Anisotropic nonlinear elliptic equations

PDEs and related topics

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G. Bonanno, G. D'Aguì, A. Sciammetta, *Existence of two positive solutions for anisotropic nonlinear elliptic equations*, Advances in Differential Equations, vol. **26** (2021), 229-258.

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (D_\lambda^{\vec{p}})$$

- $\Omega \subset \mathbb{R}^N$ with a boundary of class C^1 and with $N \geq 2$;
- $\vec{p} = (p_1, p_2, \dots, p_N)$, $\vec{p} \in \mathbb{R}^N$;
- $p^- = \min\{p_1, p_2 \dots, p_N\} > N$;
- $p^+ = \max\{p_1, p_2 \dots, p_N\}$;
- $\lambda > 0$;
- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1−Carathéodory function, that is:
  1. $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
  2. $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in \Omega$;
  3. for every $s > 0$ there is a function $l_s \in L^1(\Omega)$ such that

$$\sup_{|\xi| \leq s} |f(x,\xi)| \leq l_s(x), \quad \text{for a.e.} \quad x \in \Omega.$$

Anisotropic $p-$Laplacian operator

$$\Delta_{\vec{p}} u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right)$$

If $p_i = 2$ for all $i = 1, \ldots, N$

$$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \Delta u, \qquad \text{Laplacian operator.}$$

If $p_i = p$ for all $i = 1, \ldots, N$

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) = \tilde{\Delta}_p u, \quad \text{pseudo}-p-\text{Laplacian operator.}$$

[1] M. Belloni, B. Kawohl, *The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$*, ESAIM Control Optim. Calc. Var. **10** (2004), 28–52.

[2] L. Brasco, G. Franzina, *An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities*, Nonlinear Differ. Equ. Appl. **20** (2013), 1795–1830.

Some references

[1] S.M. Nikol'skii, *An imbedding theorem for functions with partial derivatives considered in different metrics*, Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 321–336.

[2] J. Rákosník, *Some remarks to anisotropic Sobolev spaces I*, Beiträge zur Analysis **13** (1979) 55–68.

[3] J. Rákosník, *Some remarks to anisotropic Sobolev spaces II*, Beiträge zur Analysis **15** (1981), 127–140.

[4] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

Let $\alpha \in \mathbb{N}^N$ be multiindices such that $\alpha = (\alpha_1, \dots, \alpha_N)$. The length of α is $|\alpha| = \alpha_1 + \dots + \alpha_N$.

$$D^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}, \tag{1}$$

$$D^0 u := u.$$

$E = \left\{\alpha \in \mathbb{N}_0^N : |\alpha| \leq 1\right\}$ and $\vec{p} = (p_0, p_1, \dots, p_N)$ with $p_0 \geq p_i \geq 1$ for $i = 1, \dots, N$.

$$W^{E,\vec{p}}(\Omega) = \{u = u(x) : D^\alpha u \in L^{p_\alpha}(\Omega), \text{ for } \alpha \in E\}, \tag{2}$$

is a reflexive Banach space if it is equipped with the norm

$$\|u\|_{W^{E,\vec{p}}(\Omega)} := \sum_{\alpha \in E} \|D^\alpha u\|_{L^{p_\alpha}(\Omega)} . \tag{3}$$

We denote by $W_0^{E,\vec{p}}(\Omega)$ as closure of $C_0^\infty(\Omega)$ in the topology of $W^{E,\vec{p}}(\Omega)$.

Anisotripic Sobolev spaces

Consider the following $N+1$ multiindices of $N-$tuple

$$E = \{(0,0,\dots,0),(1,0,\dots,0),(0,1,\dots,0),\dots(0,0,\dots,1)\},$$

and consiter $\vec{p} = (p_0, p_1, p_2, \dots, p_N)$ with $p_i \geq 1$ for all $i = 1, \dots N$.
Then, the set (2) becomes

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), \text{ for } i = 1,\dots,N \right\}. \tag{4}$$

in which we consider the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^{p_0}(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \tag{5}$$

We define $W_0^{1,\vec{p}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (5). On $W_0^{1,\vec{p}}(\Omega)$ we can also define the following norm

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \tag{6}$$

Remark

We observe also that if $\vec{p}$ is constant (that is $p_i = p$ for all $i = 1, \dots, N$) we get

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \right\}.$$

Other references

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[4] R. Di Nardo, F. Feo, *Existence and uniqueness for nonlinear anisotropic elliptic equations*, Arch. Math. **102** (2014), 141–153.

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Main tool

Theorem (G. Bonanno and G. D'Aguì)

Let X be a real Banach space and let Φ, $\Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{7}$$

and, for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.
Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2} \in X$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

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[1] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973) 349–381.

[2] G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*, Z. Anal. Anwend. **35** (2016), 449–464.

Preliminary results

$\left(W_0^{1,\vec{p}}(\Omega), \|\cdot\|_{W_0^{1,\vec{p}}(\Omega)}\right)$ is a Banach space, where $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with

$$\|u\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} .$$

Proposition

$W_0^{1,\vec{p}}(\Omega)$ is compactely embedded in $C^0(\bar{\Omega})$ and for each $u \in W_0^{1,\vec{p}}(\Omega)$

$$\|u\|_{C^0(\bar{\Omega})} \leq \underbrace{2^{\frac{(N-1)(p^- -1)}{p^-}} m_{p^-} \max_{1\leq i\leq N}\{|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\}}_{=T_0} \|u\|_{W_0^{1,\vec{p}}(\Omega)}$$

Proof: $p^- > N$, $W_0^{1,p^-}(\Omega)$ is continuously embedded in $C^0(\bar{\Omega})$, the embedding is compact and

$$\|u\|_{C^0(\bar{\Omega})} \leq m_{p^-} \|u\|_{W_0^{1,p^-}(\Omega)} \leq 2^{\frac{(N-1)(p^- -1)}{p^-}} m_{p^-} \max_{1\leq i\leq N}\{|\Omega|^{\frac{p_i-p^-}{p_ip^-}}\} \|u\|_{W_0^{1,\vec{p}}(\Omega)} .$$

$$m_{p^-} = \frac{N^{-\frac{1}{p^-}}}{\sqrt{\pi}} \left[\Gamma\left(1+\frac{N}{2}\right)\right]^{\frac{1}{N}} \left(\frac{p^- -1}{p^- - N}\right)^{1-\frac{1}{p^-}} |\Omega|^{\frac{1}{N}-\frac{1}{p^-}}$$

[1] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

Preliminary results

Proposition

Fix $r > 0$. Then for each $u \in W_0^{1,\vec{p}}(\Omega)$ such that

$$\sum_{i=1}^{N} \frac{1}{p_i} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} < r,$$

one has

$$\|u\|_{C^0(\bar{\Omega})} < T \max\{r^{1/p^-}; r^{1/p^+}\},$$

where $T = T_0 \sum_{i=1}^{N} {p_i}^{1/p_i}$.

Variational approach

$\Phi, \Psi : W_0^{1,\vec{p}}(\Omega) \to \mathbb{R},$ $\quad F(x,t) = \int_0^t f(x,\xi)d\xi$ for all $(x,t) \in \Omega \times \mathbb{R}$.

$$\underbrace{I_\lambda(u) = \underbrace{\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx}_{\Phi(u)} - \lambda \underbrace{\int_\Omega F(x,u(x))dx}_{\Psi(u)}}_{\text{Energy functional}}.$$

Definition

A function $u : \Omega \to \mathbb{R}$ is a weak solution of problem $(D_\lambda^{\vec{p}})$ if $u \in X$ satisfies the following condition for all $v \in X$

$$\underbrace{\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx}_{\Phi'(u)(v)} = \lambda \underbrace{\int_\Omega f(x,u(x))v(x)dx}_{\Psi'(u)(v)}.$$

(AR) *There exist constants $\mu > p^+$ and $M > 0$ such that, $0 < \mu F(x,t) \le tf(x,t)$ for all $x \in \Omega$ and for all $|t| \ge M$.*

Lemma 1

Assume that the $(AR)-$condition holds. Then I_λ satisfies the $(PS)-$condition and it is unbounded from below.

The sign of solutions

$$f^+(x,t) = \begin{cases} f(x,0), & \text{if} \quad t < 0, \\ f(x,t), & \text{if} \quad t \geq 0, \end{cases} \tag{8}$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f^+(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (D^{\vec{p}}_{\lambda,f^+})$$

Lemma 2

Assume that

$$f(x,0) \geq 0 \quad \textit{for a.e. } x \in \Omega.$$

Then, any weak solution of $(D^{\vec{p}}_{\lambda,f^+})$ is nonnegative and it is also a weak solution of $(D^{\vec{p}}_{\lambda})$.

Lemma 3

Assume that

$$f(x,t) \geq 0 \quad \textit{for a.e. } x \in \Omega, \quad \textit{for all } t \geq 0.$$

Then, any non-zero weak solution of $(D^{\vec{p}}_{\lambda,f^+})$ is positive and it is also a weak solution of $(D^{\vec{p}}_{\lambda})$.

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Main result

$$R := \sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) \;\Rightarrow\; \exists x_0 \in \Omega \;\text{ such that }\; B(x_0, R) \subseteq \Omega$$

$$\omega_R := |B(x_0,R)| = \frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})} R^N, \quad \mathcal{K} = \frac{1}{\left[\sum_{i=1}^{N} \frac{1}{p_i}\left(\frac{2}{R}\right)^{p_i}\right] \omega_R \left(\frac{2^N-1}{2^N}\right) \max\left\{T^{p^-}; T^{p^+}\right\}}$$

Theorem

Assume that the (AR)-condition holds and $\exists c, d > 0$, with $\max\left\{d^{p^-};d^{p^+}\right\}<\min\left\{c^{p^-};c^{p^+}\right\}$, s.t.

$$F(x,t) \geq 0, \quad \text{for all} \quad (x,t) \in \Omega \times [0,d]\,, \tag{9}$$

$$\frac{\int_{\Omega} \max_{|\xi|\leq c} F(x,\xi)dx}{\min\left\{c^{p^-}; c^{p^+}\right\}} < \mathcal{K}\, \frac{\int_{B(x_0,\frac{R}{2})} F\left(x,d\right)dx}{\max\left\{d^{p^-}; d^{p^+}\right\}}\,. \tag{10}$$

Then, for each

$$\lambda \in \tilde{\Lambda} := \left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{\max\left\{d^{p^-};d^{p^+}\right\}}{\int_{B(x_0,\frac{R}{2})} F\left(x,d\right)dx}, \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}} \frac{\min\left\{c^{p^-};c^{p^+}\right\}}{\int_{\Omega} \max_{|\xi|\leq c} F(x,\xi)dx}\right[,$$

problem $(D_\lambda^{\vec{p}})$ has at least two non-zero weak solutions.

Sketch of Proof

- $X = W_0^{1,\vec{p}}(\Omega)$ and $\lambda \in \tilde{\Lambda}$.
- $I_\lambda = \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - \lambda \int_\Omega F(x, u(x))dx = \Phi(u) - \lambda \Psi(u).$
- from $(AR)-$condition $\overset{\text{Lemma 1}}{\Rightarrow}$ I_λ satisfies the (PS)-condition
  I_λ is unbounded from below.
- Put $r = \min\{\left(\frac{c}{T}\right)^{p^-} ; \left(\frac{c}{T}\right)^{p^+}\}$ and

$$\tilde{u}(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R), \\ \dfrac{2d}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ d & \text{if } x \in B\left(x_0, \frac{R}{2}\right). \end{cases}$$

Clearly, $\tilde{u} \in W_0^{1,\vec{p}}(\Omega)$. From $\max\left\{d^{p^-} ; d^{p^+}\right\} < \min\left\{c^{p^-} ; c^{p^+}\right\}$ + (10) $\Rightarrow 0 < \Phi(\tilde{u}) < r$

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \max\left\{T^{p^-};T^{p^+}\right\}\mathcal{K}\frac{\int_{B(x_0,\frac{R}{2})} F(x,d)dx}{\max\left\{d^{p^-};d^{p^+}\right\}} > \max\left\{T^{p^-};T^{p^+}\right\}\frac{\int_\Omega \max_{|\xi|\leq c} F(x,\xi)dx}{\min\left\{c^{p^-};c^{p^+}\right\}} \geq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r}$$

- $\lambda \in \tilde{\Lambda} \subseteq \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right[$

Some consequences

Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

$$(AR^+)\ \exists\ \mu > p^+ \text{ and } M > 0 \text{ such that } 0 < \mu F(x,t) \leq tf(x,t)\ \forall\, x \in \Omega \text{ and } \forall\, t \geq M.$$

Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that

$$\frac{\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K}\frac{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}{d^{p^-}}.$$

Then, for each $\lambda \in \left]\frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx}, \frac{1}{\max\left\{T^{p^-};T^{p^+}\right\}}\frac{c^{p^-}}{\int_\Omega F(x,c)dx}\right[$,

problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Sketch of Proof

- from $(AR^+)-$condition $\overset{\text{Lemma 1}}{\Rightarrow}$ $I_\lambda^+ := \Phi - \lambda\Psi^+$ satisfies the $(PS)-$condition
  I_λ^+ is unbounded from below
- From Lemma 3, any non-zero weak solution of $(D_{\lambda,f^+}^{\vec{p}})$ is a positive weak solution of $(D_\lambda^{\vec{p}})$.

Some consequences

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Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

$$(AR^+)\ \exists\ \mu > p^+ \text{ and } M > 0 \text{ such that } 0 < \mu F(x,t) \leq tf(x,t)\ \forall\, x \in \Omega \text{ and } \forall\, t \geq M.$$

Moreover, assume that there are two positive constants c and d, with $d < c \leq 1$, such that

$$\frac{\int_\Omega F(x,c)dx}{c^{p^+}} < \mathcal{K} \frac{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}{d^{p^-}}.$$

Then, for each $\lambda \in \left] \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{d^{p^-}}{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{c^{p^+}}{\int_\Omega F(x,c)dx} \right[$, problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Sketch of Proof

- from $(AR^+)-$condition $\overset{\text{Lemma 1}}{\Longrightarrow}$ $I_\lambda^+ := \Phi - \lambda \Psi^+$ satisfies the $(PS)-$condition; I_λ^+ is unbounded from below
- From Lemma 3, any non-zero weak solution of $(D_{\lambda,f^+}^{\vec{p}})$ is a positive weak solution of $(D_\lambda^{\vec{p}})$.

Some consequences

Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x,t) \geq 0$ for a.e. $x \in \Omega$ and for all $t \geq 0$. Assume that

$$(AR^+)\ \exists\ \mu > p^+ \text{ and } M > 0 \text{ such that } 0 < \mu F(x,t) \leq t f(x,t)\ \forall\ x \in \Omega \text{ and } \forall\ t \geq M.$$

Moreover, assume that there are two positive constants c and d, with $1 \leq d < c$, such that

$$\frac{\int_\Omega F(x,c)dx}{c^{p^-}} < \mathcal{K} \frac{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}{d^{p^+}}.$$

Then, for each $\lambda \in \left] \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{\mathcal{K}} \frac{d^{p^+}}{\int_{B\left(x_0, \frac{R}{2}\right)} F(x,d)\, dx}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{c^{p^-}}{\int_\Omega F(x,c)dx} \right[$, problem $(D_\lambda^{\vec{p}})$ has at least two positive weak solutions.

Sketch of Proof

- from $(AR^+)-$condition $\overset{\text{Lemma 1}}{\Longrightarrow}$ $I_\lambda^+ := \Phi - \lambda\Psi^+$ satisfies the $(PS)-$condition
  I_λ^+ is unbounded from below
- From Lemma 3, any non-zero weak solution of $(D_{\lambda,f^+}^{\vec{p}})$ is a positive weak solution of $(D_\lambda^{\vec{p}})$.

Example 1: $N = 3$, $\Omega = B(0,2)$, $p_1 = 4$, $p_2 = 5$, $p_3 = 6$, $c = 1$ and $d = 10^{-14}$

$$\begin{cases} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = 10^{-12}(x^2+y^2+z^2)u^8 + 10^{-12}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

$$f(x,y,z,t) = (x^2+y^2+z^2)t^8 + t^2 \;\Rightarrow\; F(x,y,z,t) = (x^2+y^2+z^2)\frac{t^9}{9} + \frac{t^3}{3}.$$

We have that $(AR^+)-$condition holds and

$$m_{p^-} = \sqrt[4]{\frac{3^3}{2\pi}}, \quad T_0 = \sqrt[3]{\frac{2^5\cdot 3^2}{\sqrt{\pi}}}, \quad T = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})\sqrt[3]{\frac{2^5\cdot 3^2}{\sqrt{\pi}}},$$

$$\max\left\{T^{p^-}; T^{p^+}\right\} = T^6 = (\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6\frac{(2^5\cdot 3^2)^2}{\pi}, \quad \mathcal{K} = \frac{5}{2^{10}\cdot 3^2\cdot 7\cdot 37(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6}.$$

$$\frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}}\frac{1}{\mathcal{K}}\frac{d^{p^-}}{\int_{B\left(x_0,\frac{R}{2}\right)} F(x,d)\,dx} = \frac{7\cdot 37}{5}\frac{1}{\frac{2^2}{5}d^5 + \frac{2^2}{d}} \le \frac{7\cdot 37}{4}d = \frac{7\cdot 37}{4}10^{-14} < 10^{-12}$$

$$< \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}}\frac{c^{p^-}}{\int_\Omega F(x,c)dx} = \frac{5}{(\sqrt{2}+\sqrt[5]{5}+\sqrt[6]{6})^6 2^{15}3^4}$$

Some consequences

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$AD^{\vec{p}}_{\lambda}$}$$

Put

$$\mathcal{K}^* = \frac{\omega_R}{2^N |\Omega|} \mathcal{K}.$$

(AR_1^+) *there exist constants* $\mu > p^+$ *and* $M > 0$ *such that,* $0 < \mu F(t) \leq t f(t)$ *for all* $t \geq M$.

Theorem

Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the (AR_1^+)−condition holds. Moreover, assume that there are two positive constants c and d, with $d < 1 \leq c$, such that

$$\frac{F(c)}{c^{p^-}} < \mathcal{K}^* \frac{F(d)}{d^{p^-}} . \tag{12}$$

Then, for each

$$\lambda \in \tilde{\Lambda}_1 := \left] \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \frac{1}{\mathcal{K}^*} \frac{d^{p^-}}{F(d)}, \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \frac{c^{p^-}}{F(c)} \right[,$$

the problem $(AD^{\vec{p}}_{\lambda})$ has at least two positive weak solutions.

Some consequences

$$\begin{cases} -\Delta_{\vec{p}} u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (AD_{\lambda}^{\vec{p}})$$

(AR_1^+) *There exist constants $\mu > p^+$ and $M > 0$ such that, $0 < \mu F(t) \leq t f(t)$ for all $t \geq M$.*

Theorem

Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function such that the $(AR_1^+)-$condition holds. Assume that

$$\limsup_{t \to 0^+} \frac{F(t)}{t^{p^-}} = +\infty. \qquad (13)$$

Put $\lambda^* = \dfrac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \dfrac{1}{|\Omega|} \sup\limits_{c \geq 1} \dfrac{c^{p^-}}{F(c)}$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem $(AD_{\lambda}^{\vec{p}})$ admits at least two positive weak solutions.

Remark

$$\lambda^* = \frac{1}{\max\left\{T^{p^-}; T^{p^+}\right\}} \frac{1}{|\Omega|} \max\left\{\sup_{c \geq 1} \frac{c^{p^-}}{F(c)}; \sup_{0 < c < 1} \frac{c^{p^+}}{F(c)}\right\}.$$

Some consequences

Theorem

Fix s, q such that $0 \leq s < p^- - 1$ and $p^+ - 1 < q$. Put

$$\eta^* = \min\left\{ \frac{1-\frac{p^+}{q+1}}{\frac{p^+}{s+1}-1}, \left[\frac{(s+1)(q+1)}{\max\left\{T^{p^-};T^{p^+}\right\}|\Omega|} \frac{\left(\frac{p^+}{s+1}-1\right)^{\frac{p^+-(s+1)}{q-s}} \left(1-\frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1-\frac{p^+}{q+1}\right)+(s+1)\left(\frac{p^+}{s+1}-1\right)} \right]^{\frac{q-s}{(q+1)-p^+}} \right\}.$$

Then, for each $\eta \in]0, \eta^*[$ the problem

$$\begin{cases} -\Delta_{\vec{p}} u = \eta u^s + u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{$AD_\eta^{\vec{p}}$}$$

has at least two positive weak solutions.

Example 2: $N = 2$, $\Omega = B(0,1)$, $p_1 = 3$ and $p_2 = 4$

For each $\eta \in \left]0, \frac{3}{2^8(2^{\frac{1}{2}}+3^{\frac{1}{3}})^8}\right[$, the problem

$$\begin{cases} -\dfrac{\partial}{\partial x_1}\left(\left|\dfrac{\partial u}{\partial x_1}\right|\dfrac{\partial u}{\partial x_1}\right) - \dfrac{\partial}{\partial x_2}\left(\left|\dfrac{\partial u}{\partial x_2}\right|^2\dfrac{\partial u}{\partial x_2}\right) = \eta u + u^5 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits at least two positive weak solutions.
Indeed

$$m_{p^-} = \left(\frac{2}{\pi}\right)^{\frac{1}{3}}, \quad T_0 = \frac{2}{\pi^{\frac{1}{4}}}, \quad T = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})\frac{2}{\pi^{\frac{1}{4}}},$$

$$\max\left\{T^{p^-}; T^{p^+}\right\}|\Omega| = (3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4, \quad (s+1)(q+1) = 12,$$

$$\frac{\left(\frac{p^+}{s+1} - 1\right)^{\frac{p^+-(s+1)}{q-s}}\left(1 - \frac{p^+}{q+1}\right)^{\frac{(q+1)-p^+}{q-s}}}{(q+1)\left(1 - \frac{p^+}{q+1}\right) + (s+1)\left(\frac{p^+}{s+1} - 1\right)} = \frac{1}{3^{\frac{1}{2}}4},$$

$$\eta^* = \min\left\{\frac{1}{3}; \left[\frac{3^{\frac{1}{2}}}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^4 2^4}\right]^2\right\} = \frac{3}{(3^{\frac{1}{3}} + 4^{\frac{1}{4}})^8 2^8}.$$

Thank you for your kind attention