Approximation of elliptic PDE solutions with high contrast diffusion coefficients

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Motivation

Consider $u(y)$ solution to

$$\begin{cases} -\operatorname{div}(a\nabla u) & = & f & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{cases}$$

with

$$a(y) = \bar{a} + \sum_{j=1}^{d} y_j \,\psi_j$$

Model order reduction : Approximate all $u(y)$, $y \in Y \subset \mathbb{R}^d$, by functions from $V_n \subset H_0^1(\Omega)$ with $\dim(V_n) = n$

Reduced bases $V_n = \operatorname{Span}\{u(y^1), \dots, u(y^n)\}$

Rate of convergence of $u_n(y) = P_{V_n} u(y)$ towards $u(y)$?

Setting

Parameter domain $Y \subset \mathbb{R}^d$, solution manifold

$$\mathcal{M} := \{u(y), y \in Y\}$$

included in $V := H_0^1(\Omega)$

Kolmogorov widths

$$d_n(\mathcal{M})_V = \inf_{\dim V_n = n} \sup_{u \in \mathcal{M}} \inf_{v \in V_n} \|u - v\|_V$$

Goal

Approximate all $u \in \mathcal{M}$ by some space $V_n \subset V$

$$\forall u \in \mathcal{M}, \|u - u_n\|_{H_0^1(\Omega)} \leqslant \varepsilon_n \Longleftrightarrow d_n(\mathcal{M})_V \leqslant \varepsilon_n$$

Exponential convergence in bounded contrast

Theorem (Bachmayr, Cohen, 2017)

Let $Y = [-1,1]^d$ and $a(y) = \bar{a} + \sum_{j=1}^{d} y_j \, \psi_j$ such that $\sum_{j=1}^{d} |\psi_j(x)| \leqslant \delta \, \bar{a}(x)$

For all $k \geqslant 0$, there exists V_n of dimension $n \sim k^d$ such that

$$\Big\| u(y) - u_n(y) \Big\|_{H_0^1} \leqslant \|f\|_{H^{-1}} \, \delta^k, \quad y \in Y$$

Uniform Ellipticity Assumption : $\forall x \in \Omega, \ \forall y \in Y, \ r \leqslant a(x,y) \leqslant R$

Corollary

Under (UEA), for $\delta = \frac{R-r}{R+r}$, *we get* $\|u - u_n\|_{H_0^1} \leqslant C e^{-n^{1/d}}$

Polynomial rates in large dimension (Beck et al. 2010, Tran et al. 2017)

Sketch of proof

Define $\bar{A}, \Psi_j : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\langle \bar{A}u, v \rangle_{H^{-1},H_0^1} = \int_\Omega \bar{a}\, \nabla u \cdot \nabla v$$

and

$$\langle \Psi_j u, v \rangle_{H^{-1},H_0^1} = \int_\Omega \psi_j\, \nabla u \cdot \nabla v$$

then $(\bar{A} + \sum_{j=1}^d y_j \Psi_j) u = f$ so

$$u(y) = \left(I + \sum_{j=1}^d y_j \bar{A}^{-1} \Psi_j \right)^{-1} \bar{A}^{-1} f \approx \sum_{\ell=0}^k \left(- \sum_{j=1}^d y_j \bar{A}^{-1} \Psi_j \right)^\ell \bar{A}^{-1} f$$

is a Taylor series expansion with $\binom{k+d}{d} \sim k^d$ terms

High contrast regime

What happens when $a(y)$ tends to 0 or $+\infty$ at some points ?

The contrast

$$\kappa(y) := \frac{\max_{x\in\Omega} a(x,y)}{\min_{x\in\Omega} a(x,y)} \leqslant \frac{R}{r}$$

tends to infinity, so we lose the approximation bound. Other problem :

$$\|u(y) - G_{V_n} u(y)\|_{H_0^1} \leqslant \sqrt{\kappa(y)}\, \|u(y) - u_n(y)\|_{H_0^1}$$

Assumption : Piecewise constant diffusion coefficient

$$a(y) = \sum_{j=1}^{d} y_j \chi_{\Omega_j}$$

on a partition $\Omega_1 \cup \cdots \cup \Omega_d$ of Ω

Soft inclusions problem

Homogeneity

$$u(ty) = t^{-1} u(y)$$

If $y_i \to 0$ for some i such that $f_{|\Omega_i} \neq 0$, then $\|u(y)\|_{H_0^1(\Omega)} \to \infty$

Outside of Ω_i, $u(y)$ converges to

Soft inclusion problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(a\nabla u) & = & f & \text{in } \ \Omega_i^c \\ \partial_n u & = & \partial_n v & \text{on } \ \partial\Omega_i, \\ u & = & 0 & \text{on } \ \partial\Omega \end{array} \right. \qquad v \in H_0^1(\Omega_i), \quad -\Delta v = f_{|\Omega_i} \text{ in } \Omega_i$$

Stiff inclusions problem

Let $i \in \{1, \ldots, d\}$ and consider

$$V_i = \{v \in H^1_0(\Omega), \nabla v = 0 \text{ on } \Omega_i\}$$

Then there exists a unique $u_i(y) \in V_i$ such that

$$\sum_{j \neq i} y_j \int_{\Omega_j} \nabla u_i \cdot \nabla v = \langle f, v \rangle_{H^{-1}, H^1_0}, \quad v \in V_i$$

It is characterised by

Stiff inclusion problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}(a \nabla u) & = & f & \text{in } \Omega_i^c \\ \nabla u & = & 0 & \text{in } \Omega_i, \\ u & = & 0 & \text{on } \partial\Omega \end{array} \right. \qquad \int_{\partial\Omega_i} \partial_n u = \int_{\Omega_i} f$$

Convergence of solutions

Lemma (Jikov, Kozlov, Oleinik, 2012)

$u(y) \to u_i(y)$ as $y_i \to \infty$

Idea : The sequence $(\|u(y)\|_{H_0^1})$ is bounded, extract a weakly converging subsequence $(u(y^p))$, then its limit $\bar{u}$ is solution to the stiff inclusion problem, and the convergence is strong due to the convergence of the energy norm

$$\int_\Omega a(y^p)|\nabla u(y^p)|^2 = \langle f, u(y^p)\rangle_{H^{-1},H_0^1} \to \langle f, \bar{u}\rangle_{H^{-1},H_0^1} = \int_{\Omega_j^c} a(y)|\nabla \bar{u}|^2$$

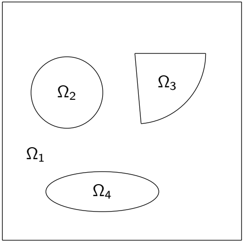
Corollary

For $a(y) = \sum_{j=1}^{d} y_j \chi_{\Omega_j}$ *and* $Y = [1, +\infty]^d$, *the set* $\mathcal{M} = \{u(y), y \in Y\}$ *is compact, hence*

$$d_n(\mathcal{M})_V \underset{n\to\infty}{\longrightarrow} 0$$

Rate of convergence

Disjoint inclusions : The Ω_j are Lipschitz and do not intersect for $j \geqslant 2$



Lemma

Assume that $a(y) = \sum_{j=1}^{d} y_j \chi_{\Omega_j}$ with (Ω_j) disjoint inclusions, then

$$\|u(y) - u_i(y)\|_{H_0^1} \leqslant C y_i^{-1}$$

Exponential convergence for unbounded contrast

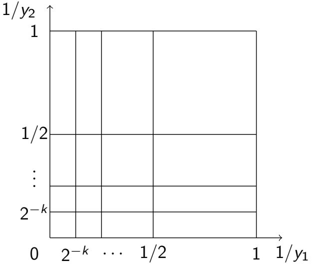
Theorem (Cohen, D., Somacal, 2021)

Assume $a(y) = \sum_{j=1}^{d} y_j \chi_{\Omega_j}$ with (Ω_j) disjoint inclusions, $Y = [1, +\infty]^d$

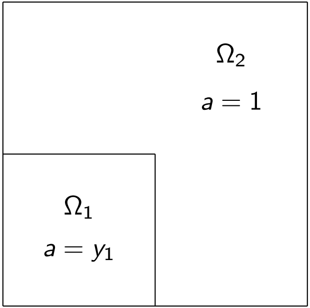
There exists spaces $V_n^{(1)}, \dots, V_n^{(k^d)}$ of dimension $n \sim k^d$ such that

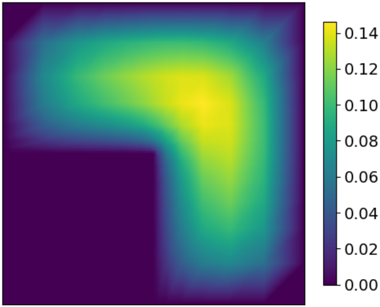
$$\forall y \in Y, \quad \exists r, \quad \|u(y) - P_{V_n^{(r)}}(y)\|_{H_0^1} \leqslant C2^{-k}$$

In particular $$d_n(\mathcal{M})_V \leqslant Cn^{-1/2d}$$



Numerical illustration





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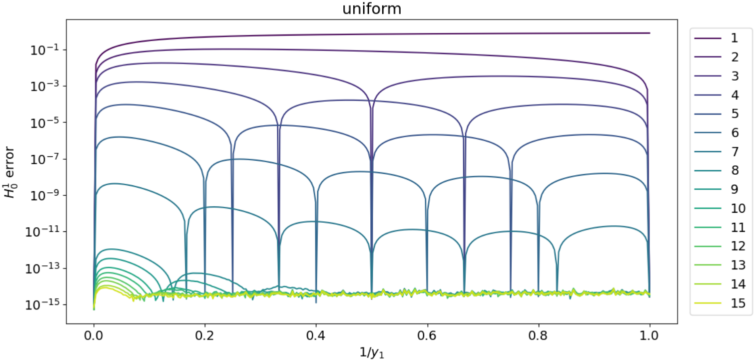

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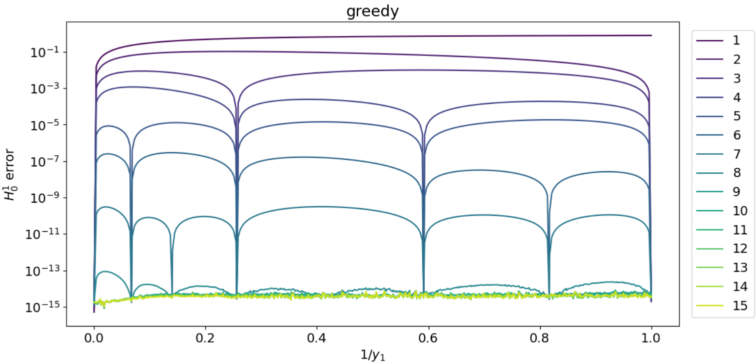
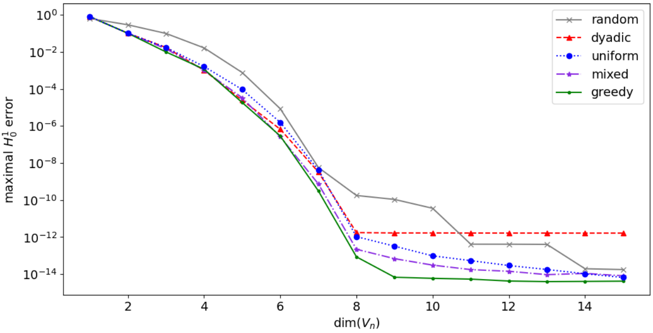
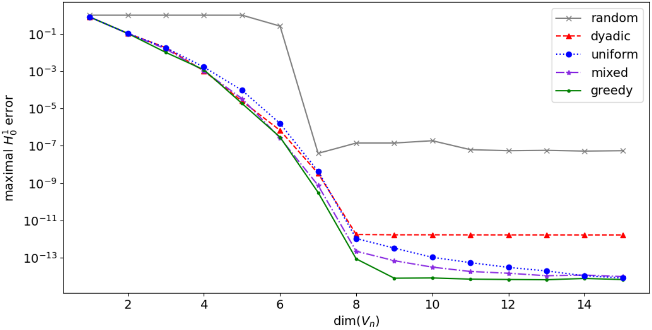


Error of H_0^1 projection



Error of galerkin projection



Perspectives

We proved exponential convergence of reduced order model approximation for an elliptic PDE with unbounded diffusion coefficient

- Does the result extend to reduced bases, and if so, how to choose the samples ?
- Can we improve the rate $\exp(-n^{1/2d})$?
- Could the assumption on disjoint inclusions be removed ?

Thank you for your attention !

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