Conditions Limites Absorbantes d'ordre faible pour l'équation de Helmholtz convectée

16 2022
CANUM 2020+2

Nathan Rouxelin (LMI – INSA Rouen)
Joint work with: H. Barucq, S. Tordeux (MAKUTU – Inria, Univ Pau, TotalEnergies)

Inria Makutu – e2s-UPPA – LMAP UMR CNRS 5142 – TotalEnergies

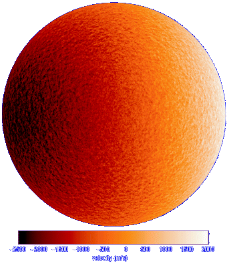
Outline

Context

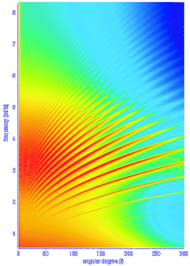
Context of this work

Helioseismology in a nutshell:

- Aims at imaging the solar interior thanks to surface observations
- Surfacic «acoustic waves» can be measured thanks to the Doppler effect
- The full models are too complicated for numerical simulation, two main approaches for computational helioseismology:
  - Aeroacoustics: magnetic effects are neglected
  - Magnetoacoustics: hydrodynamics effects are neglected



(a) Dopplergram



(b) Power spectrum

Context of this work

Helioseismology in a nutshell:

- Aims at imaging the solar interior thanks to surface observations
- Surfacic «acoustic waves» can be measured thanks to the Doppler effect
- The full models are too complicated for numerical simulation, two main approaches for computational helioseismology:
  - Aeroacoustics: magnetic effects are neglected
  - Magnetoacoustics: hydrodynamics effects are neglected

If you want to know more about helioseismology:

J. Christensen-Dalsgaard.
Lecture Notes on Stellar Oscillations.
University of Aarhus

Model problem and discretization

Admissible cases

To construct ABCs for the convected Helmholtz equation, we limit ourselves to the following configurations:

- the physical parameters can vary inside of the computational domain,
- but they become uniform far from the source.

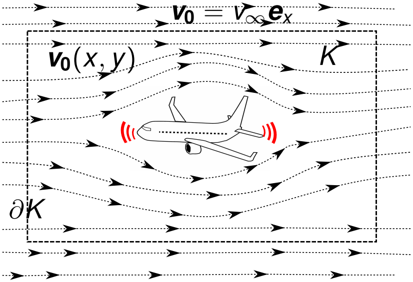


Figure: Admissible configuration

Convected Helmholtz equation

Model problem :

$$\rho_0 \left(-\omega^2 p - 2i\omega \mathbf{v_0} \cdot \nabla p + \mathbf{v_0} \cdot \nabla(\mathbf{v_0} \cdot \nabla p)\right) - \mathrm{div}\left(\rho_0 {c_0}^2 \nabla p\right) = s$$

where

- p: acoustic potential ($\sim$ pressure perturbation)
- ρ_0: mass density
- $\mathbf{v_0}$: velocity field
- c_0: sound speed
- s: acoustic source

We only consider the **subsonic case** where

$$M = \frac{|\mathbf{v_0}|}{c_0} < 1, \ \ \forall x \in \mathcal{O}.$$

Convected Helmholtz equation

Model problem :

$$\rho_0\left(-\omega^2 p - 2i\omega \mathbf{v_0}\cdot\nabla p\right) - \mathrm{div}\left(\mathbf{K_0}\nabla p\right) = s$$

where $\mathbf{K_0} = \rho_0 {c_0}^2 \mathrm{Id} - \rho_0 \mathbf{v_0}\mathbf{v_0}^T$.

Convected Helmholtz equation

Model problem : We solve the *total flux* formulation

$$\begin{aligned} \boldsymbol{\sigma} + \boldsymbol{K_0}\nabla p + 2i\omega p \rho_0 \boldsymbol{v_0} &= 0, \\ -\rho_0\omega^2 p + \operatorname{div}(\boldsymbol{\sigma}) &= s, \end{aligned}$$

where $\boldsymbol{\sigma} = -\boldsymbol{K_0}\nabla p - 2i\omega p \rho_0 \boldsymbol{v_0}$ using a HDG method.

H. Barucq, N. Rouxelin, S. Tordeux.
HDG and HDG+ methods for harmonic wave problems with convection.
https://hal.inria.fr/hal-03253415

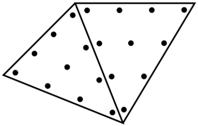
Convected Helmholtz equation

Model problem : We solve the *total flux* formulation

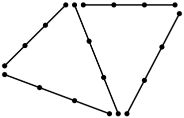
$$\boldsymbol{\sigma} + \boldsymbol{K_0} \nabla p + 2 i \omega p \rho_0 \boldsymbol{v_0} = 0,$$
$$-\rho_0 \omega^2 p + \operatorname{div}(\boldsymbol{\sigma}) = s,$$

where $\boldsymbol{\sigma} = -\boldsymbol{K_0} \nabla p - 2 i \omega p \rho_0 \boldsymbol{v_0}$ using a HDG method.

H. Barucq, N. Rouxelin, S. Tordeux.
HDG and HDG+ methods for harmonic wave problems with convection.
https://hal.inria.fr/hal-03253415



(a) DG



(b) HDG

Figure: Degrees of freedom for degree 3

Convected Helmholtz equation

Model problem : We solve the *total flux* formulation

$$\boldsymbol{\sigma} + \boldsymbol{K_0}\nabla p + 2i\omega p \rho_0 \boldsymbol{v_0} = 0,$$
$$-\rho_0\omega^2 p + \operatorname{div}(\boldsymbol{\sigma}) = s,$$

where $\boldsymbol{\sigma} = -\boldsymbol{K_0}\nabla p - 2i\omega p \rho_0 \boldsymbol{v_0}$ using a HDG method.

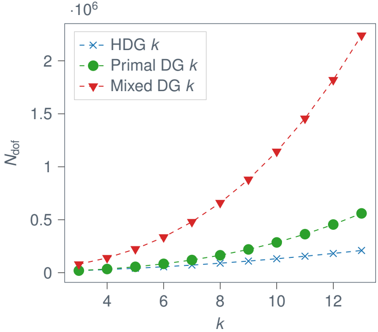


Figure: Sizes of the system to solve

Domain truncation

Domain truncation

To simulate wave propagation in infinite domains, we need to truncate the domain

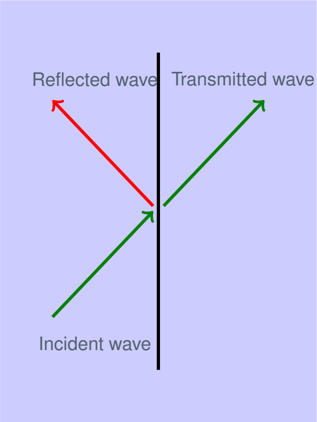


Figure: Reflection at the artificial boundary

Domain truncation

To simulate wave propagation in infinite domains, we need to truncate the domain

- *Perfectly Matched Layers*: absorbing layer surrounding the domain

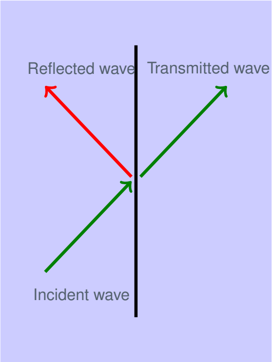
JP. Berenger.
A perfectly matched layer for the absorption of electromagnetic waves.
Journal of Computational Physics - 1994

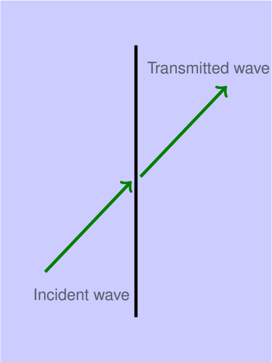
P. Marchner, H. Beriot, X. Antoine, C. Geuzaine.
Stable Perfectly Matched Layers with Lorentz transformation for the convected Helmholtz equation.
Journal of Computational Physics - 2021

Domain truncation

To simulate wave propagation in infinite domains, we need to truncate the domain

- *Perfectly Matched Layers*: absorbing layer surrounding the domain
- *Absorbing Boundary Conditions*

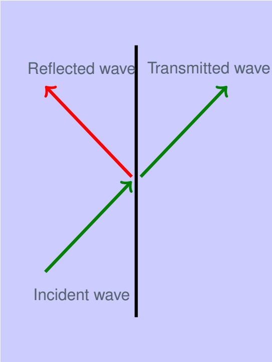


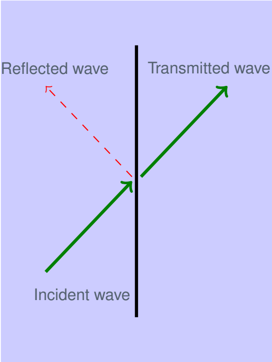


Domain truncation

To simulate wave propagation in infinite domains, we need to truncate the domain

- *Perfectly Matched Layers*: absorbing layer surrounding the domain
- *Absorbing Boundary Conditions*





Domain truncation

To simulate wave propagation in infinite domains, we need to truncate the domain

- *Perfectly Matched Layers*: absorbing layer surrounding the domain
- *Absorbing Boundary Conditions*

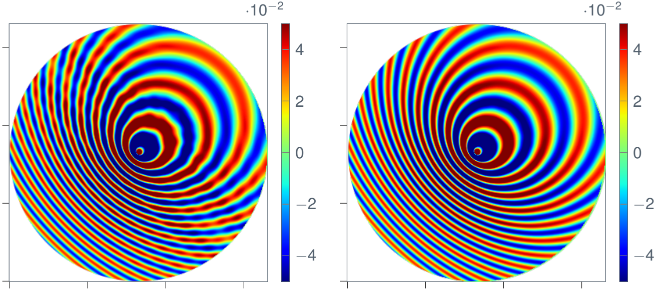
B. Engquist, A. Majda.
Absorbing Boundary Conditions for the Numerical Simulation of Waves.
Mathematics of Computation - 1977

A. Bayliss, E. Turkel.
Radiation boundary conditions for wave-like equations.
Communications on Pure and Applied Mathematics - 1980

H. Barucq, N. Rouxelin, S. Tordeux.
Prandtl-Glauert-Lorentz based ABCs for the convected Helmholtz equation.
Submitted – 2021

First idea

Usual idea: Construct an ABC that selects outgoing planewaves that are locally orthogonal to the boundary.



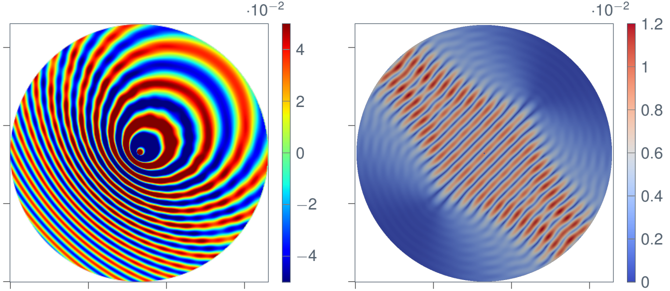
(a) Numerical solution (b) Exact solution

Figure: Planewave-based ABC with $M = 0.6$

The flow generates spurious oscillations !

First idea

Usual idea: Construct an ABC that selects outgoing planewaves that are locally orthogonal to the boundary.



(a) Numerical solution (b) Local error

Figure: Planewave-based ABC with $M = 0.6$

The flow generates spurious oscillations !

Construction of ABCs

Standard Helmholtz

$$-\widetilde{\omega}^2\widetilde{p} - {c_0}^2\widetilde{\Delta}\widetilde{p} = \widetilde{s}$$

Convected Helmholtz

$$-\omega^2 p - 2i\omega \boldsymbol{v_0}\cdot\nabla p - \mathrm{div}\left(\boldsymbol{K_0}\nabla p\right) = s$$

Prandtl-Glauert-Lorentz transformation

$$\widetilde{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} = \left(\mathrm{Id} + \frac{1}{\alpha(1+\alpha)}\frac{\boldsymbol{v_0}{\boldsymbol{v_0}}^T}{{c_0}^2}\right)\boldsymbol{x}, \;\; \widetilde{\omega} = \frac{\omega}{\alpha}, \;\; \alpha = \sqrt{1 - \frac{|\boldsymbol{v_0}|^2}{{c_0}^2}}$$

Then

$$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega}) := \alpha \exp\left[\frac{i\omega}{\alpha^2 {c_0}^2}\boldsymbol{v_0}\cdot\boldsymbol{x}\right] p(\boldsymbol{x}, \omega)$$

Only for uniform physical parameters !

Construction of ABCs

Standard Helmholtz

$$-\widetilde{\omega}^2\widetilde{p} - {c_0}^2\widetilde{\Delta}\widetilde{p} = \widetilde{s}$$

Convected Helmholtz

$$-\omega^2 p - 2i\omega \boldsymbol{v_0}\cdot\nabla p - \mathrm{div}\,(\boldsymbol{K_0}\nabla p) = s$$

Prandtl-Glauert-Lorentz transformation

$$\widetilde{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} = \left(\mathrm{Id} + \frac{1}{\alpha(1+\alpha)}\frac{\boldsymbol{v_0}{\boldsymbol{v_0}}^T}{{c_0}^2}\right)\boldsymbol{x}, \quad \widetilde{\omega} = \frac{\omega}{\alpha}, \quad \alpha = \sqrt{1 - \frac{|\boldsymbol{v_0}|^2}{{c_0}^2}}$$

Then

$$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega}) := \alpha \exp\left[\frac{i\omega}{\alpha^2 {c_0}^2}\boldsymbol{v_0}\cdot\boldsymbol{x}\right] p(\boldsymbol{x}, \omega)$$

Furthermore, if $\widetilde{p}$ is an outgoing solution, so is p.
Limiting Amplitude Principle:

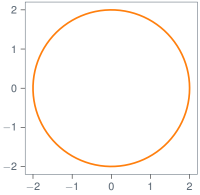
$$\lim_{t\to+\infty}\left\|p(\boldsymbol{x}, t) - p(\boldsymbol{x}, \omega)e^{-i\omega t}\right\| = 0,$$

where p is the solution of the time-domain equation with $s = g(x)e^{-i\omega t}$.

Construction of ABCs

Standard Helmholtz

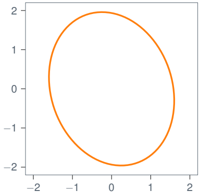
$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega})$



(a) Boundary $\widetilde{\Sigma}$

Convected Helmholtz

$p(\boldsymbol{x}, \omega)$



(b) Boundary Σ

Prandtl-Glauert-Lorentz transformation

$$\widetilde{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} = \left(\mathrm{Id} + \frac{1}{\alpha(1+\alpha)} \frac{\boldsymbol{v_0}\boldsymbol{v_0}^T}{c_0{}^2}\right)\boldsymbol{x}, \quad \widetilde{\omega} = \frac{\omega}{\alpha}, \quad \alpha = \sqrt{1 - \frac{|\boldsymbol{v_0}|^2}{c_0{}^2}}$$

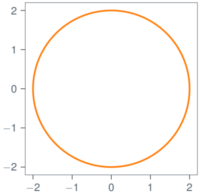
Then

$$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega}) := \alpha \exp\left[\frac{i\omega}{\alpha^2 c_0{}^2} \boldsymbol{v_0} \cdot \boldsymbol{x}\right] p(\boldsymbol{x}, \omega)$$

Construction of ABCs

Standard Helmholtz

$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega})$

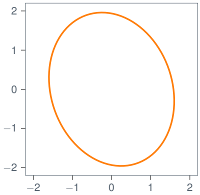


(a) Boundary $\widetilde{\Sigma}$

$$\partial_{\widetilde{\boldsymbol{n}}}\widetilde{p} + \widetilde{\mathcal{Z}}\widetilde{p} = 0$$

Convected Helmholtz

$p(\boldsymbol{x}, \omega)$



(b) Boundary Σ

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} + \mathcal{Z}p = 0$$

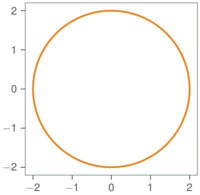
Change of ABC

$$\mathcal{Z} = -c_0{}^2 |\boldsymbol{A}^{-T}\boldsymbol{n}| \widetilde{\mathcal{Z}}(\widetilde{\boldsymbol{x}}, \widetilde{\omega}) + i\omega \boldsymbol{v_0} \cdot \boldsymbol{n}$$

Construction of ABCs

Standard Helmholtz

$$\widetilde{p}(\widetilde{\boldsymbol{x}}, \widetilde{\omega})$$

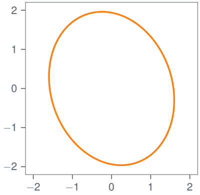


(a) Boundary $\widetilde{\Sigma}$

$$\partial_{\widetilde{\boldsymbol{n}}}\widetilde{p} + \widetilde{\mathcal{Z}}\widetilde{p} = 0$$

Convected Helmholtz

$$p(\boldsymbol{x}, \omega)$$



(b) Boundary Σ

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} + \mathcal{Z}p = 0$$

Example :

$$\widetilde{\mathcal{Z}} = -\frac{i\widetilde{\omega}}{c_0} + \frac{1}{2R} \implies \mathcal{Z} = -\frac{{c_0}^2|\boldsymbol{A}^{-T}\boldsymbol{n}|}{2R} + i\left(\frac{c_0|\boldsymbol{A}^{-T}\boldsymbol{n}|}{\alpha} + \boldsymbol{v_0}\cdot\boldsymbol{n}\right)\omega$$

Performance assessment – 1/4

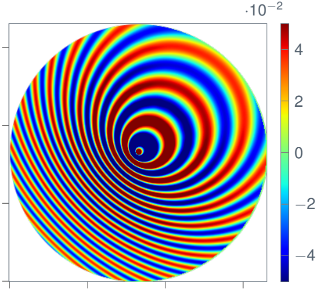


Figure: First-order PGL-based ABC with $M = 0.6$

Performance assessment – 2/4

We define the relative error as

$$\mathcal{E}_{\mathcal{O}} := \sqrt{\frac{\sum_{K,i} |p_h - p_{\text{ref}}|^2(\boldsymbol{x}_i^K)}{\sum_{K,i} |p_{\text{ref}}|^2(\boldsymbol{x}_i^K)}},$$

and the convected Helmholtz number as

$$\mathrm{He} := \frac{\omega \alpha R}{2\pi(1+M)}.$$

R	He	ABC-0	ABC-1	ABC-PW
0.5	0.75	2.14%	0.67%	8.3%
1.0	1.5	1.21%	0.62%	7.31%
1.5	2.25	0.98%	0.66%	8.02%
2.0	3.0	0.83%	0.64%	7.1%

Table: Relative error $\mathcal{E}_{\mathcal{O}}$ in the domain for $M = 0.6$

Performance assessment – 3/4

Local error

$$\mathcal{E}_{\mathrm{loc}}(\boldsymbol{x}) = |p_h - p_{\mathrm{ref}}|^2$$

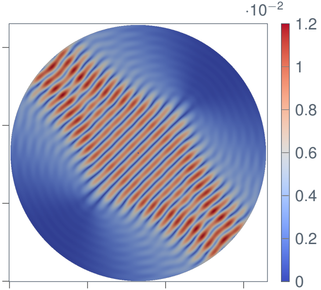


Figure: ABC-PW

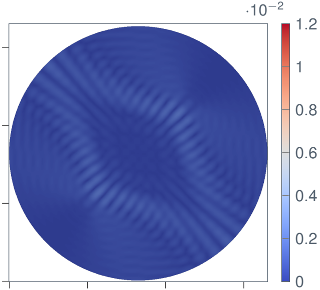


Figure: ABC-1

Performance assessment – 4/4

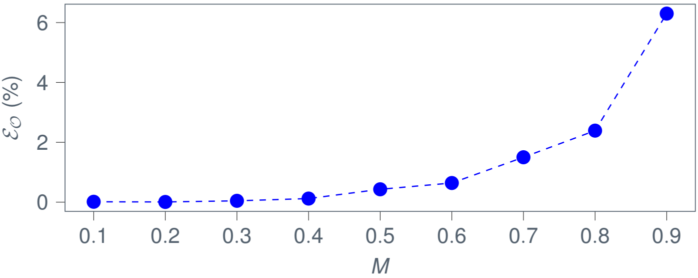


Figure: $\mathcal{E}_{\mathcal{O}}$ with respect to M

- Good numerical results for low and intermediate Mach numbers.
- Not robust to high Mach numbers: higher order ABCs or PMLs should be considered.

Higher-order ABCs ?

- In principle, it's possible...
- but very difficult with DG methods !

Higher-order ABCs ?

- In principle, it's possible...
- but very difficult with DG methods !

Indeed, the second-order boundary involves the Laplace-Beltrami operator on $\Gamma = \cup_k [s_k, s_{k+1}]$, which leads to the following term

$$\sum_k \int_{s_k}^{s_{k+1}} \partial^2_{\nu\nu} \Phi_j \Phi_i \mathrm{d}\sigma = -\sum_k \int_{s_k}^{s_{k+1}} \partial_\nu \Phi_j \partial_\nu \Phi_i \mathrm{d}\sigma + \sum_k \left[\partial_\nu \Phi_j \Phi_i\right]_{s_k}^{s_{k+1}}$$

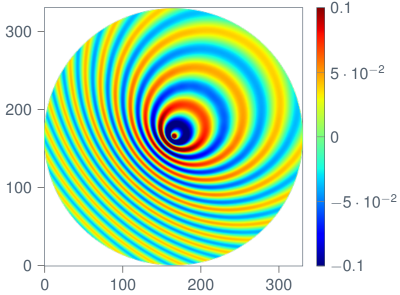
Numerical experiments

Uniform flow:

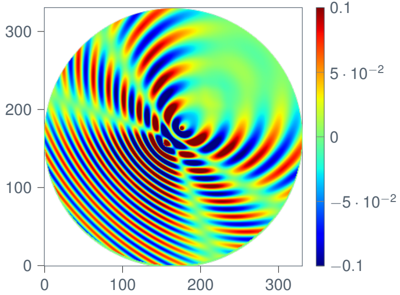
$$\mathbf{v_0} = 0.6 \begin{bmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{bmatrix}$$

Validation: comparison with analytic solutions, for both cases the *relative error*

$$\mathcal{E} := \left(\frac{\sum_i |p_h(\boldsymbol{x}_i) - p_{\text{ref}}(\boldsymbol{x}_i)|^2}{\sum_i |p_{\text{ref}}(\boldsymbol{x}_i)|^2} \right)^{\frac{1}{2}} < 1\%.$$



(a) One point-source



(b) Two point-sources

Potential flow around a circular obstacle:

$$\boldsymbol{v_0} = M_\infty \left[\left(1 - \frac{R_C^2}{r^2} \right) \cos\theta \, \boldsymbol{e}_r + \left(1 + \frac{R_C^2}{r^2} \right) \sin\theta \, \boldsymbol{e}_\theta \right],$$

where R_C is the radius of the obstacle and M_∞ the Mach number at infinity.

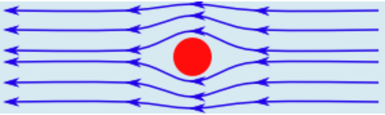


Figure: Potential flow around a circular obstacle

Illustrative examples

Potential flow around a circular obstacle:

$$\boldsymbol{v_0} = M_\infty \left[\left(1 - \frac{R_C^2}{r^2}\right) \cos\theta \, \boldsymbol{e}_r + \left(1 + \frac{R_C^2}{r^2}\right) \sin\theta \, \boldsymbol{e}_\theta \right],$$

where R_C is the radius of the obstacle and M_∞ the Mach number at infinity.

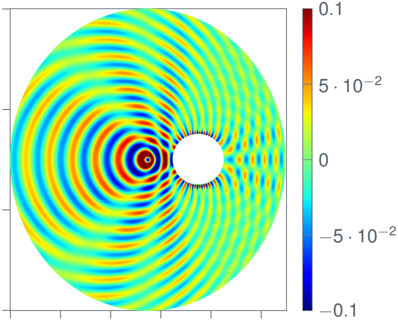


Figure: Point source in a potential flow around an obstacle using HDG

Thank you for your attention!
Any questions ?