Critical Hardy inequality on the half-space via the harmonic transplantation

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A joint work with Prof. Futoshi Takahashi (Osaka Metropolitan University).

Plan of my talk

- §1 Introduction
  - The subcritical Hardy inequality ($1 < p < N$)
  - The critical Hardy inequality ($p = N$)

- §2 Main results
  - Thm 1 (The improved subcritical Hardy inequality on $\mathbb{R}^N_+$)
  - Thm 2 (The critical Hardy inequality on $\mathbb{R}^N_+$)

- §3 Harmonic transplantation and its applications
  - The limit of the subcritical Hardy inequality as $p \nearrow N$
  (- The limit of the Sobolev inequality as $N \nearrow \infty$)

- §4 Proof of Thm1
  - Difficulty (Explicit form of Green's function on $\mathbb{R}^N_+$)

§1.1 The subcritical Hardy inequality

Let Ω be a domain in $\mathbb{R}^N$, $a \in \Omega$ and $\dot{W}_0^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\nabla\cdot\|_{L^p(\Omega)}}$.

The subcritical Hardy inequality $(1 < p < N)$

$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x-a|^p}\,dx \le \int_\Omega |\nabla u|^p\,dx \quad \left({}^\forall u \in \dot{W}_0^{1,p}(\Omega)\right)$$

Facts

- The constant $\left(\frac{N-p}{p}\right)^p$ is optimal.
- $\left(\frac{N-p}{p}\right)^p = \inf_{0\not\equiv u\in \dot{W}_0^{1,p}(\Omega)} \dfrac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega \frac{|u|^p}{|x-a|^p}\,dx}$ is not attained for any domain Ω.
- Let $a = O$. $|x|^{-\frac{N-p}{p}} \notin \dot{W}_0^{1,p}$ is "*the virtual minimizer*".
  ($|x|^{-\frac{N-p}{p}}$ is a distributional sol. to the Euler-Lagrange equation.)

§1.2 The critical Hardy inequality

Let Ω be a bounded domain in $\mathbb{R}^N$, $a = O \in \Omega$ and $R = \sup_{x\in\Omega} |x|$.

The critical Hardy inequality ($p = N$)

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N \left(\log \frac{R}{|x|}\right)^N} \, dx \le \int_\Omega |\nabla u|^N \, dx \quad \left({}^\forall u \in \dot{W}_0^{1,N}(\Omega)\right)$$

Facts (Adimurthi-Sandeep, '02, D'Ambrosio-Dipierro, '14, Ioku-Ishiwata, '15)

- The constant $\left(\frac{N-1}{N}\right)^N$ is optimal.
- The optimal constant $\left(\frac{N-1}{N}\right)^N$ is not attained for any domain Ω.
- $\left(\log \frac{R}{|x|}\right)^{\frac{N-1}{N}} \notin \dot{W}_0^{1,N}$ is "*the virtual minimizer*".

  ($\left(\log \frac{R}{|x|}\right)^{\frac{N-1}{N}}$ is a distributional sol. to the E-L equation.)

Q. What happen when Ω is unbounded?
Especially when Ω is the half-space $\mathbb{R}^N_+$.

Remark (The critical Hardy type inequality on $\dot{W}_0^{1,N}(\mathbb{R}^N)$)

There is no weight function $g > 0$ such that the inequality

$$C\left(\int_{\mathbb{R}^N} |u|^q g(x)\, dx\right)^{\frac{N}{q}} \le \int_{\mathbb{R}^N} |\nabla u|^N\, dx$$

holds for any $u \in C^1_{c,\mathrm{rad}}(\mathbb{R}^N)$ for some $C > 0$.

Let $z = (x, y) \in \mathbb{R}^{N-1} \times (0, \infty) = \mathbb{R}^N_+$ and set
$a = (0, \cdots, 0, 1) \in \mathbb{R}^N_+$.

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$a = (0, \cdots, 0, 1) \in \mathbb{R}^N_+$.

§2 Main results

Thm 1 (The improved subcritical Hardy inequality on $\mathbb{R}^N_+$)

Let $2 \le p < N$. Then the improved subcritical Hardy inequality

$$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{V_p(x,y)^{\frac{p}{2}}}{\left(|x|^2+(1-y)^2\right)^{\frac{p}{2}}} |u(x,y)|^p \, dxdy \le \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p \, dxdy$$

holds for any $u \in \dot{W}^{1,p}_0(\mathbb{R}^N_+)$ (of the form $u(x,y) = u\left(U_p(x,y)\right)$), where

$$X = \frac{|x|^2+(1-y)^2}{|x|^2+(1+y)^2} \in [0,1),$$

$$V_p(x,y) = \frac{1 + X^{\frac{N-1}{p-1}} - 2X^{\frac{N-p}{2(p-1)}} \left(|x|^2+(1+y)^2\right)^{-1} (|x|^2+y^2-1)}{\left[1 - X^{\frac{N-p}{2(p-1)}}\right]^2} \ge 1.$$

The constant $(\frac{N-p}{p})^p$ is optimal and is not attained.

$\boxed{\text{Remark}}$ Actually, symmetric condition w.r.t. U_p is not essentially needed by using the result (D'Ambrosio-Dipierro, '14) and Lemma 1 later. However, in this talk, we will use the harmonic transplantation method under the symmetric condition.

$\boxed{\text{Remark}}$ We can take a limit of the improved subcritical Hardy inequality as $p \nearrow N$ as follows. (Note that $1 - X^\varepsilon \sim \varepsilon \log \frac{1}{X}$ as $\varepsilon \to 0$)

$$
\begin{aligned}
&\left(\frac{N-p}{p}\right)^p \frac{V_p(x,y)^{\frac{p}{2}}}{(|x|^2 + (1-y)^2)^{\frac{p}{2}}} \\
&\sim \left(\frac{N-p}{p}\right)^p \frac{\left\{1 + X - 2\left(|x|^2 + (y+1)^2\right)^{-1} (|x|^2 + y^2 - 1)\right\}^{\frac{N}{2}}}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left[\frac{N-p}{2(p-1)} \log \frac{1}{X}\right]^p} \\
&\sim \left(\frac{N-1}{N}\right)^N \frac{1}{(|x|^2 + (1-y)^2)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \quad (p \nearrow N).
\end{aligned}
$$

Thm 2 (The critical Hardy inequality on $\mathbb{R}^N_+$)

Let $N \geq 2$. Then the critical Hardy inequality

$$\left(\frac{N-1}{N}\right)^N \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^N \, dxdy}{\left(|x|^2+(1-y)^2\right)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \leq \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^N \, dxdy$$

holds for any $u \in \dot{W}^{1,N}_0(\mathbb{R}^N_+)$. The constant $(\frac{N-1}{N})^N$ is optimal and is not attained.

Remark (The behavior of the potential function)

$$V_N(x,y) := \frac{1}{\left(|x|^2+(1-y)^2\right)\left(\frac{|x|^2+(1+y)^2}{4}\right)\left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^2}$$

$$V_N(x,y)^{\frac{N}{2}} \sim \left(|x|^2+(1-y)^2\right)^{-\frac{N}{2}} \left(\log \frac{2}{\sqrt{|x|^2+(1-y)^2}}\right)^{-N} \quad ((x,y) \to (0,1))$$

$$V_N(x,y)^{\frac{N}{2}} \sim y^{-N} \quad \left(|x|^2+(y-1)^2 \to \infty \text{ or } y \to 0\right)$$

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holds for any $u \in \dot{W}^{1,N}_0(\mathbb{R}^N_+)$. The constant $(\frac{N-1}{N})^N$ is optimal and is not attained.

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§3.1 Harmonic transplantation

Let $B \subset \mathbb{R}^N$ be the unit ball, $\Omega\,(\subset \mathbb{R}^N)$ be a domain, $a \in \Omega$, $N \geq 2$, $1 < p \leq N$ and $G = G_{\Omega,a}$ be a sol. of

$$\begin{cases} -\Delta_p G_{\Omega,a} = \delta_a & \text{in } \Omega, \\ \qquad G_{\Omega,a} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where } \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

For a radial function v on B, define the function u on Ω as follows.

Harmonic transplantation (Ref. Hersch, 1969)

$$\text{(H.T.)} \quad u(y) = v(|x|), \text{ where } G_{\Omega,a}(y) = G_{B,O}(|x|).$$

(Cf. The Riemann mapping theorem)

Proposition（Preservation of $\|\nabla(\cdot)\|_p$）

It holds that $\|\nabla u\|_{L^p(\Omega)} = \|\nabla v\|_{L^p(B)}$.

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§3.2 Several applications of (H.T.)

Harmonic transplantation (Again)

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Domains or operators or dimensions of two Green's functions $G_{\Omega,a}$ are different from each other.

Harmonic transplantation (Generalization of (H.T.))

$$\text{(H.T.)} \quad u(y) = v(x)\,(= w(|z|)), \text{ where } G_{\Omega_1,a_1}(y) = \tilde{G}_{\Omega_2,a_2}(x)\,(= G_{B,O}(|z|)).$$

Ref. (Classification of various transformations) Moser, 1971, [Flucher, 1992, Csató-Roy, 2015], Bandle-Brilland-Flucher, 1998, Zographopoulous, 2010, Horiuchi-Kumlin, 2012, S.-Takahashi, 2017 (Equivalence of the subcritical and the critical Sobolev spaces), Ioku, 2019 (§3.3, $p \nearrow N$), S., 2020 (§3.4, $N \nearrow \infty$) etc.

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§3.3 Improved subcritical Hardy inequality on B and its limit : $p \nearrow N$

Let $p < N$. Consider the following (H.T.).

$$\boxed{\text{(H.T.)} \quad u(|x|) = v(|y|), \text{ where } \quad G_{\mathbb{R}^N,O}(|x|) = G_{B,O}(|y|)}$$

(Subcritical Hardy ineq.) $$\left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p}\, dx \leq \int_{\mathbb{R}^N} |\nabla u|^p\, dx$$

$$\| \qquad \downarrow \qquad \|$$

(Improved ineq.) $$\left(\frac{N-p}{p}\right)^p \int_B \frac{|v|^p}{|y|^p \left(1 - |y|^{\frac{N-p}{p-1}}\right)^p}\, dy \leq \int_B |\nabla v|^p\, dy$$

Since $1 - X^\varepsilon = \varepsilon \log \frac{1}{X} + o(\varepsilon)\,(\varepsilon \to 0)$, we obtain the critical Hardy ineq. on B as a limit of the improved Hardy ineq. as $p \nearrow N$.

$$\left(\frac{N-1}{N}\right)^N \int_B \frac{|v|^N}{|y|^N \left(\log \frac{1}{|y|}\right)^N}\, dy \leq \int_B |\nabla v|^N\, dy$$

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§3.4 Infinite dimensional form of the Sobolev inequality : $N \nearrow \infty$

Let $1 < p < m < N$. Consider the following (H.T.).

(H.T.) $u(|x|) = v(|y|)$, where $G_{\mathbb{R}^N,O}(|x|) = G_{\mathbb{R}^m,O}(|y|)$

(N-dim. Sobolev ineq.) $$S_{N,p}\left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}}\,dx\right)^{\frac{N-p}{N}} \le \int_{\mathbb{R}^N} |\nabla u|^p\,dx$$

$$\| \qquad \downarrow \qquad \|$$

$$S_{N,p}\left(\frac{|\mathbb{S}^{m-1}|}{|\mathbb{S}^{N-1}|}\right)^{\frac{p}{N}}\left(\frac{m-p}{N-p}\right)^{p-\frac{p}{N}}\left(\int_{\mathbb{R}^m} \frac{|v|^{\frac{Np}{N-p}}}{|y|^{\frac{N-m}{N-p}p}}\,dy\right)^{\frac{N-p}{N}} \le \int_{\mathbb{R}^m} |\nabla v|^p\,dy$$

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$$\left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|v|^p}{|y|^p}\,dy \le \int_{\mathbb{R}^m} |\nabla v|^p\,dy$$

Remark $\dot{W}_0^{1,p} \subset L^{p^*,p} \subset L^{p^*}$ (i.e. Hardy ineq. $\Rightarrow$ Sobolev ineq.)

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$$(N\text{-dim. Sobolev ineq.}) \quad S_{N,p} \left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}} \, dx \right)^{\frac{N-p}{N}} \leq \int_{\mathbb{R}^N} |\nabla u|^p \, dx$$

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$$S_{N,p} \left(\frac{|\mathbb{S}^{m-1}|}{|\mathbb{S}^{N-1}|} \right)^{\frac{p}{N}} \left(\frac{m-p}{N-p} \right)^{p-\frac{p}{N}} \left(\int_{\mathbb{R}^m} \frac{|v|^{\frac{Np}{N-p}}}{|y|^{\frac{N-m}{N-p}p}} \, dy \right)^{\frac{N-p}{N}} \leq \int_{\mathbb{R}^m} |\nabla v|^p \, dy$$

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$\boxed{\text{Remark}}$ $\dot{W}_0^{1,p} \subset L^{p^*,p} \subset L^{p^*}$ (i.e. Hardy ineq. $\Rightarrow$ Sobolev ineq.)

We want to apply (H.T.) on $\mathbb{R}^N_+$ in the same way as §3.3 on B.

Difficulty of Thm 1 (Δ_p is nonlinear operator)

Let $1 < p < N$. Green's function on $\mathbb{R}^N_+$:

$$G_{\mathbb{R}^N_+,(0,1)}(x,y) = \frac{p-1}{N-p}|\mathbb{S}^{N-1}|^{-\frac{1}{p-1}}\left[\left(|x|^2+(1-y)^2\right)^{-\frac{N-p}{2(p-1)}} - \psi_p(x,y)\right]$$

Although $\psi_2(x,y) = \left(|x|^2+(1+y)^2\right)^{-\frac{N-2}{2}}$, we do not know the explicit form of $\psi_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N_+)$ in general.

(★) Modified (H.T.)

$u(z) = v(\tilde{z})$, where $G_{\mathbb{R}^N,O}(\tilde{z}) = U_p(z)$ ($\neq G_{\mathbb{R}^N_+,(0,1)}(z)$ if $p \neq 2$)

$$:= \frac{p-1}{N-p}|\mathbb{S}^{N-1}|^{-\frac{1}{p-1}}\left[\left(|x|^2+(1-y)^2\right)^{-\frac{N-p}{2(p-1)}} - \left(|x|^2+(1+y)^2\right)^{-\frac{N-p}{2(p-1)}}\right]$$

§4 Proof of Thm 1

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$$-\Delta_p U_p = \frac{(N-p)(p-2)}{(p-1)^2\, \omega_{N-1}^{\frac{2}{p-1}}} |\nabla U_p|^{p-4} U_p \left[|x|^2 + (y-1)^2\right]^{-\frac{N-p}{2(p-1)}-1} \left[|x|^2 + (y+1)^2\right]^{-\frac{N-p}{2(p-1)}-1}$$
$$\times \left[N - p + (N+p-2)\frac{(|x|^2 + y^2 - 1)^2}{\{|x|^2 + (y-1)^2\}\{|x|^2 + (y+1)^2\}}\right].$$

Lemma 1 (Superharmonicity of U_p)

If $p \geq 2$, then $-\Delta_p U_p \geq 0$ in $\mathbb{R}^N_+ \setminus \{(0,1)\}$.

Lemma 2 (1-dim. weighted Hardy ineq.)

Let $p \geq 2$. Then, for any radial function $v \in C^1_c(\mathbb{R}^N)$, the inequality

$$\left(\frac{N-p}{p}\right)^p \int_0^\infty |v(t)|^p t^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(t)\right) dt \leq \int_0^\infty |v'(t)|^p t^{N-1} F_p\left(G_{\mathbb{R}^N,O}(t)\right) dt$$

holds, where $F_p(s) = \int_{[U_p > s]} (-\Delta_p U_p)\, dz \geq 0$.

§4 Proof of Thm 1

$$\int_{\mathbb{R}^N_+} |\nabla u(x,y)|^p \, dxdy$$

$$\overset{(\bigstar)}{=} \int_{\mathbb{R}^N} |\nabla v|^p \, d\tilde{z} + |\mathbb{S}^{N-1}| \int_0^\infty |v'|^p t^{N-1} F_p\left(G_{\mathbb{R}^N,O}(t)\right) \, dt$$

$$> \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N} \frac{|v|^p}{|\tilde{z}|^p} \, d\tilde{z} \quad \left(\because \text{ Lemma 2 \& Hardy ineq. on } \mathbb{R}^N\right)$$

$$+ \left(\frac{N-p}{p}\right)^p |\mathbb{S}^{N-1}| \int_0^\infty |v|^p t^{N-1-p} F_p\left(G_{\mathbb{R}^N,O}(t)\right) \, dt$$

$$\overset{(\bigstar)}{=} \left(\frac{N-p}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^p}{(|x|^2 + (1-y)^2)^{\frac{p}{2}}} V_p(x,y)^{\frac{p}{2}} \, dxdy$$

Thank you very much for your kind attention!

§3 Möbius transformation

Definition (Möbius transformation)

Let $O(N)$ be the orthogonal group in dimension N. Set

$$\begin{aligned} T_b(z) &= z + b \quad \text{(translation)}, \\ S_\lambda(z) &= \lambda z \quad \text{(scaling)}, \\ R(z) &= Rz \quad \text{(rotation)}, \\ J(z) &= z^* = \frac{z}{|z|^2} \quad \text{(reflection)} \end{aligned}$$

for $b \in \mathbb{R}^N, \lambda > 0, R \in O(N)$. A *Möbius transformation* $M : \mathbb{R}^N \to \mathbb{R}^N$ is a finite composition of T_b, S_λ, R and J.

$$(M.T.) \quad (M^\# f)(z) = |\det M'(z)|^{\frac{N-p}{Np}} f(M(z)), \quad z \in \mathbb{R}^N.$$

§3 Property of (M.T.)

$$(M.T.) \quad (M^\# f)(z) = |\det M'(z)|^{\frac{N-p}{Np}} f(M(z)), \quad z \in \mathbb{R}^N.$$

Remark

$$(T_b^\# f)(z) = f(z+b),$$
$$(S_\lambda^\# f)(z) = \lambda^{\frac{N-p}{p}} f(\lambda z),$$
$$(R^\# f)(z) = f(Rz),$$
$$(J^\# f)(z) = |z|^{\frac{2}{p}(p-N)} f\left(\frac{z}{|z|^2}\right).$$

Proposition 2 (Preservation of $\|\nabla(\cdot)\|_p$)

For any $f \in C_c^1(\mathbb{R}^N \setminus \{0\})$,

$$\|\nabla(M^\# f)\|_{L^p(\mathbb{R}^N)} = \|\nabla f\|_{L^p(\mathbb{R}^N)}. \text{ (only when } p = 2, N \text{ if } M \text{ includes } J)$$

§3 Difference between (M.T.) and (H.T.)

$\boxed{p = N \text{ and } \Omega = B_R \subset \mathbb{R}^N}$

$$\text{(H.T.)} \quad u(|y|) = v(|x|), \text{ where } G_{B_R,O}(|y|) = G_{B,O}(|x|),$$

$$\text{namely, } \log \frac{R}{|y|} = \log \frac{1}{|x|}.$$

$$\text{(M.T.)} \quad (M^{\#} f)(y) = f(x), \text{ where } x = M(y) = \frac{y}{R}.$$

$\boxed{p < N \text{ and } \Omega = B_R \subset \mathbb{R}^N}$

$$\text{(H.T.)} \quad u(|y|) = v(|x|), \text{ where } G_{B_R,O}(|y|) = G_{B,O}(|x|),$$

$$\text{namely, } |y|^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} = |x|^{-\frac{N-p}{p-1}} - 1.$$

$$\text{(M.T.)} \quad (M^{\#} f)(y) = R^{-\frac{N-p}{p}} f(x), \text{ where } x = M(y) = \frac{y}{R}.$$

§3.4 Infinite dimensional form of the Sobolev inequality : $N \nearrow \infty$

Let $1 < p < m < N$. Consider the following (H.T.).

$$\boxed{\text{(H.T.)} \quad u(|x|) = v(|y|), \text{ where } G_{\mathbb{R}^N,O}(|x|) = G_{\mathbb{R}^m,O}(|y|)}$$

$$(N\text{-dim. Sobolev ineq.}) \quad S_{N,p} \left(\int_{\mathbb{R}^N} |u|^{\frac{Np}{N-p}}\, dx \right)^{\frac{N-p}{N}} \leq \int_{\mathbb{R}^N} |\nabla u|^p\, dx$$

$$\| \qquad \downarrow \qquad \|$$

$$S_{N,p} \left(\frac{|\mathbb{S}^{m-1}|}{|\mathbb{S}^{N-1}|} \right)^{\frac{p}{N}} \left(\frac{m-p}{N-p} \right)^{p-\frac{p}{N}} \left(\int_{\mathbb{R}^m} \frac{|v|^{\frac{Np}{N-p}}}{|y|^{\frac{N-m}{N-p}p}}\, dy \right)^{\frac{N-p}{N}} \leq \int_{\mathbb{R}^m} |\nabla v|^p\, dy$$

We obtain the Hardy ineq. as a limit of the Sobolev ineq. as $N \nearrow \infty$.

$$\left(\frac{m-p}{p} \right)^p \int_{\mathbb{R}^m} \frac{|v|^p}{|y|^p}\, dy \leq \int_{\mathbb{R}^m} |\nabla v|^p\, dy$$

§3.4 Infinite dimensional form of the Sobolev inequality : $N \nearrow \infty$

Since $\Gamma(t) \sim \sqrt{2\pi}\, t^{t-\frac{1}{2}}\, e^{-t}$ $(t \to \infty)$ and $|\mathbb{S}^{N-1}| = \dfrac{N\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}$, we have

$$S_{N,p}\left(\frac{|\mathbb{S}^{m-1}|}{|\mathbb{S}^{N-1}|}\right)^{\frac{p}{N}}\left(\frac{m-p}{N-p}\right)^{p-\frac{p}{N}}$$

$$= \pi^{\frac{p}{2}} N\left(\frac{N-p}{p-1}\right)^{p-1}\left(\frac{\Gamma\left(\frac{N}{p}\right)\Gamma\left(\frac{N}{p'}+1\right)}{\Gamma(N)\Gamma\left(1+\frac{N}{2}\right)}\right)^{\frac{p}{N}}\left(\frac{m\pi^{\frac{m}{2}}\Gamma\left(1+\frac{N}{2}\right)}{N\pi^{\frac{N}{2}}\Gamma\left(1+\frac{m}{2}\right)}\right)^{\frac{p}{N}}\left(\frac{m-p}{N-p}\right)^{p-\frac{p}{N}}$$

$$\sim \frac{(m-p)^p}{(p-1)^{p-1}}\left(\frac{\left(\frac{N}{p}\right)^{\frac{N}{p}-\frac{1}{2}} e^{-\frac{N}{p}}\left(\frac{N}{p'}+1\right)^{\frac{N}{p'}+1-\frac{1}{2}} e^{-\frac{N}{p'}-1}}{N^{N-\frac{1}{2}} e^{-N}}\right)^{\frac{p}{N}}$$

$$\sim \frac{(m-p)^p}{(p-1)^{p-1}}\cdot\frac{1}{N^p}\cdot\frac{N}{p}\left(\frac{p-1}{p}N+1\right)^{p-1} \sim \left(\frac{m-p}{p}\right)^p \quad (N \nearrow \infty).$$

Let $1 < p = N < m$. Consider

$$\boxed{\text{(H.T.)} \quad u(|x|) = v(|y|), \text{ where } G_{\mathbb{R}^m,O}(|x|) = G_{B,O}(|y|), \ B \subset \mathbb{R}^N}$$

$$(m\text{-dim. Hardy ineq.}) \quad \left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|u|^p}{|x|^p}\,dx \le \int_{\mathbb{R}^m} |\nabla u|^p\,dx$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$

$$(\text{Critical Hardy ineq.}) \quad \left(\frac{N-1}{N}\right)^N \int_B \frac{|v|^N}{|y|^N \left(\log \frac{1}{|y|}\right)^N}\,dy \le \int_B |\nabla v|^N\,dy$$

$\boxed{\text{Remark}}$ We can rewrite (H.T.) as follows.

$$\left(\frac{p-1}{m-p}\right)^{\frac{p-1}{p}} |\mathbb{S}^{m-1}|^{-\frac{1}{p}} |x|^{-\frac{m-p}{p}} = |\mathbb{S}^{N-1}|^{-\frac{1}{N}} \left(\log \frac{1}{|y|}\right)^{\frac{N-1}{N}} \quad \text{(Virtual minimizers)}$$

§4.2 Proof of Thm 2

Let $p = N$ and $\Omega = \mathbb{R}^N_+$. Consider (H.T.) :

$$u(z) = v(\tilde{z}), \text{ where } |\mathbb{S}^{N-1}|^{-\frac{1}{N-1}} \log \sqrt{\frac{|x|^2 + (1+y)^2}{|x|^2 + (1-y)^2}} = G_{\mathbb{R}^N_+,(0,1)}(z)$$
$$= G_{B,O}(\tilde{z})$$

which coincides with the Möbius transformation $\mathbf{B}$:

$$(\tilde{x}, \tilde{y}) = \mathbf{B}(x, y) = \left(\frac{2x,\ 1 - |x|^2 - y^2}{(1+y)^2 + |x|^2} \right), \quad z = (x, y) \in \mathbb{R}^N_+,\ \tilde{z} = (\tilde{x}, \tilde{y}) \in B,$$

$$\mathbf{B}(z) = R \circ J \circ T_{(0,1)} \circ S_2 \circ J \circ T_{(0,-1)}(z), \text{ where } R = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

§4.2 Proof of Thm 2

$$|\mathbf{B}(z)| = \frac{|x|^2 + (1-y)^2}{|x|^2 + (1+y)^2}, \ \det \mathbf{B}'(z) = -\left\{ \frac{2}{(1+y)^2 + |x|^2} \right\}^N .$$

$$\int_B \frac{|v(\tilde{x}, \tilde{y})|^N}{\{|\tilde{x}|^2 + |\tilde{y}|^2\}^{\frac{N}{2}} \left(\log \frac{1}{\sqrt{|\tilde{x}|^2 + |\tilde{y}|^2}}\right)^N} \, d\,\tilde{x}\, d\,\tilde{y}$$

$$= \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^N}{|\mathbf{B}(x,y)|^{\frac{N}{2}} \left(\log \frac{1}{|\mathbf{B}(x,y)|}\right)^N} |\det \mathbf{B}'(x,y)| \, dx\, dy$$

$$= \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^N}{\{|x|^2 + (1-y)^2\}^{\frac{N}{2}} \left\{\frac{|x|^2+(1+y)^2}{4}\right\}^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \, dxdy.$$

§4.2 Proof of Thm 2

(Critical Hardy ineq. on B) $$\left(\frac{N-1}{N}\right)^N \int_B \frac{|v|^N}{|\tilde{z}|^N \left(\log \frac{1}{|\tilde{z}|}\right)^N} \, d\tilde{z} \le \int_B |\nabla v|^N \, d\tilde{z}$$

$$\| \qquad \downarrow \qquad \|$$ (Proposition 2)

$$\left(\frac{N-1}{N}\right)^N \int_{\mathbb{R}^N_+} \frac{|u(x,y)|^N \, dxdy}{\left(|x|^2 + (1-y)^2\right)^{\frac{N}{2}} \left(\frac{|x|^2+(1+y)^2}{4}\right)^{\frac{N}{2}} \left(\log \sqrt{\frac{|x|^2+(1+y)^2}{|x|^2+(1-y)^2}}\right)^N} \le \int_{\mathbb{R}^N_+} |\nabla u(x,y)|^N \, dxdy$$