Analysis of a domain decomposition method for a convected Helmholtz like equation

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In this work, we are interested in solving an equation of the form

$$-\mathrm{div}\,(A\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ equipped with, for instance, homogeneous Dirichlet boundary conditions. In this equation, A is a symmetric positive definite matrix, $\mathbf{a}$ a vector of $\mathbb{R}^2$ and $\mu \geq 0$. For the analysis, we will assume that A, $\mathbf{a}$ and μ are constant. This type of equation occurs in several contexts such as the convected Helmholtz equation [5], the wave-ray equation [6], or in a minimization process for solving the Gröss-Pitaevskii equation [2].

For the convected Helmholtz equation, it is well-known that using an appropriate change of variables, one can reformulate equation (1) as a "classical" Helmholtz equation. In particular, this enables to derive Perfectly Matched Layers (PML) formulations [5] or Absorbing Boundary Conditions (ABC) [1]. Using a similar idea, we propose a change of variables that enables us to reformulate (1) as a Helmholtz equation. This allows us to derive a PML formulation as well as a low order ABC for the slightly more general equation (1).

We can then also easily study the convergence properties of an iterative Schwarz algorithm with overlap, based on the study of a similar Schwarz algorithm for the Helmholtz equation [3] (besides, note also for the non-overlapping case the recent work [4]). In particular, one can derive optimized transmission conditions from the optimized transmission conditions for the Helmholtz problem. We also study the impact on the convergence rate of using PML to simulate unbounded domains. In particular, we observe that the convergence rate becomes much better.

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