbb{R}^m, \quad y(0),\ y(1) \quad \textit{given} \end{cases}$$

- Implementation of multiple shooting for solving
  - Optimal control problems with Bang-Bang solution
  - Optimal control problems with singular arcs
- Used a stiff integrator to compute the shooting function
- Numerical comparisons with codes for solving stiff Boundary Value Problem : `COLNEW` from U. Ascher and al., `HAGRON` from J. R. Cash and M. H. Wright
- Comparison with direct methods

[1] J. R. Cash and M. H. Wright, A Differed correction method for nonlinear two-points boundary value problems: implementation and numerical evaluation, SIAM Journal on Scientific Computing, Vol. 12 (1991), no. 4, 971–989.

[2] H. K. Khalil Non-Linear systems, Prentice-Hall: Upper Saddle River, second edn 1996.

[3] P.V. Kokotovic, H. K. Khalil and J. O'Reilly, Singular Perturbation Methods in Control, Birkhäuser, Mathematics: Theory and Applications, second edn 1988.

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