Méthode de Newton pour le calcul de carènes optimales basée sur la formule de Michell pour des vitesses aléatoires

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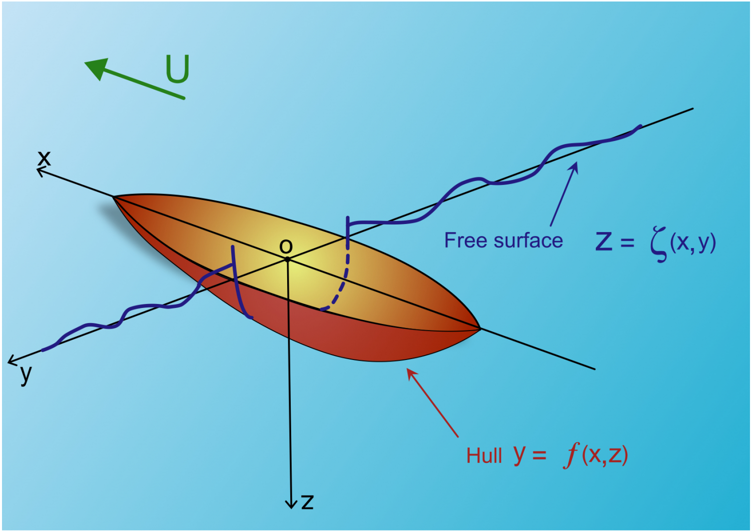
The setting



The setting

The fluid is assumed to be

1. **incompressible, inviscid** : We start from the Euler equations :

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{\rho}\nabla P + \boldsymbol{g} \quad \text{in} \quad W = \text{(Domain of the water)} \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases}$$

2. **irrotational** :
   - $\nabla \times \boldsymbol{v} = 0 \quad \text{in} \quad W \quad .$
   - $\exists \Phi,\ \boldsymbol{v} = \nabla \Phi$ (Helmholtz-Hodge Theorem).

3. A steady state has been reached so that we have

$$\Delta \Phi = 0 \qquad \text{in } W.$$

The problem

If the ship hull is represented by $y = f(x, z)$, and $z = \zeta(x, y)$ is the unknown free surface of the water, the boundary conditions are :

- Hull boundary condition :

  $\nabla\Phi \cdot n_f = 0 \quad$ on $y = f(x, z)$

- Free surface conditons :

  i. Kinematic condition :
  $\nabla\Phi \cdot n_\zeta = 0 \quad$ on $z = \zeta(x, y)$,

  ii. Dynamic condition (Bernoulli's equation) :
  $|\nabla\Phi|^2 - 2g\zeta + \frac{2}{\rho}P = \frac{2}{\rho}P_0 \quad$ on $z = \zeta(x, y)$

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- $\Phi = -Ux + \phi$ (ϕ small) .
- Thin-ship assumptions :

  $|\partial_x f| << 1$; $|\partial_z f| << 1$

- Michell condition on the hull :

  $\partial_y\phi - U\partial_x f(x,z) = 0$ on $y = 0^+$

- Linearized free surface conditions :

  i. Kinematic condition :

  $\partial_z\phi + U\partial_x\zeta = 0$ on $z = 0$

  ii. Dynamic condition :

  $\partial_{xx}\phi + \frac{g}{U}\partial_x\zeta = 0$ on $z = 0$

$$\begin{cases} \Delta\phi = 0, & \text{in } \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+ \quad \text{(quarter space)} \\ \partial_{xx}\phi - \frac{g}{U^2}\partial_z\phi = 0, & \text{on } z = 0 \\ \partial_y\phi - U\partial_x f = 0, & \text{on } y = 0^+ \end{cases}$$

Solution and wave resistance

- If δp is the increase of fluid pressure due to the disturbance Φ, by Bernoulli's equation, the wave resistance is given by

$$R_{Michell} = -2\int\int \delta p \partial_x f dxdz = 2\rho U \int\int \partial_x \phi \partial_x f dxdz,$$
$$= \frac{4\rho g^2}{\pi U^2}\int_1^\infty (\mathcal{I}^2(\lambda) + \mathcal{J}^2(\lambda))\frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

where, if D denotes the domain of definition of f, we have :

$$\mathcal{I}(\lambda) = \int_D \partial_x f(x,z)\exp(-\lambda^2 gz/U^2)\cos(\lambda gx/U^2)dxdz,$$
$$\mathcal{J}(\lambda) = \int_D \partial_x f(x,z)\exp(-\lambda^2 gz/U^2)\sin(\lambda gx/U^2)dxdz.$$

- A standard approximation of the viscous drag for small ∇f is given by a linearization of the area functional and reads

$$R_{viscous} = \frac{1}{2}\rho U^2 C_F \left(2|D| + \int_D |\nabla f(x,z)|^2 dxdz\right)$$

Normalized water resistance[1]

The normalized total water resistance writes : $J_{total} = J_{Michell} + J_{viscous}$, with

$$J_{viscous} = \int_D |\nabla f(x,z)|^2 \, dxdz$$
$$J_{Michell} = \int_{D\times D} k(U,x,z,x',z') f(x,z) f(x',z') dxdzdx'dz'$$

where

$$\begin{cases} k(U,x,z,x',z') &= \frac{4U^4}{\pi C_F} w(\nu(U)(x-x'), \nu(U)(|z|+|z'|)) \\ w(X,Z) &= \int_1^{\infty} e^{-\lambda^2 Z} cos(\lambda X) \dfrac{\lambda^4}{\sqrt{\lambda^2-1}} d\lambda \\ \nu(U) &= \frac{g}{U^2} \end{cases}$$

[1] J. Dambrine and M. Pierre, "Regularity of optimal ship forms based on Michell's wave resistance", Appl. Math. Optim. **82**, 23–62 (2020).

The hull parametrization f

The problem

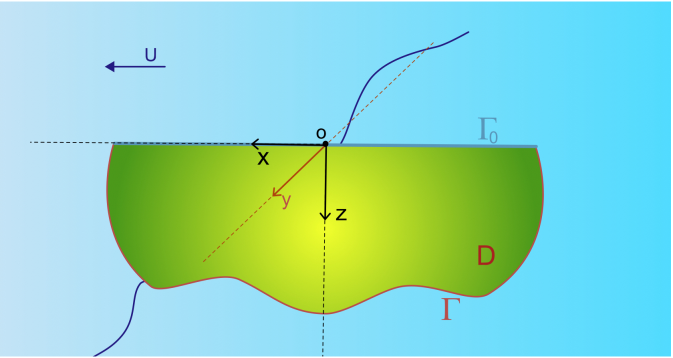
Expectation of the water resistance

Newton's method

Numerical results

The hull parametrization $f(x, z)$ is computed as a solution to the following problem :

$$\left\{\begin{array}{l}\text{Find a function } f_D \text{ which minimizes } J_{total}(f) \text{ in the set} \\ \left\{f : \overline{D} \to \mathbb{R},\ f = 0 \text{ on } \Gamma,\ f \geqslant 0 \text{ in } D \text{ and } 2\int_D f(x,z)dxdz = V\right\}.\end{array}\right.$$



Expectation of the water resistance

Assume that the variables g, ρ and C_F are known physical constants, and V is the fixed volume of the hull. Given a speed distribution $U(.,\omega)$, where ω is an event in a probability space $(\mathcal{O}, \mathcal{F}, \mathbb{R})$, the expectation of the water resistance writes :

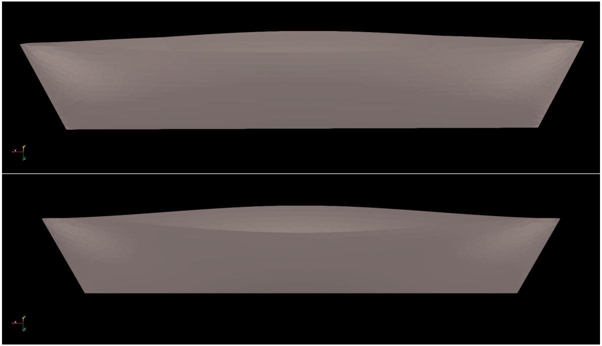
$$\begin{aligned}
\mathbb{E}(J_{total}) &= \mathbb{E}(J_{viscous} + J_{wave}) \\
&= \int_D |\nabla f(x,z)|^2 \, dxdz + \int_{\mathcal{O}} \left(\int_{D\times D} k(U(.,\omega), x, z, x', z') f(x,z) f(x',z') dxdzdx'dz' \right) d\mathbb{P}(\omega). \\
&= \int_D |\nabla f(x,z)|^2 \, dxdz + \int_{D\times D} \left(\int_{\mathcal{O}} k(U(.,\omega), x, z, x', z') d\mathbb{P}(\omega) \right) f(x,z) f(x',z') dxdzdx'dz'.
\end{aligned}$$

i.e the hull parametrisation that minimizes the expectation is solution to :

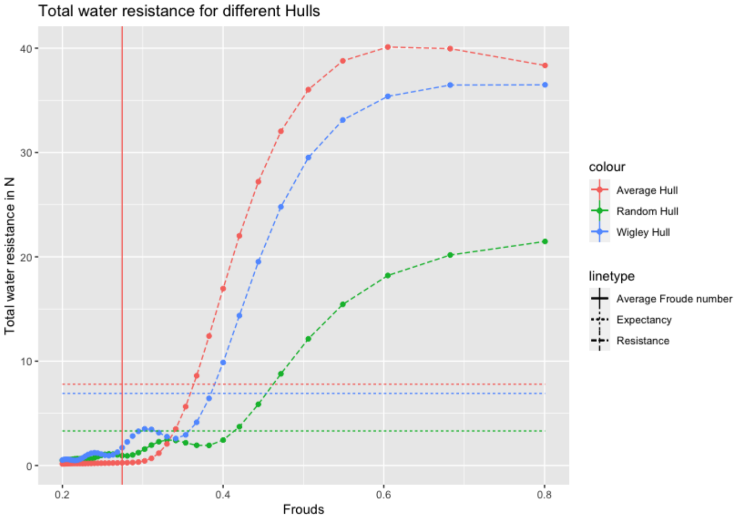
$$\begin{cases}
-\Delta f_D(x,z) + \int_D \left(\int_{\mathcal{O}} k(U(.,\omega), x, z, x', z') d\mathbb{P} \right) f_D(x',z') dx'dz' = C & (x,z) \in D \\
f_D(x,z) = 0 & \text{on } \Gamma \\
2 \int_D \hat{f}_D(x,z) dxdz = V. &
\end{cases}$$

Numerical results : optimal hulls

- We consider a uniform distribution of velocities $\nu(\hat{U}(\omega)) \in [0.2\sqrt{gL}, \sqrt{gL}]$, where $L = 2$ is the length of the ship. i.e $Fr = \frac{U}{\sqrt{gL}} \in [0.2, 1]$. D is taken as the fixed rectangle of length L and draft $T = 0.4$.



Numerical results : Total water resistance



Newton's method for the shape optimization problem

We now fix an area a of D, the optimal design problem reads :

$$\begin{cases} \text{Find the domain } D^\star \text{ which minimizes } \mathbb{E}(J_{total}(f_D)) \\ \text{among all regular open subsets } D \text{ of the lower half-plane} \\ \text{ such that } |D| = a \end{cases}$$

- A. Novruzi and J. R. Roche, "Newton's method in shape optimisation: a three-dimensional case", BIT **40**, 102–120 (2000)
- H. Harbrecht, "A Newton method for Bernoulli's free boundary problem in three dimensions", Computing **82**, 11–30 (2008)
- J.-L. Vie, "Second-order derivatives for shape optimization with a level-set method", 2016PESC1072, PhD thesis (2016)

A simple example : Functionals without PDE

Let $f \in C^2(\mathbb{R}^2;\mathbb{R})$, and $E(D) = \int_D f(x)dx$. We consider the minimization problem

$$\inf_{D \in \mathcal{O}_2} E(D) = D^* = \{x \in \mathbb{R}^N | f(x) < 0\}$$

We search for a descent direction by solving the Newton problem : find $\theta, \xi \in C^{2,\infty}(\mathbb{R}^N;\mathbb{R}^N)$ such that

$$E''(D;\theta,\xi) = -E'(D;\xi),$$

$$E'(\Omega;\xi) = \int_\Gamma (\xi \cdot n) f,$$

$$E''(\Omega;\theta,\xi) = \int_\Gamma (\theta \cdot n)(\xi \cdot n)\big(\mathcal{H}f + \partial_n f\big) + \int_\Gamma Z_{\theta,\xi} f.$$

The proposed discretization

The problem

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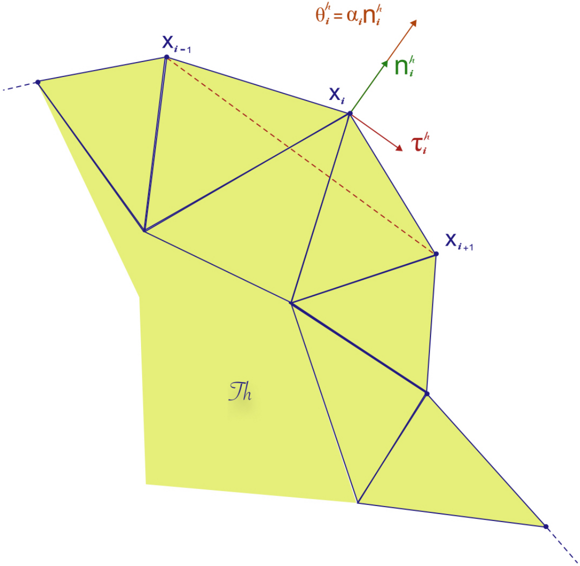
Newton's method

Numerical results



The proposed discretization

- In its discrete form, Newton's equation writes as the problem of finding $\theta^h = (\theta_1^h, \theta_2^h) \in \mathcal{V}_h \times \mathcal{V}_h$ such that $\forall\, \xi^h = (\xi_1^h, \xi_2^h) \in \mathcal{V}_h \times \mathcal{V}_h$

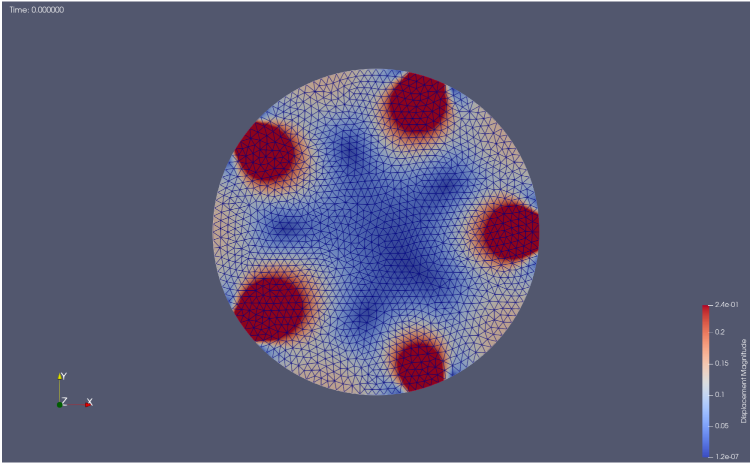
$$\int_\Gamma (\theta^h \cdot n^h)(\xi^h \cdot n^h)\left(\mathcal{H}^h f^h + \nabla f^h \cdot n^h\right) + \int_\Gamma Z^h_{\theta,\xi} f^h = -\int_\Gamma (\xi^h \cdot n^h) f^h.$$

- If we consider a normal basis, such that $\theta^h = \sum_{i=0}^{nbe} \alpha_i n_i^h$, and $\xi^h = n^h$, then at every vertex x_i, a descent direction can be obtained by solving Newton's equation, which is now reduced to

$$\alpha_i \int_\Gamma \mathcal{H}_i^h f_i^h + \nabla f_i^h \cdot n_i^h = -\int_\Gamma f_i^h \qquad \forall i \in \{0, 1, \ldots, nbe\}$$

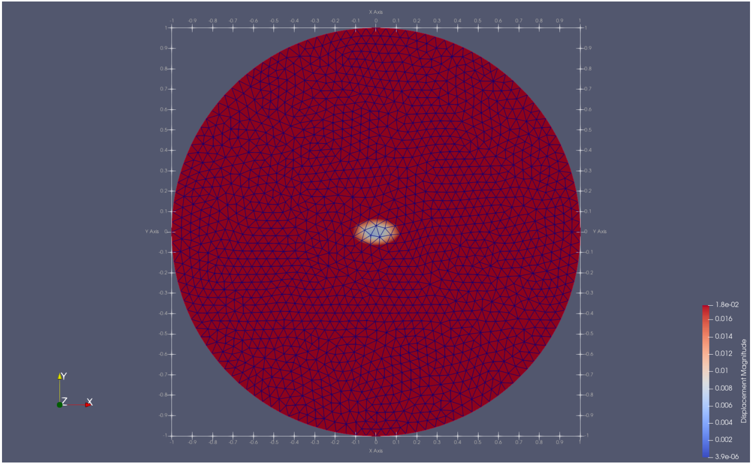
- We thus compute a descent direction on the whole boundary by solving the system $AT = B$, where :
  - A is the **diagonal** matrix of size $nbe \times nbe$, with entries $A_{ii} = E''(n_i^h, n_i^h)_{|x_i}$.
  - B is the vector of size nbe with entries $B_i = -E'(n_i^h)_{|x_i}$

Examples



$$f(x,y) = (x^2+y^2)^5 - 2\times 0.95^5(x^5-10x^3)y^2 + 5xy^4 + 0.95^{10} - 0.953^{10}$$

Examples



$$f(x,y) = ((x-0.5)^2 + y^2)((x+0.5)^2 + y^2) - 0.51^4$$

PDE dependent functionals with constraints

- Consider the set of admissible domains $\mathcal{A} = \{\Omega : \Omega \subset \mathcal{D}, \Omega \text{ open }, |\Omega| = 1\}$,
- Consider the Dirichlet energy :

$$E(\Omega) = \min\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega u dx \ : \ u \in H_0^1(\Omega)\right\}$$
$$= -\frac{1}{2}\int_\Omega w_\Omega dx,$$

where w_Ω is the weak solution of the equation $-\Delta w_\Omega = 1 \qquad w_\Omega \in H_0^1(\Omega)$

- Consider the shape optimization problem $\min\left\{E(\Omega) \ : \ \Omega \in \mathcal{A}\right\}$
- to handle the volume constraint we define the Lagrangian : $L(\Omega) = E(\Omega) + \lambda C(\Omega) = E(\Omega) + \lambda(V(\Omega) - V_0)$
- The shape derivative $w'_\Omega(\theta)$ of w_Ω is solution to the following problem :

$$\begin{cases} -\Delta w'_\Omega &= 0 & \text{in } \Omega \\ w'_\Omega &= -(\theta\cdot n)\partial_n w_\Omega & \text{on } \Gamma \end{cases}$$

Discretization

- The Newton method defines the next iterate $(\Omega_{k+1}, \lambda_{k+1})$ by :

$$\Omega_{k+1} = (Id + \theta_k)(\Omega_k) \qquad \text{and} \qquad \lambda_{k+1} = \lambda_k + \mu_k$$

where (θ_k, μ_k) are a solution to

$$\begin{pmatrix} L''_k(\Omega;\theta,\xi) & C'^T(\Omega;\theta) \\ C'(\Omega;\theta) & 0 \end{pmatrix} \begin{pmatrix} \theta_k \\ \mu_k \end{pmatrix} = - \begin{pmatrix} L'_k(\Omega;\xi) \\ c_k \end{pmatrix}$$

- Once again taking θ^h, ξ^h normal to the boundary gives the following discretization :
  - i $L''_k(\Omega, n^h, n^h)$ is the diagonal matrix of size $nbe \times nbe$ with diagonal entries $L''(\Omega, n_i^h, n_i^h)_{|x^i}$
  - ii $C'(\Omega, n^h)$ is the nbe vector with entries $C'(\Omega, n_i^h)_{|x^i}$
  - iii $L'(\Omega, n^h)$ is the nbe vector with entries $L'(\Omega, n_i^h)_{|x^i}$
  - iv c_k is the constraint.

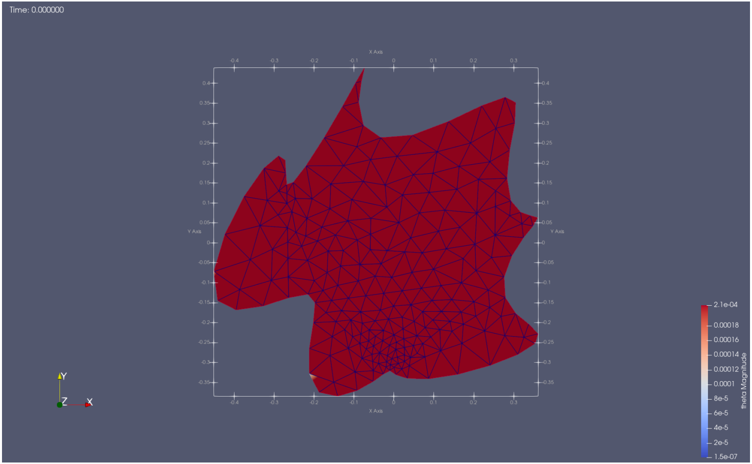
Simulation

The problem

Expectation of the water resistance

Newton's method

Numerical results



minimisation of the Dirichlet energy with volume constraint.

Back to the water resistance

The problem

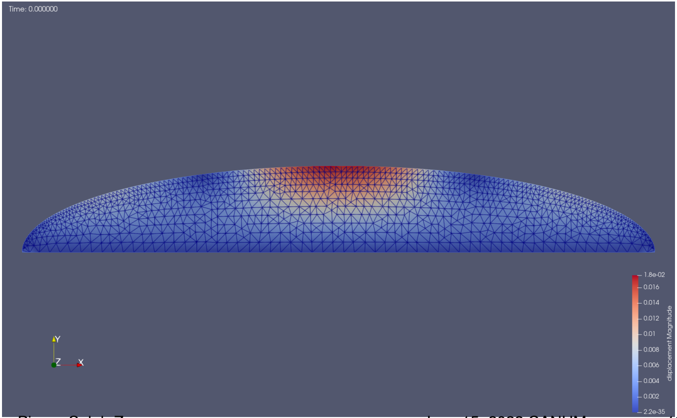
Expectation of the water resistance

Newton's method

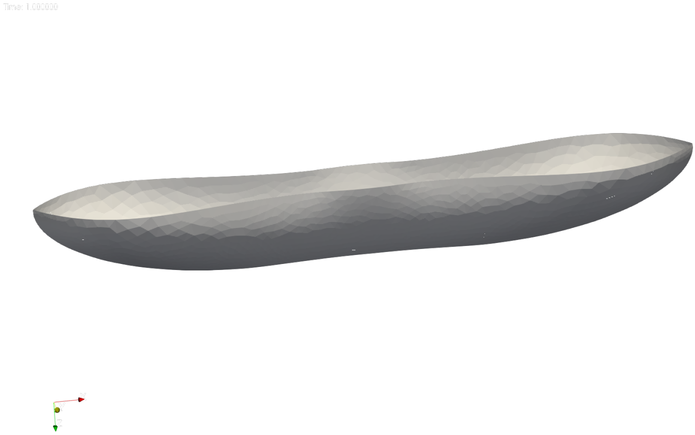
Numerical results

Theorem (J.Dambrine, Mo.Pierre)

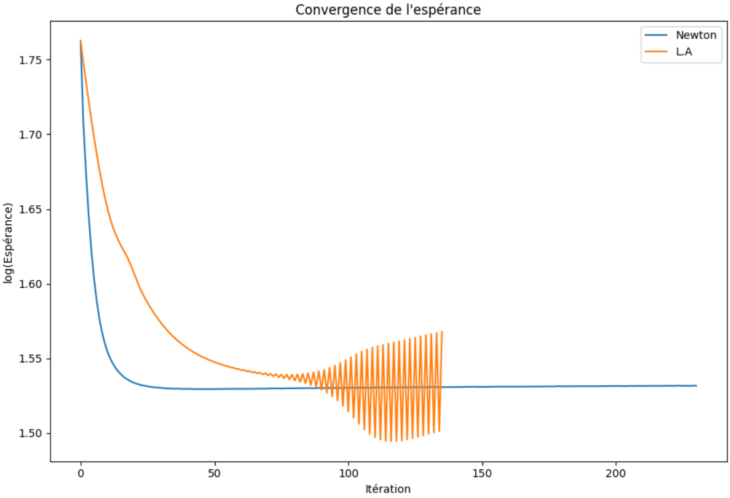
Michell's normalized wave resistance kernel k belongs to $L^q(D \times D)$ for all $1 \leqslant q < \frac{5}{4}$. Moreover, if D contains an open disc centered on the $x-$axis then k does not belong to $L^{5/4}(D \times D)$.



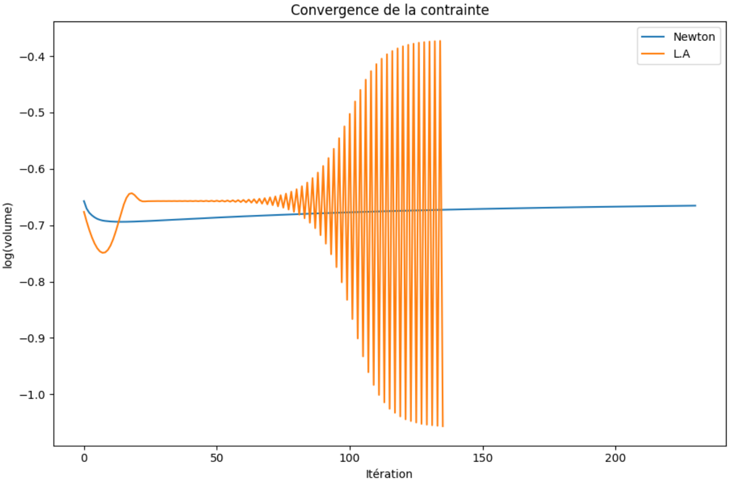
Optimal Hull



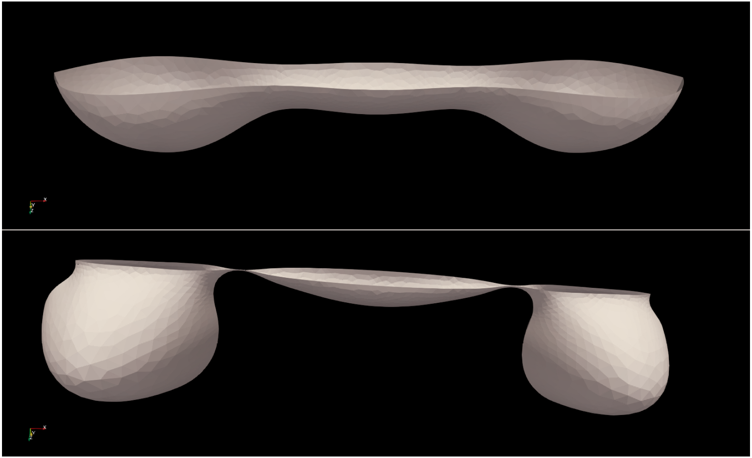
Numerical results : Convergence of the water resistance



Numerical results : Convergence of the constraint

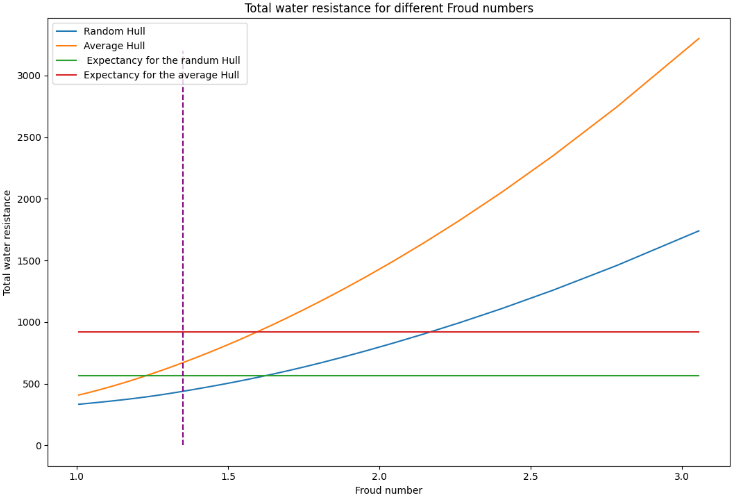


Numerical results : Optimal hulls



top : optimal hull for the expectancy, bottom : optimal hull for the average speed.

Numerical results : Comparison for different Froud numbers



Thank you