Numerical simulations and statistical description of the quantum turbulence modelled by HVBK equations
(ANR Project QUTE-HPC)

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Outline

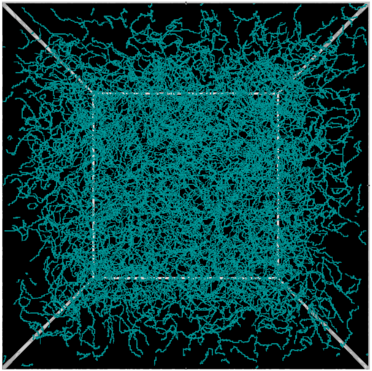
Quantum turbulence in helium II

• Quantum turbulence: quantum vortex lines randomly distributed in the space. The circulation is quantized in unit circulation of $\frac{\hbar}{m}$, with $\hbar$ is the Plank's constant and m is the atom mass, thus

$$\kappa = \oint \boldsymbol{v}_s dr = \frac{\hbar}{m}$$

A macroscopic approximation Feynman rule:

$$\kappa \mathcal{L} = \omega$$



HVBK two-fluid model

Helium II two-fluid model (Tisza & Landau 1941)

- Normal fluid: ρ^n, $\boldsymbol{u}_n$, ν^n, carries entropy
- Superfluid: ρ^s, $\boldsymbol{u}_s$, free of entropy
- Blow T_λ mixture of two fluid $\rho = \rho^n + \rho^s$, for $T > T_\lambda$, $\rho^s/\rho = 0$; for $T = 0K$, $\rho^n/\rho = 0$

The mutual friction force writes:

$$\boldsymbol{F}_M = -B\frac{\rho^s\rho^n}{\rho}\frac{\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge (\boldsymbol{u}_s - \boldsymbol{u}_n))}{|\boldsymbol{\omega}|} - B'\frac{\rho^s\rho^n}{\rho}\boldsymbol{\omega} \wedge (\boldsymbol{u}_s - \boldsymbol{u}_n) \tag{1}$$

where B and $B^{'}$ are two temperature-dependent parameters, and

$$\kappa\mathcal{L} = \omega = \nabla \times \boldsymbol{u}_s$$

The governing equations write

$$\nabla.\boldsymbol{u}_n = 0, \ \nabla.\boldsymbol{u}_s = 0 \tag{2}$$

$$\frac{\partial \boldsymbol{u}_n}{\partial t} + \nabla.(\boldsymbol{u}_n \otimes \boldsymbol{u}_n) = -\nabla p_n + \frac{1}{\rho^n}\boldsymbol{F}_M + \nu^n\Delta\boldsymbol{u}_n + \boldsymbol{f}_{ext} \tag{3}$$

$$\frac{\partial \boldsymbol{u}_s}{\partial t} + \nabla.(\boldsymbol{u}_s \otimes \boldsymbol{u}_s) = -\nabla p_s - \frac{1}{\rho^s}\boldsymbol{F}_M + \nu^s\Delta\boldsymbol{u}_s + \boldsymbol{f}_{ext} \tag{4}$$

The numerical simulations

The 3dt-HVBK solver

A solver of 3D box periodic boundary condition HIT using spectral method (p3dfft, pencil 2D decomposition), large eddy forcing, standard de-aliasing, two-fluid HVBK model.

The truncated HVBK two-fluid model

- $\nu^n/\nu^s = 10$ with $\nu^n = cst$ (independent of temperature)
- Simplified mutual friction expression, with $B = 1.5$ invariant to T

$$\boldsymbol{F}_M = B\frac{\rho^s \rho^n}{\rho}|\omega|.(\boldsymbol{u}_s - \boldsymbol{u}_n)$$

- Unique variable $\rho^n/\rho^s = 10,\ 1,\ 0.1$, for high, intermediate and low temperature.
- External forcing f^n_{ext} and f^s_{ext} with constant energy injection rate per unite masse ε^*.

Turbulence parameters

- The large scales are invariant with temperature (u', L)
- The small scales are temperature-dependent (λ, η)
- The energy flows averagely from the superfluid to the normal fluid and reaches a balance.

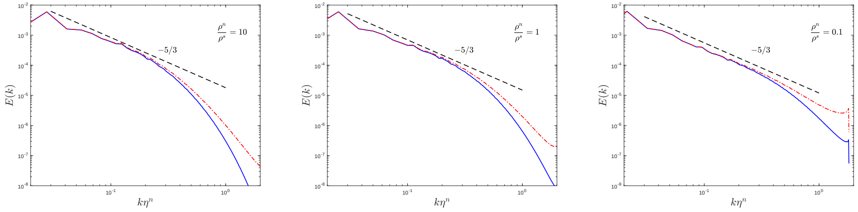
	ρ^n/ρ^s	Re_L	Re_λ	L/η	η/Δ	$\varepsilon/\varepsilon^*$	ϕ/ε^*
Normal fluid	10	580.74	90.37	60.10	1.12	1.06	0.07
	1	570.78	74.09	65.49	1.02	1.58	0.59
	0.1	568.68	54.12	76.00	0.87	3.01	2.03
Superfluid	10	5.78e+03	783.29	202.46	0.33	0.15	-0.85
	1	5.65e+03	600.84	227.32	0.28	0.26	-0.83
	0.1	5.59e+03	369.07	286.42	0.22	0.70	-0.30

Table: $Re_L = \frac{L.u'}{\nu}$, $Re_\lambda = \frac{\lambda.u'}{\nu}$ with $\lambda = \sqrt{15\nu(u')^2/\varepsilon}$, ε is the average energy dissipation rate, $L = ((u')^2)^{3/2}/\varepsilon$ is the scale of the large eddies, η is the Kolmogorov length scales, Δ is the mesh size $\varepsilon^* = \overline{U_i^s f_i^s} = \overline{U_i^n f_i^n}$ is is the energy injection rate at the large scales. $\phi^n = \frac{\rho^s}{\rho}\overline{U_i^n F_i^{ns}}$ is the average energy flux exerted by the friction force on the normal fluid, $\phi^s = -\frac{\rho^n}{\rho}\overline{u_i^s F_i^{ns}}$ is the average energy flux applied by the friction force on the superfluid.

The turbulence statistics

The energy spectrum

- -5/3 in the inertial range
- When temperature decreases: the inertial range extends for both the normal fluid and the superfluid.
- When temperature decreases: the Kolmogorov scales η decrease monotonously.



(a) (b) (c)

Figure: The kinetic energy spectrum for different temperature (a) $\rho^n/\rho^s = 10$, (b) $\rho^n/\rho^s = 1$, (c) $\rho^n/\rho^s = 0.1$. (–) is the normal fluid, $(-\cdot)$ is the superfluid, (- -) marks the $-5/3$ slope. Wavenumbers are normalised with the normal fluid Kolmogorov scales, $1/\eta^n$.

The energy exchange through the mutual friction

- Knowing that the high enstrophy area are the same, the strong energy exchange only occurs at the strong enstrophy area

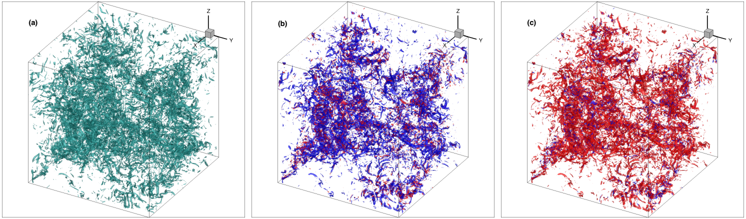


Figure: (a) The snapshot of iso-contour $|\omega_s| = 7\sqrt{\langle|\omega_s|^2\rangle}$, (b) the snapshot of the iso-contour of $u_n.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_n.F_{ns})^2\rangle}$, (c) the snapshot of the iso-contour of $u_s.F_{ns}$. (blue) energy gain $5\sqrt{\langle(u_s.F_{ns})^2\rangle}$, (red) energy loss $-3\sqrt{\langle(u_s.F_{ns})^2\rangle}$,

The scale-by-scale energy budget equation

We now consider two points of the flow, $\vec{x}^+$ and $\vec{x}^-$, separated by the increment $\vec{r}$ such as $\vec{x}^+ = \vec{x}^- + \vec{r}$. The following abbreviations are applied

$$Note: \; Var^{\pm} = Var(\vec{x}^{\pm}) \; and \; U_i = \overline{U_i} + u_i$$

.

$$\partial_t U_i^+ + \partial_j^+ (U_j^+ U_i^+) = -\frac{1}{\rho} \partial_i^+ P^+ + \partial_j^+ (\nu^+ \tau_{ij}^+) \tag{5a}$$

$$\partial_t U_i^- + \partial_j^- (U_j^- U_i^-) = -\frac{1}{\rho} \partial_i^- P^- + \partial_j^- (\nu^- \tau_{ij}^-) \tag{5b}$$

(5a) - (5b)

$$\begin{gathered} \tfrac{D}{Dt} \delta u_i + (\partial_j^+ u_j^+ + \partial_j^- u_j^-) \delta u_i + \delta \left(u_j \partial_j \overline{U}_i \right) \\ = \\ -\tfrac{1}{\rho} (\partial_i^+ + \partial_i^-) \delta P + (\partial_j^+ + \partial_j^-) \delta(\nu \tau_{ij}) \qquad , \end{gathered} \tag{6}$$

The equation (6) is then multiplied by $2\delta u_i$ and averaged then integral over 0 to r. Taking the simplified for in IHT viz.

$$-\overline{\delta u_{\|} (\delta u_i)^2} + 2\nu \frac{d}{dr} \overline{(\delta u_i)^2} = \frac{4}{3} \overline{\varepsilon} r, \tag{7}$$

The scale-by-scale energy budget

Applying in the context of HVBK model

$$-\overline{\delta u_{\parallel}^n (\delta u_i^n)^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^n ds + 2\nu^n \frac{d}{dr}\overline{(\delta u_i^n)^2} = \frac{4}{3}\overline{\epsilon}^n r, \tag{8}$$

$$-\overline{\delta u_{\parallel}^s (\delta u_i^s)^2} - \frac{1}{r^2}\int_0^r s^2 \mathcal{L}^s ds + 2\nu^s \frac{d}{dr}\overline{(\delta u_i^s)^2} = \frac{4}{3}\overline{\epsilon}^s r, \tag{9}$$

with

$$\mathcal{L}^n \equiv -2\frac{\rho^s}{\rho}\overline{(\delta u_i^n)(\delta F_i^{ns})} - 2\overline{(\delta u_i^n)(\delta f_i^n)}, \tag{10}$$

$$\mathcal{L}^s \equiv 2\frac{\rho^n}{\rho}\overline{(\delta u_i^s)(\delta F_i^{ns})} - 2\overline{(\delta u_i^s)(\delta f_i^s)}, \tag{11}$$

where $u_{\parallel}$ is the velocity component parallel to vector $\vec{r}$, $r = |\vec{r}|$, $\delta f = f(\vec{x}+\vec{r}) - f(\vec{x})$ and for isotropic turbulence δf depends only on r.

The scale-by-scale energy budget

$$\underbrace{-\overline{\delta u_{\|}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu \frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \qquad (12)$$

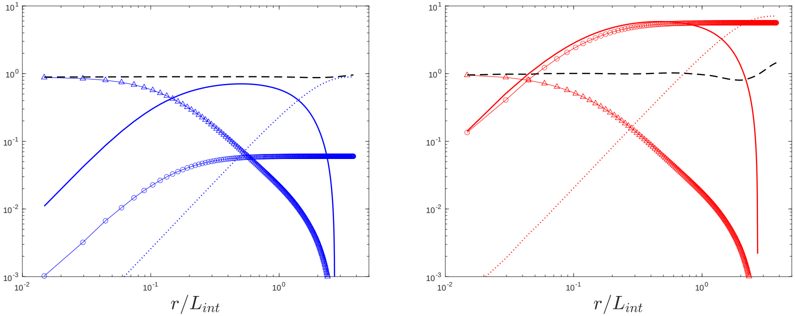


Figure: The case $\rho^n/\rho^s = 10$. (—) is the convection term $-S3_{ii}^{n,s}$, (○) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdots$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale for the total fluid.

The scale-by-scale energy budget

$$\underbrace{-\overline{\delta u_{\parallel}(\delta u_i)^2}}_{S3_{ii}} + LF + L_{ext} + \underbrace{2\nu\frac{d}{dr}\overline{(\delta u_i)^2}}_{dS2_{ii}} = \frac{4}{3}\varepsilon r \qquad (13)$$

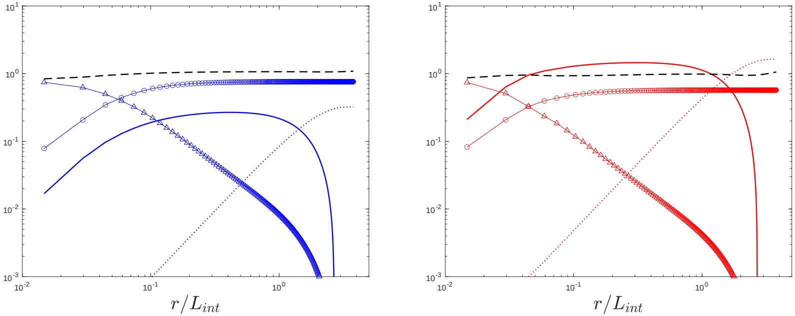


Figure: The case $\rho^n/\rho^s = 0.1$. (—-) is the convection term $-S3_{ii}^{n,s}$, (○) the mutual friction term $LF^{n,s}$, ($-\triangle$) the viscous term $dS2_{ii}^{n,s}$, ($\cdots$) the external forcing term $L_{ext}^{n,s}$. All terms are normalised by $4/3r\varepsilon^{n,s}$, ($--$) marks the sum of all the terms. (left column) for the normal fluid, (right column) for the superfluid, L_{int} is the integral scale of the total fluid.

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \tag{14}$$

with $\partial_r \equiv \partial/\partial r$,

$$\begin{aligned} D_{111} &= \overline{(\delta u)^3}; \\ D_{1111} &= \overline{(\delta u)^4}; \\ D_{1122} &= \overline{(\delta u)^2 (\delta v)^2}; \\ C(r,t) &= -\frac{4}{r^2} D_{111}(r,t) + \frac{4}{r}\partial_r D_{111} + \partial_r \partial_r D_{111}; \\ Z_{111} &= \overline{3\delta u \left[\left(\frac{\partial u}{\partial x_l}\right)^2 + \left(\frac{\partial u'}{\partial x'_l}\right)^2\right]}, \end{aligned} \tag{15}$$

where $\delta u = u(x+r) - u(x)$ is the longitudinal velocity increment, $\delta v = v(x+r) - v(x)$ is the transverse velocity increment and double indices indicate summation and a prime denotes variables at point $x + r$. Finally,

$$T_{111} = \overline{3(\delta u)^2\, \delta \left(\frac{\partial p}{\partial x}\right)}. \tag{16}$$

The vortex stretching and the internal intermittency

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (17)$$

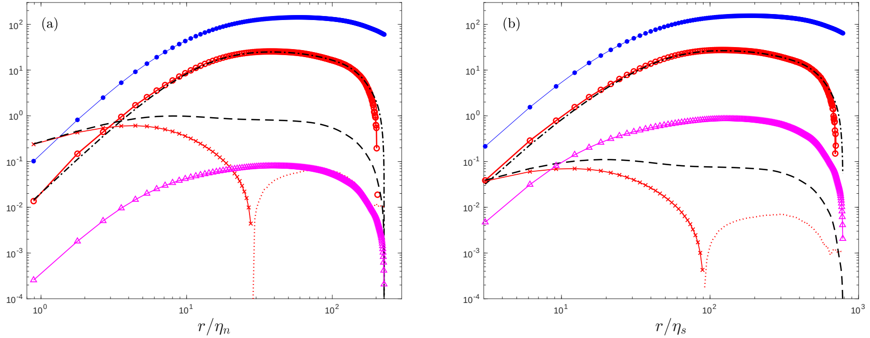


Figure: Balances of different terms in equations for the normal fluid (left) and the superfluid (right) for density ratios $\rho_n/\rho = 0.91$. ($\bullet-$) $(\partial_r + 2/r)D1111$, ($\circ-$) $(\partial_r + 2/r)D1111 - 6/r.D1122$, (black $-\cdot$) $-T111$, ($\times-$) $-2\nu C$ ($\cdots$) positive part of $2\nu C$, (- -) $-2\nu Z111$, ($\triangle-$) coupling terms. The plots are dimensionless.

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu C - 2\nu Z_{111} + MF + Fext, \qquad (18)$$

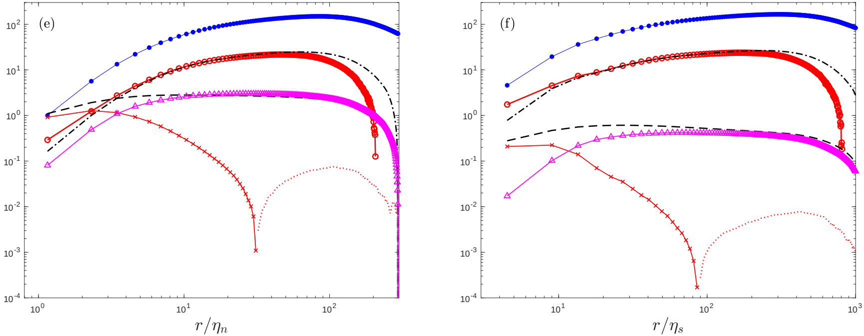


Figure: Balances of different terms in equations for the normal fluid (left) and the superfluid (right) for density ratios $\rho_n/\rho = 0.09$. ($\bullet-$) $(\partial_r + 2/r)D1111$, ($\circ-$) $(\partial_r + 2/r)D1111 - 6/r.D1122$, (black $-\cdot$) $-T111$, ($\times-$) $-2\nu C$ ($\cdots$) positive part of $2\nu C$, (- -) $-2\nu Z111$, ($\triangle-$) coupling terms. The plots are dimensionless.

The intermittency of small scales

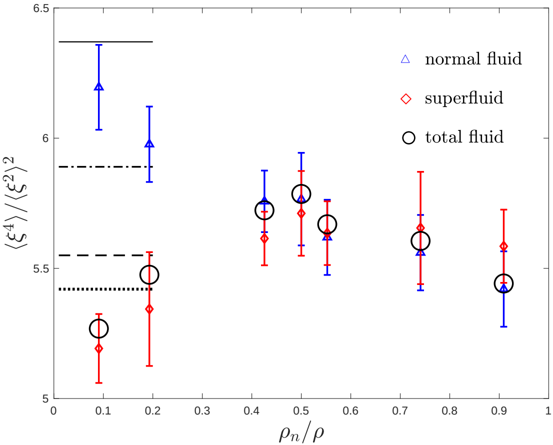


Figure: Flatness factors of the longitudinal velocity gradient $\xi = \partial_x u$ versus density ratio ρ_n/ρ for the normal fluid ($\triangle$), the superfluid ($\diamond$) and total fluid ($\bigcirc$). Error bars are the root-mean-square values of the flatness factors computed from 20 to 50 snapshots, with 10^8 data points for each snapshot. Horizontal lines indicate the flatness factor computed from DNS of classical turbulence (e.g., Ishihara 2007). (· ·) $Re_\lambda = 94.6$, (- -) $R_\lambda = 94.4$, (- ·) $R_\lambda = 167$. (–) $R_\lambda = 173$.

Conclusions

A picture of the energy exchange

• The mutual friction exchanges momentum between the two fluid components, averagely it extracts energy from the superfluid and adds energy to the normal fluid.
• The largest energy exchange mainly occurs at regions with strongest vorticity, which is an intermittent effect.

The scale-by-scale energy budget

• Although the mutual friction has a very important impact on the energy balance, the way of energy cascading in the inertial sub-range is unchanged (classical way).
• The local dissipation rate is balanced by both the large-eddy energy injection, **and** the energy exchange through the mutual friction (mainly at small scales).

The intermittency

• The mutual friction shows no effect on internal intermittency across the inertial range
• The energy source term due to the mutual friction modifies the small scale intermittency