Analysis of a domain decomposition method for the convected Helmholtz equation

CANUM 2020 - Evian les Bains

13 - 17 June 2022

M.J. Gander,[1] A. Tonnoir[2]

UNIVERSITÉ DE GENÈVE[1]

LMI - EA 3226[2]

Motivation

In this work, we are interested in solving an Helmholtz like equation:

$$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

where:
* A is a 2x2 symmetric positive definite matrix,
* $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
* μ is a real constant.

Motivation

In this work, we are interested in solving an **Helmholtz like equation**:

$$-\mathrm{div}(A\,\nabla \mathbf{u}) + i\mathbf{a}\cdot\nabla\mathbf{u} + \mu\mathbf{u} = \mathsf{f}$$

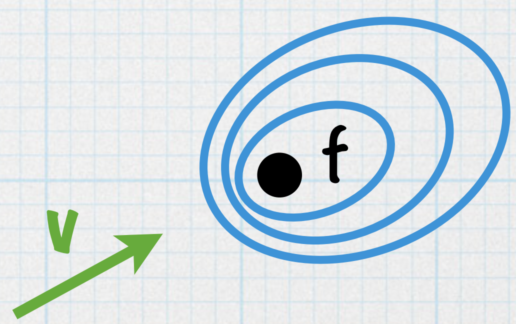
where:
- A is a 2x2 symmetric positive definite matrix,
- $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
- μ is a real constant.

This type of equation occurs in several contexts:

- The **convected Helmholtz** equation $\left(A = \mathbf{c}_0 I - \mathbf{v}\mathbf{v}^t,\ \mathbf{a} = -2\omega\mathbf{v},\ \mu = -\omega^2 \right)$

$$-\mathrm{div}\left(\left(c_0 - \mathbf{v}\mathbf{v}^t\right)\nabla\mathbf{u}\right) - 2i\omega\mathbf{v}\cdot\nabla\mathbf{u} + \omega^2\mathbf{u} = \mathsf{f}$$

H. Barucq et al, **HDG and HDG+ methods for harmonic wave problems with convections, 2021**



Motivation

In this work, we are interested in solving an **Helmholtz like equation**:

$$-\mathrm{div}(A\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

where:

* A is a 2x2 symmetric positive definite matrix,
* $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
* μ is a real constant.

This type of equation occurs in several contexts:

* The **convected Helmholtz** equation ($A = c_0 I - \mathbf{v}\mathbf{v}^t$, $\mathbf{a} = -2\omega\mathbf{v}$, $\mu = -\omega^2$)
* The **Gröss-Pitaevskii** equation (computation of the ground states)

I. Danaila et al, Computation of ground states of the Gröss-Pitaevskii functional via Riemannian optimization, 2017

Motivation

In this work, we are interested in solving an **Helmholtz like equation:**

$$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

where:
* A is a 2x2 symmetric positive definite matrix,
* $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
* μ is a real constant.

This type of equation occurs in several contexts:

* The **convected Helmholtz** equation ($A = c_0 I - \mathbf{v}\mathbf{v}^t$, $\mathbf{a} = -2\omega\mathbf{v}$, $\mu = -\omega^2$)
* The **Gröss-Pitaevskii** equation (computation of the ground states)
* The **wave-ray** equation ($A = I$, $\mathbf{a} = \mathbf{v}$, $\mu = 0$)

  $-\Delta u + i\mathbf{a}\cdot\nabla u = f$

P. Verburg et al, **Multi-level wave-ray method for 2d Helmholtz equation, 2010**

Motivation

In this work, we are interested in solving an **Helmholtz like equation**:

$$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

where:
- A is a 2x2 symmetric positive definite matrix,
- $\mathbf{a}$ is a vecteur of $\mathbb{R}^2$,
- μ is a real constant.

Goal: Propose an **efficient iterative algorithm** of resolution

→ In short, it is **as difficult as solving the Helmholtz equation** !!

O.G. Ernst et al, **Why is it difficult to solve the Helmholtz equation ?** 2012

In the next...

1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

Link with the Helmholtz equation

Let us consider $\mathbf{u}$ solution to

$$-\mathrm{div}(A\,\nabla \mathbf{u}) + i\mathbf{a}\cdot\nabla \mathbf{u} + \mu \mathbf{u} = \mathrm{f}$$

Then, setting $\mathbf{u} = e^{ik\cdot x}\mathbf{v}'$ with $k = \frac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\,\nabla \mathbf{v}') + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathbf{v}' = \mathrm{f}'$$

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

Then, setting $\mathsf{u} = e^{ik\cdot x}\mathbf{v}'$ with $k = \frac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\,\nabla \mathbf{v}') + \boxed{\left(\mu - \frac{\|\mathbf{a}\|_{A^{-1}}^2}{4}\right)}\mathbf{v}' = \mathsf{f}'$$

Remark: Even if $\mu \geq 0$, we see that the problem is **not coercive** if $\|\mathbf{a}\|$ is large.

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

Then, setting $\mathsf{u} = e^{ik\cdot x}\mathsf{v}'$ with $k = \dfrac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\,\nabla \mathsf{v}') + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v}' = \mathsf{f}'$$

Now, taking the change of variables $(x, y) \leftarrow T(x, y)$ where T is a matrix, we get

$$-\mathrm{div}(TAT^t\,\nabla \mathsf{v}) + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v} = \widetilde{\mathsf{f}}'$$

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla\mathsf{u} + \mu\mathsf{u} = \mathsf{f}$$

Then, setting $\mathsf{u} = e^{ik\cdot x}\mathsf{v}'$ with $k = \dfrac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\,\nabla\mathsf{v}') + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v}' = \mathsf{f}'$$

Now, taking the change of variables $(x, y) \leftarrow T(x, y)$ where T is a matrix, we get

$$-\mathrm{div}(TAT^t\,\nabla\mathsf{v}) + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v} = \widetilde{\mathsf{f}}'$$

A simple idea then to obtain the Helmholtz equation is to take $T = G^{-1}$ where $A = GG^t$.

Link with the Helmholtz equation

Let us consider u solution to

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla\mathsf{u} + \mu\mathsf{u} = \mathsf{f}$$

Then, setting $\mathsf{u} = e^{ik\cdot x}\mathsf{v}'$ with $k = \dfrac{1}{2}A^{-1}\mathbf{a}$ one get that

$$-\mathrm{div}(A\,\nabla \mathsf{v}') + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v}' = \mathsf{f}'$$

Now, taking the change of variables $(x,y) \leftarrow T(x,y)$ where T is a matrix, we get

$$-\mathrm{div}(TAT^t\,\nabla \mathsf{v}) + \left(\mu - \frac{\|\mathbf{a}\|^2_{A^{-1}}}{4}\right)\mathsf{v} = \widetilde{\mathsf{f}}'$$

A simple idea then to obtain the Helmholtz equation is to take $T = G^{-1}$ where $A = GG^t$.

Remark: The choice of the transformation is not unique !

F.Q. Hu et al, On the use of Prandtl-Glauert-Lorentz transformation for acoustic scattering by rigid bodies with a uniform flow, 2019

Y. Gao et al, Wave scattering in layered orthotropic media I: a stable PML and a high-accuracy boundary integral equation method, 2021

Link with the Helmholtz equation

Cartesian PML formulation:

$-\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$

Convected Helmholtz equation

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f} \xrightarrow{\text{Coordinates transformation}} -\Delta \mathsf{v} + \widetilde{\omega}\mathsf{u} = \mathsf{f}$$

Convected Helmholtz equation

Helmholtz equation

Link with the Helmholtz equation

Cartesian PML formulation:

$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla\mathsf{u} + \mu\mathsf{u} = \mathsf{f}$

Convected Helmholtz equation

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla\mathsf{u} + \mu\mathsf{u} = \mathsf{f} \xrightarrow{\text{Coordinates transformation}} -\Delta\mathsf{v} + \widetilde{\omega}\mathsf{u} = \mathsf{f}$$

$-\Delta\mathsf{v} + \widetilde{\omega}\mathsf{u} = \mathsf{f}$

Helmholtz equation

$$\Big\downarrow \text{Complex stretching}$$

$-\mathrm{div}(D_{PML}\,\nabla\mathsf{v}_{PML}) + \rho_{PML}\widetilde{\omega}\,\mathsf{u}_{PML} = \mathsf{f}$

PML Helmholtz equation

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathbf{u}) + i\mathbf{a}\cdot\nabla\mathbf{u} + \mu\mathbf{u} = \mathsf{f}$$

Convected Helmholtz equation

$\xrightarrow{\text{Coordinates transformation}}$

$$-\Delta \mathbf{v} + \widetilde{\omega}\mathbf{u} = \mathsf{f}$$

Helmholtz equation

$\Big\downarrow$ Complex stretching

$$-\mathrm{div}(D_{PML}\,\nabla \mathbf{v}_{PML}) + \rho_{PML}\widetilde{\omega}\mathbf{u}_{PML} = \mathsf{f}$$

PML Helmholtz equation

$\xleftarrow{\text{Coordinates transformation}}$

$$-\mathrm{div}(A_{PML}\,\nabla \mathbf{u}_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla\mathbf{u}_{PML} + \frac{i}{2}\mathrm{div}(\mathbf{a}_{PML}\mathbf{u}_{PML}) + \mu_{PML}\mathbf{u}_{PML} = \mathsf{f}$$

PML Convected Helmholtz equation

P. Marchner et al, Stable Perfectly Matched Layers with Lorentz transformation for the convected Helmholtz equation, 2019

E. Becache et al, Perfectly matched layers for the convected Helmholtz equation, 2004

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

Convected Helmholtz equation

Coordinates transformation →

$$-\Delta \mathsf{v} + \widetilde{\omega}\mathsf{u} = \mathsf{f}$$

Helmholtz equation

↓ Complex stretching

$$-\mathrm{div}(D_{PML}\nabla \mathsf{v}_{PML}) + \rho_{PML}\widetilde{\omega}\mathsf{u}_{PML} = \mathsf{f}$$

PML Helmholtz equation

← Coordinates transformation

$$-\mathrm{div}(A_{PML}\nabla \mathsf{u}_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla \mathsf{u}_{PML}$$
$$+\frac{i}{2}\mathrm{div}(\mathbf{a}_{PML}\mathsf{u}_{PML}) + \mu_{PML}\mathsf{u}_{PML} = \mathsf{f}$$

PML Convected Helmholtz equation

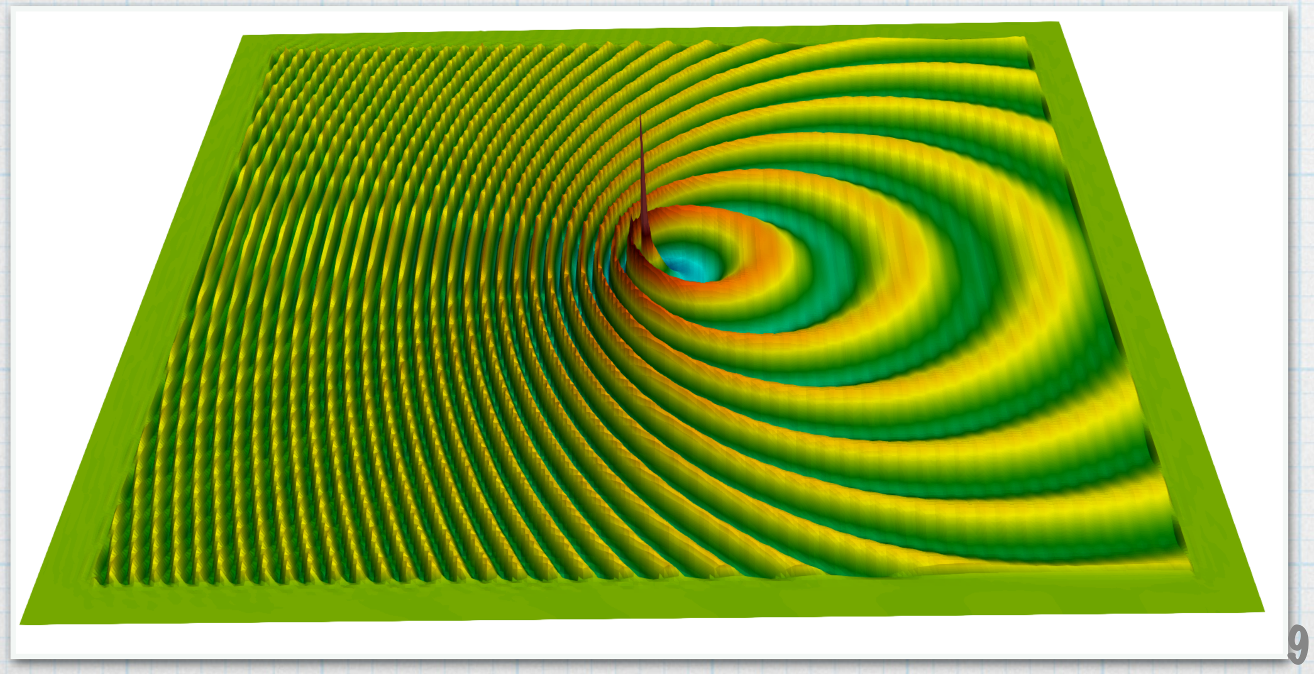


Illustration (Convected Hemlholtz):

$\mathbf{a} = 2\omega\mathbf{v}, \quad \omega = 20$

$\mathbf{v} = [0.8,\ 0]^t$

$A = Id - \mathbf{v}\mathbf{v}^t$

Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\,\nabla \mathsf{u}) + i\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f}$$

Convected Helmholtz equation

Coordinates transformation

$$-\Delta \mathsf{v} + \widetilde{\omega}\,\mathsf{u} = \mathsf{f}$$

Helmholtz equation

~~Complex stretching~~

Complex stretching

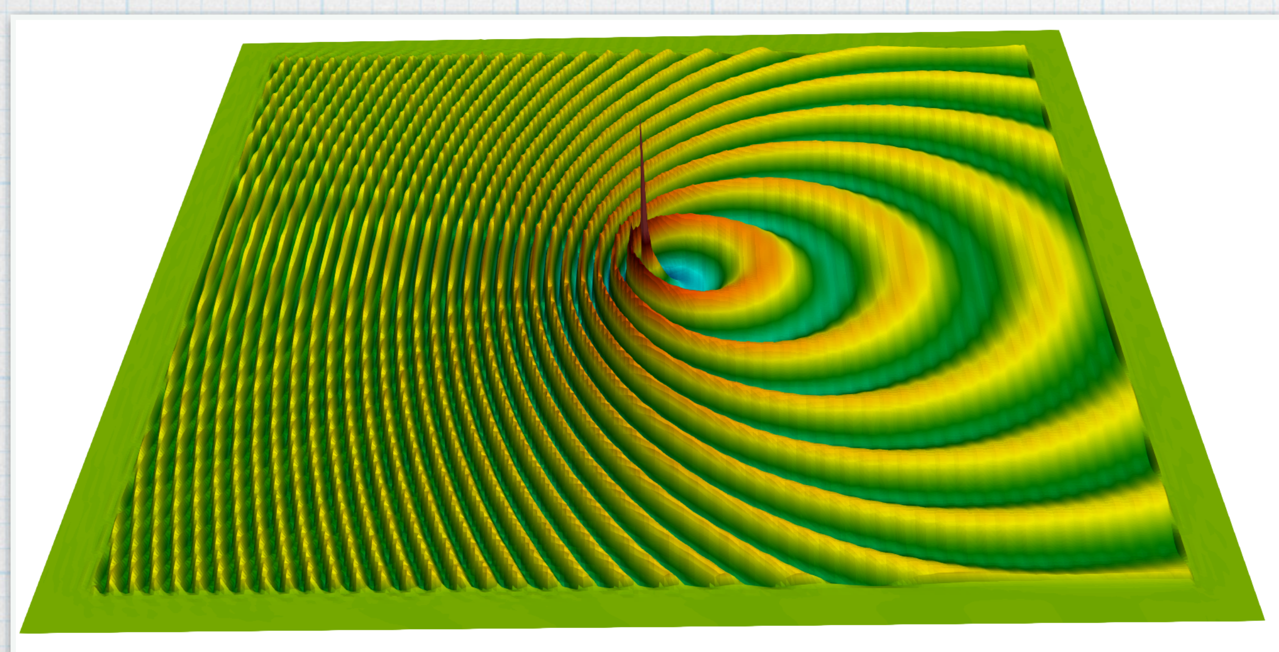
$$-\mathrm{div}(D_{PML}\nabla \mathsf{v}_{PML}) + \rho_{PML}\,\widetilde{\omega}\,\mathsf{u}_{PML} = \mathsf{f}$$

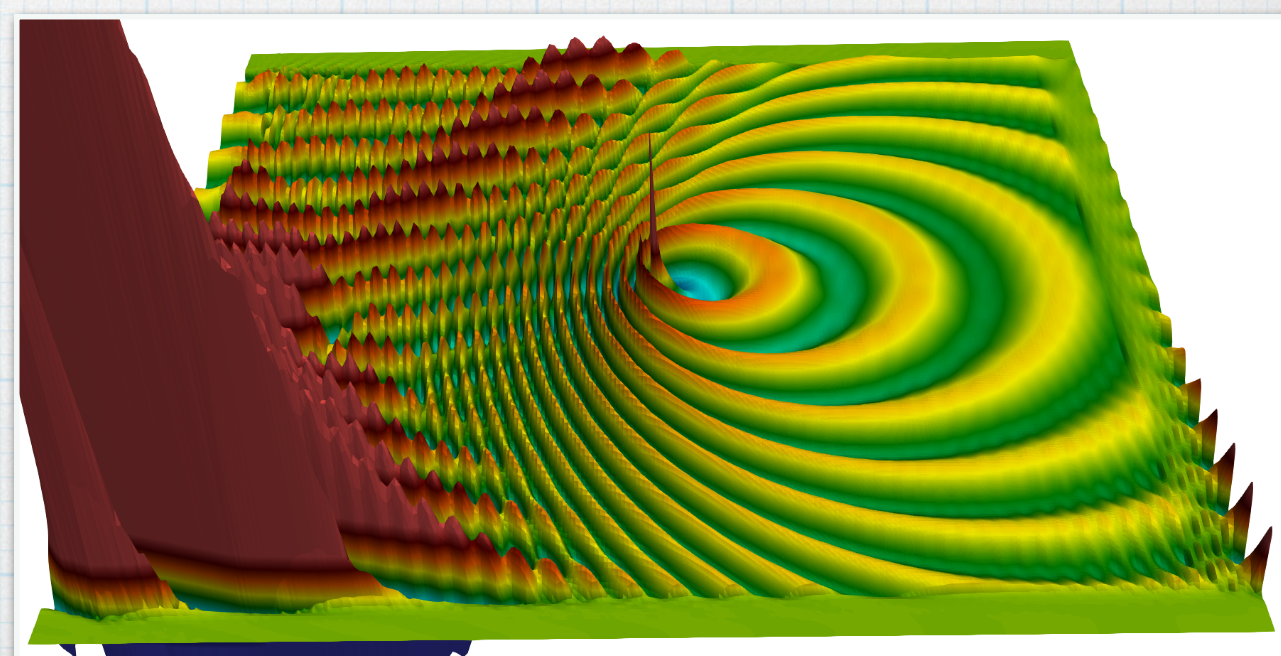
PML Helmholtz equation

Coordinates transformation

$$-\mathrm{div}(A_{PML}\nabla \mathsf{u}_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla \mathsf{u}_{PML} + \frac{i}{2}\mathrm{div}(\mathbf{a}_{PML}\mathsf{u}_{PML}) + \mu_{PML}\mathsf{u}_{PML} = \mathsf{f}$$

PML Convected Helmholtz equation





Link with the Helmholtz equation

Cartesian PML formulation:

$$-\mathrm{div}(A\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f$$

Convected Helmholtz equation

Coordinates transformation

$$-\Delta v + \widetilde{\omega} u = f$$

Helmholtz equation

~~Complex stretching~~

Complex stretching

$$-\mathrm{div}(D_{PML}\nabla v_{PML}) + \rho_{PML}\widetilde{\omega} u_{PML} = f$$

PML Helmholtz equation

Coordinates transformation

$$-\mathrm{div}(A_{PML}\nabla u_{PML}) + \frac{i}{2}\mathbf{a}_{PML}\cdot\nabla u_{PML} + \frac{i}{2}\mathrm{div}(\mathbf{a}_{PML}u_{PML}) + \mu_{PML}u_{PML} = f$$

PML CH equation

ABC (Absorbing Boundary Conditions):

N. Rouxelin et al, **Prandtl-Glauert-Lorentz based Absorbing Boundary Conditions for the convected Helmholtz equation, 2021**

In the next...

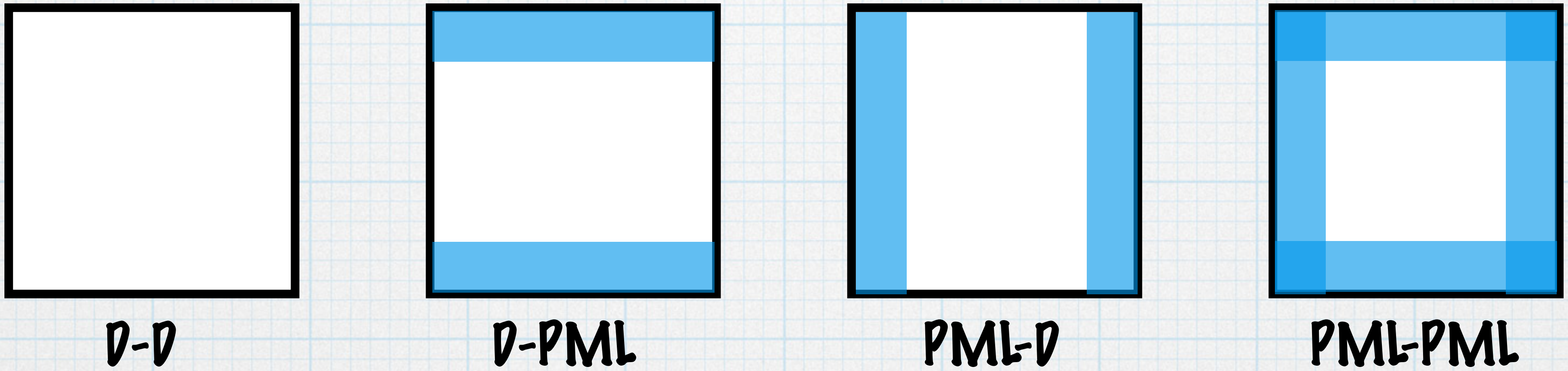
1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

Convergence analysis on a toy problem

Formulation of the problem

$$\begin{cases} -\text{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We will consider four configurations :

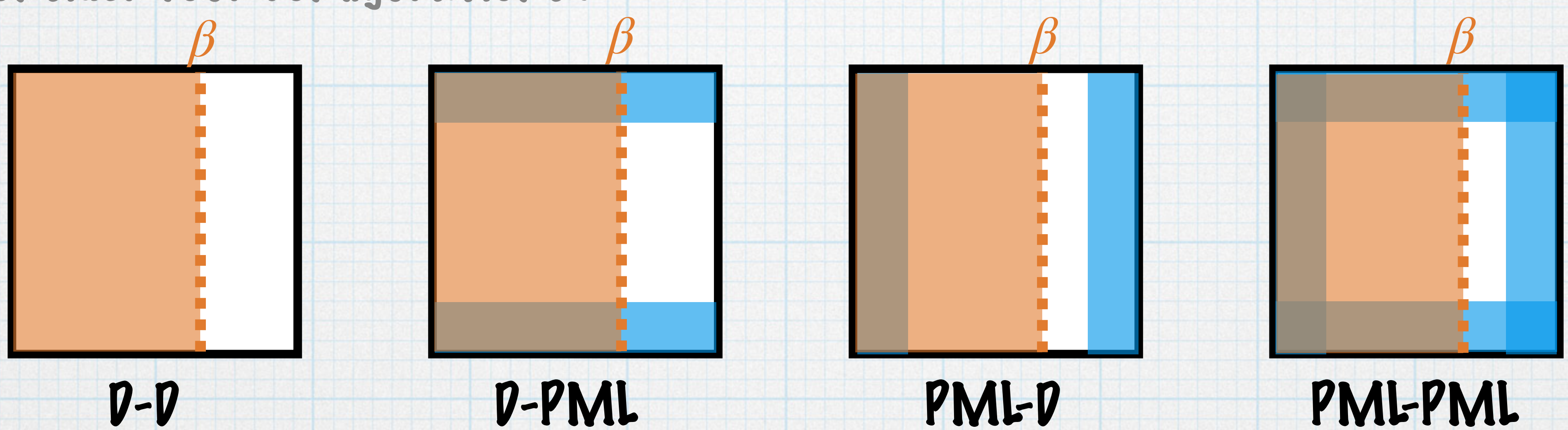


Convergence analysis on a toy problem

Formulation of the problem

$$\begin{cases} -\mathrm{div}(A\,\nabla \mathsf{u}) + i\,\mathbf{a}\cdot\nabla \mathsf{u} + \mu \mathsf{u} = \mathsf{f} & \text{in} \quad \Omega \\ \mathsf{u} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We will consider four configurations :



In each case, we will consider a **Schwarz iterative algorithm** of resolution with **2 subdomains**.

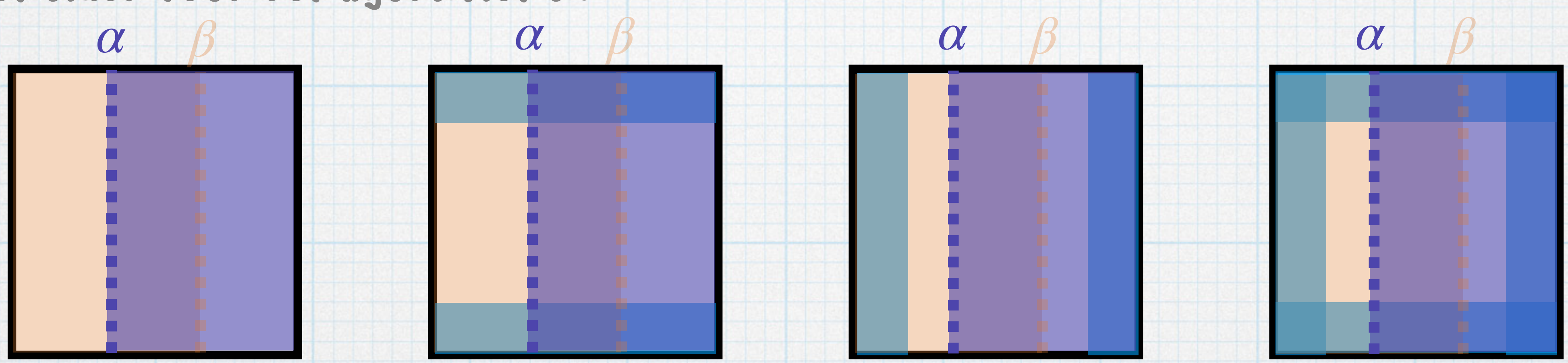
$$\begin{cases} \mathscr{L}_{CH}\mathsf{u}^{1,n} = \mathsf{f}_1 & \text{in} \quad \Omega_1 \\ \mathsf{u}^{1,n} = 0 & \text{on} \quad \partial\Omega \\ (\partial_n + p_{1,2})\mathsf{u}^{1,n} = (\partial_n + p_{1,2})\mathsf{u}^{2,n-1} & \text{on} \quad \Gamma_{12} \end{cases}$$

Convergence analysis on a toy problem

Formulation of the problem

$$
\left|\begin{array}{lll} -\mathrm{div}(A\,\nabla u) + i\mathbf{a}\cdot\nabla u + \mu u = f & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array}\right.
$$

We will consider four configurations :



D-D D-PML PML-D PML-PML

In each case, we will consider a Schwarz iterative algorithm of resolution with 2 subdomains.

$$
\left|\begin{array}{lll} \mathscr{L}_{CH}u^{1,n} = f_1 & \text{in} & \Omega_1 \\ u^{1,n} = 0 & \text{on} & \partial\Omega \\ (\partial_n + p_{1,2})u^{1,n} = (\partial_n + p_{1,2})u^{2,n-1} & \text{on} & \Gamma_{12} \end{array}\right.
\qquad
\left|\begin{array}{lll} \mathscr{L}_{CH}u^{12,n} = f_2 & \text{in} & \Omega_2 \\ u^{2,n} = 0 & \text{on} & \partial\Omega \\ (\partial_n + p_{2,1})u^{2,n} = (\partial_n + p_{2,1})u^{1,n-1} & \text{on} & \Gamma_{21} \end{array}\right.
$$

Convergence analysis on a toy problem

Schwarz algorithm:

$$\begin{cases} \mathscr{L}_{CH}\mathbf{u}^{1,n} = \mathbf{f}_1 & \text{in} \quad \Omega_1 \\ \mathbf{u}^{1,n} = 0 & \text{on} \quad \partial\Omega \\ (\partial_x + p_{1,2})\mathbf{u}^{1,n} = (\partial_x + p_{1,2})\mathbf{u}^{2,n-1} & \text{on} \quad \Gamma_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_{CH}\mathbf{u}^{12,n} = \mathbf{f}_2 & \text{in} \quad \Omega_2 \\ \mathbf{u}^{2,n} = 0 & \text{on} \quad \partial\Omega \\ (\partial_x + p_{2,1})\mathbf{u}^{2,n} = (\partial_x + p_{2,1})\mathbf{u}^{1,n-1} & \text{on} \quad \Gamma_{21} \end{cases}$$

Equivalent Schwarz algorithm for Helmholtz equation:

$$\begin{cases} \mathscr{L}_{H}\mathbf{v}^{1,n} = \widetilde{\mathbf{f}}_1 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_{H}\mathbf{v}^{2,n} = \widetilde{\mathbf{f}}_2 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Convergence analysis on a toy problem

Schwarz algorithm:

$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathsf{u}^{1,n} = \mathsf{f}_1 & \text{in} & \Omega_1 \\ \mathsf{u}^{1,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{1,2})\mathsf{u}^{1,n} = (\partial_x + p_{1,2})\mathsf{u}^{2,n-1} & \text{on} & \Gamma_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathsf{u}^{12,n} = \mathsf{f}_2 & \text{in} & \Omega_2 \\ \mathsf{u}^{2,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{2,1})\mathsf{u}^{2,n} = (\partial_x + p_{2,1})\mathsf{u}^{1,n-1} & \text{on} & \Gamma_{21} \end{array}\right.$$

Equivalent Schwarz algorithm for Helmholtz equation:

$$\left|\begin{array}{lll} \mathscr{L}_{H}\mathsf{v}^{1,n} = \widetilde{\mathsf{f}}_1 & \text{in} & \widetilde{\Omega}_1 \\ \mathsf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathsf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathsf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathscr{L}_{H}\mathsf{v}^{2,n} = \widetilde{\mathsf{f}}_2 & \text{in} & \widetilde{\Omega}_2 \\ \mathsf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathsf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathsf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Remarks: * The convergence analysis can be done only for the Helmholtz equation.

Convergence analysis on a toy problem

Schwarz algorithm:

$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathbf{u}^{1,n} = \mathrm{f}_1 & \text{in} & \Omega_1 \\ \mathbf{u}^{1,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{1,2})\mathbf{u}^{1,n} = (\partial_x + p_{1,2})\mathbf{u}^{2,n-1} & \text{on} & \Gamma_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_{CH}\mathbf{u}^{12,n} = \mathrm{f}_2 & \text{in} & \Omega_2 \\ \mathbf{u}^{2,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{2,1})\mathbf{u}^{2,n} = (\partial_x + p_{2,1})\mathbf{u}^{1,n-1} & \text{on} & \Gamma_{21} \end{array}\right.$$

Equivalent Schwarz algorithm for Helmholtz equation:

$$\left|\begin{array}{lll} \mathscr{L}_{H}\mathbf{v}^{1,n} = \widetilde{\mathrm{f}}_1 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_{H}\mathbf{v}^{2,n} = \widetilde{\mathrm{f}}_2 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Remarks:

* The convergence analysis can be done only for the Helmholtz equation.
* To preserve the separable geometry in the Helmholtz case, we need to assume that A is diagonale.

Convergence analysis on a toy problem

Schwarz algorithm:

$$\left|\begin{array}{lll} \mathscr{L}_{CH} u^{1,n} = f_1 & \text{in} & \Omega_1 \\ u^{1,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{1,2}) u^{1,n} = (\partial_x + p_{1,2}) u^{2,n-1} & \text{on} & \Gamma_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_{CH} u^{12,n} = f_2 & \text{in} & \Omega_2 \\ u^{2,n} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{2,1}) u^{2,n} = (\partial_x + p_{2,1}) u^{1,n-1} & \text{on} & \Gamma_{21} \end{array}\right.$$

Equivalent Schwarz algorithm for Helmholtz equation:

$$\left|\begin{array}{lll} \mathscr{L}_{H} v^{1,n} = \widetilde{f}_1 & \text{in} & \widetilde{\Omega}_1 \\ v^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2}) v^{1,n} = (\partial_x + \widetilde{p}_{1,2}) v^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_{H} v^{2,n} = \widetilde{f}_2 & \text{in} & \widetilde{\Omega}_2 \\ v^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1}) v^{2,n} = (\partial_x + \widetilde{p}_{2,1}) v^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Remarks: * Optimized TC can be derived from optimized TC for the Helmholtz equation

M.J. Gander et al, Optimized schwarz methods with overlap for the helmholtz equation 2016

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Remark: For simplicity, we will assume that $A = Id$ s.t. $\widetilde{\Omega} = [0,1]^2$, $\widetilde{\Gamma}_{1,2} = \{\beta\} \times [0,1]$ and $\widetilde{\Gamma}_{2,1} = \{\beta\} \times [0,1]$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi\pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \boxed{\mathbf{v}^{1,n} = 0} & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \boxed{\mathbf{v}^{2,n} = 0} & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi \pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi \pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies:

$$\mathbf{A}^{1,n}(\xi) = \frac{e^{i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) + \widetilde{p}_{1,2}) + e^{2i\mathcal{S}(\xi)}e^{-i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) - \widetilde{p}_{1,2})}{e^{i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) + \widetilde{p}_{1,2}) + e^{-i\mathcal{S}(\xi)\beta}(i\mathcal{S}(\xi) - \widetilde{p}_{1,2})}\mathbf{A}^{2,n-1}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \frac{e^{i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) + \widetilde{p}_{2,1}) + e^{-i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) - \widetilde{p}_{2,1})}{e^{i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) + \widetilde{p}_{2,1}) + e^{2i\mathcal{S}(\xi)}e^{-i\mathcal{S}(\xi)\alpha}(i\mathcal{S}(\xi) - \widetilde{p}_{2,1})}\mathbf{A}^{1,n-1}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi\in\mathbb{N}} N_\xi \sin(\xi\pi y)\Big(\mathbf{A}^{i,n}(\xi)e^{i\mathscr{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathscr{S}(\xi)x}\Big) \quad \text{where } \mathscr{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathscr{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies: $\mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\mathbf{A}^{2,n-1}(\xi)$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathcal{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathcal{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi \pi y)\left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\xi)x}\right) \quad \text{where } \mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -e^{2i\mathcal{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies: $\mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\mathbf{A}^{2,n-1}(\xi) \longrightarrow \mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\rho_2^{DD}(\xi)\mathbf{A}^{1,n-2}(\xi)$

$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \rho_1^{DD}(\xi)\rho_2^{DD}(\xi)\mathbf{A}^{2,n-2}(\xi)$

Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \sin(\xi\pi y)\Big(\mathbf{A}^{i,n}(\xi)e^{i\mathscr{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathscr{S}(\xi)x}\Big) \quad \text{where } \mathscr{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}.$$

The boundary conditions implies: $\mathbf{B}^{1,n}(\xi) = -\mathbf{A}^{1,n}(\xi)$ and $\mathbf{B}^{2,n}(\xi) = -\,e^{2i\mathscr{S}(\xi)}\mathbf{A}^{2,n}(\xi)$

The TC implies: $\mathbf{A}^{1,n}(\xi) = \rho_1^{DD}(\xi)\mathbf{A}^{2,n-1}(\xi) \longrightarrow \mathbf{A}^{1,n}(\xi) = \boxed{\rho_1^{DD}(\xi)\rho_2^{DD}(\xi)}\mathbf{A}^{1,n-2}(\xi)$

$\mathbf{A}^{2,n}(\xi) = \rho_2^{DD}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \boxed{\rho_1^{DD}(\xi)\rho_2^{DD}(\xi)}\mathbf{A}^{2,n-2}(\xi)$

$:= \rho^{DD}(\xi)$

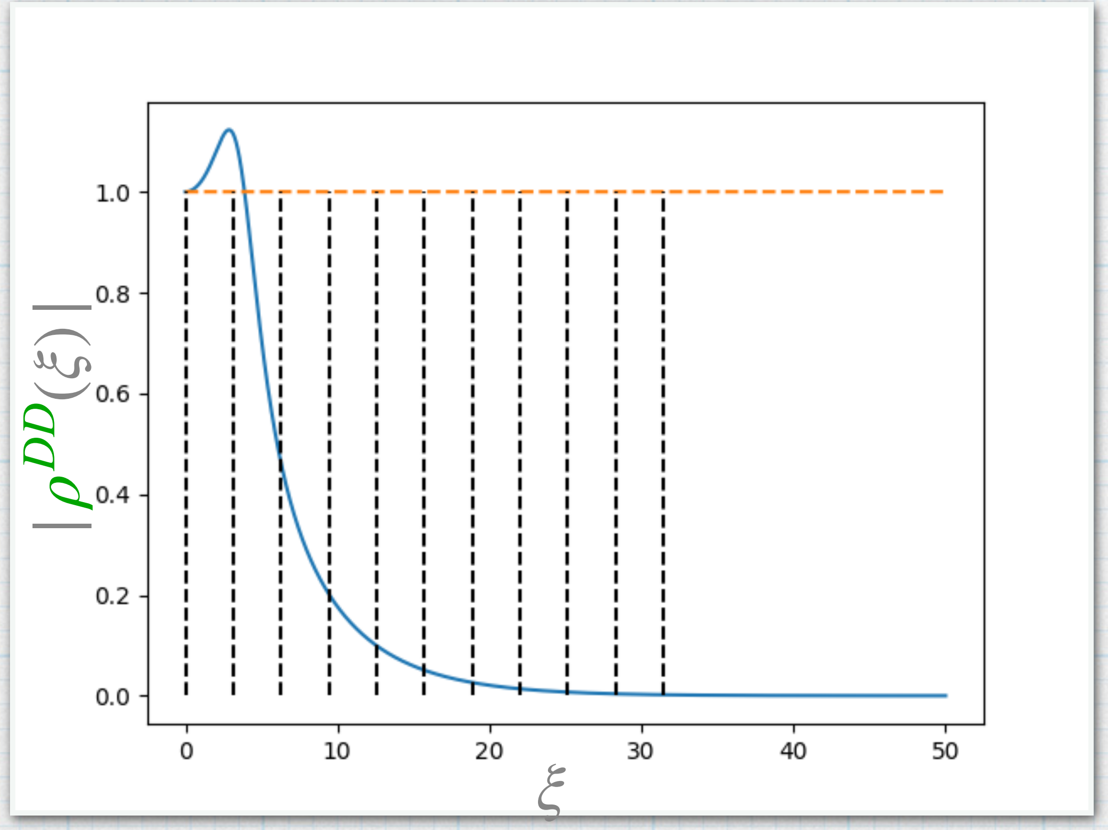
Convergence analysis on a toy problem

Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in} \quad \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} \quad \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in} \quad \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} \quad \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} \quad \widetilde{\Gamma}_{21} \end{cases}$$

Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)



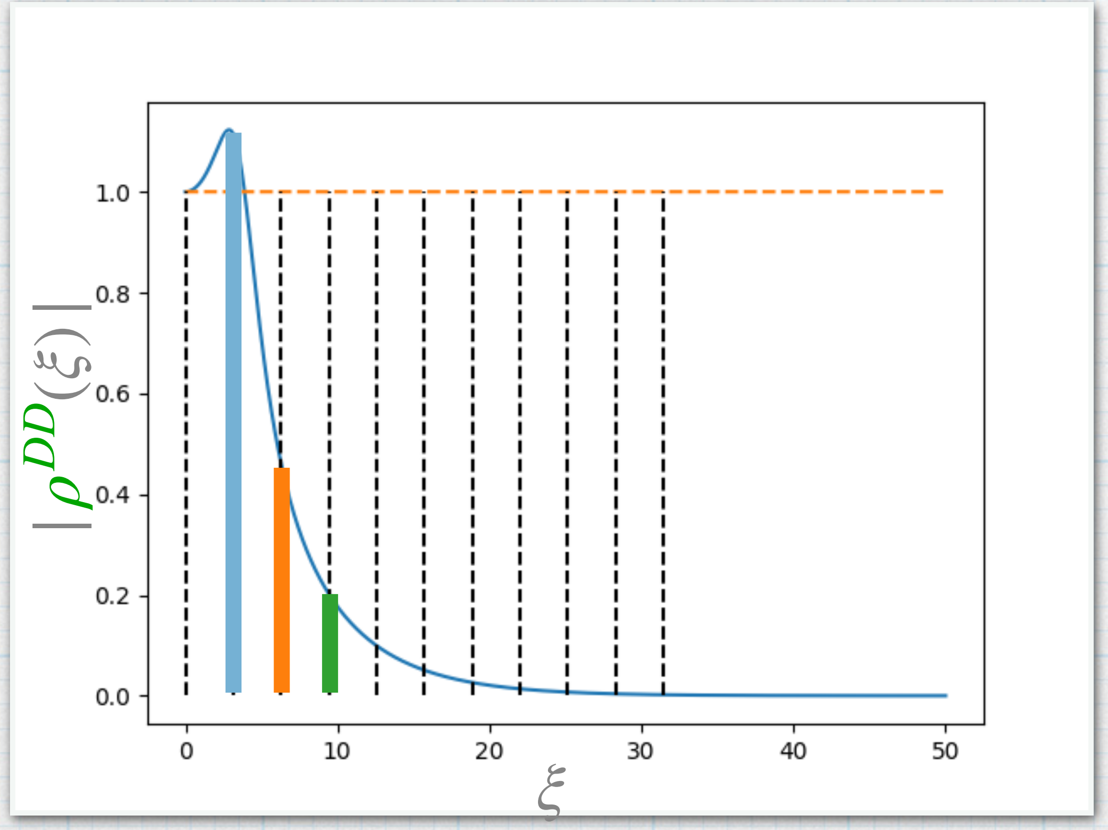
Convergence analysis on a toy problem

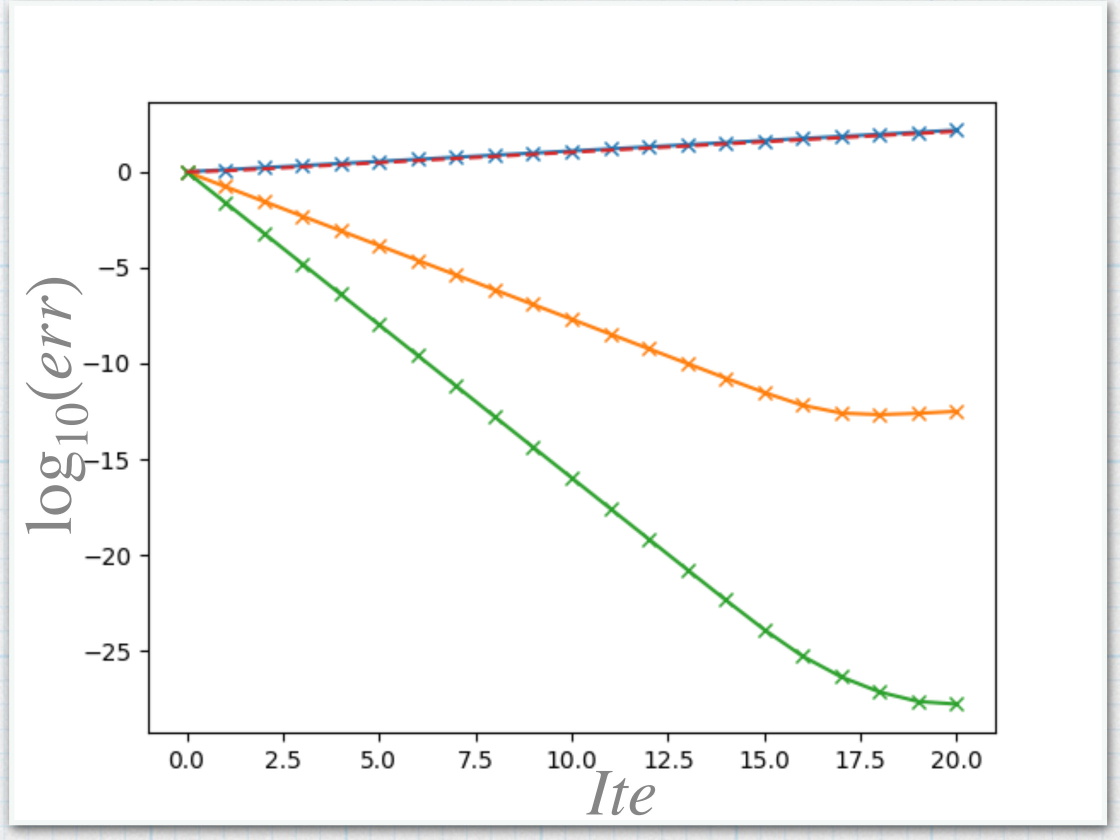
Convergence analysis (D-D case)

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases}$$

$$\begin{cases} \mathscr{L}_H \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)





Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathcal{L}_H^{PML} \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathcal{L}_H^{PML} \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \Psi_\xi(y) \left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\lambda_\xi)x} \right)$$

where $\left(\Psi_\xi, \lambda_\xi\right)$ are the eigenfunctions and eigenvalues of

$$\begin{cases} -\partial_{\widetilde{y}}^2 \Psi_\xi = \lambda_\xi^2 \Psi_\xi, & y \in [0,1] \\ \Psi_\xi = 0, & y \in \{0,1\} \end{cases}$$

where $\widetilde{y}$ is the complex stretched coordinate.

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\begin{cases} \mathscr{L}_H^{PML} \mathbf{v}^{1,n} = 0 & \text{in } \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on } \widetilde{\Gamma}_{12} \end{cases} \qquad \begin{cases} \mathscr{L}_H^{PML} \mathbf{v}^{2,n} = 0 & \text{in } \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on } \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on } \widetilde{\Gamma}_{21} \end{cases}$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \Psi_\xi(y) \left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\lambda_\xi)x} \right)$$

where $\left(\Psi_\xi, \lambda_\xi\right)$ are the eigenfunctions and eigenvalues of

$$\begin{cases} -\partial_{\widetilde{y}}^2 \Psi_\xi = \lambda_\xi^2 \Psi_\xi, & y \in [0,1] \\ \Psi_\xi = 0, & y \in \{0,1\} \end{cases}$$

where $\widetilde{y}$ is the complex stretched coordinate.

Remark: It is not clear that the eigenfunctions $\left(\Psi_\xi\right)_\xi$ form a basis of $L^2([0,1])$!

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\left|\begin{array}{lll} \mathcal{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right. \qquad \left|\begin{array}{lll} \mathcal{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi \in \mathbb{N}} N_\xi \Psi_\xi(y) \left(\mathbf{A}^{i,n}(\xi) e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi) e^{-i\mathcal{S}(\lambda_\xi)x} \right).$$

Similar calculations as before show that

$$\mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi)\mathbf{A}^{2,n-1}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DPML}(\xi)\mathbf{A}^{1,n-1}(\xi)$$

Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\left|\begin{array}{lll} \mathcal{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathcal{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Using separation of variables methods, we get that

$$\mathbf{v}^{i,n} = \sum_{\xi\in\mathbb{N}} N_\xi \Psi_\xi(y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\lambda_\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\lambda_\xi)x}\right).$$

Similar calculations as before show that

$$\mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi)\mathbf{A}^{2,n-1}(\xi) \longrightarrow \mathbf{A}^{1,n}(\xi) = \rho_1^{DPML}(\xi)\rho_2^{DPML}(\xi)\mathbf{A}^{1,n-2}(\xi)$$

$$\mathbf{A}^{2,n}(\xi) = \rho_2^{DPML}(\xi)\mathbf{A}^{1,n-1}(\xi) \longrightarrow \mathbf{A}^{2,n}(\xi) = \underbrace{\rho_1^{DPML}(\xi)\rho_2^{DPML}(\xi)}_{:= \rho^{DPML}(\xi)}\mathbf{A}^{2,n-2}(\xi)$$

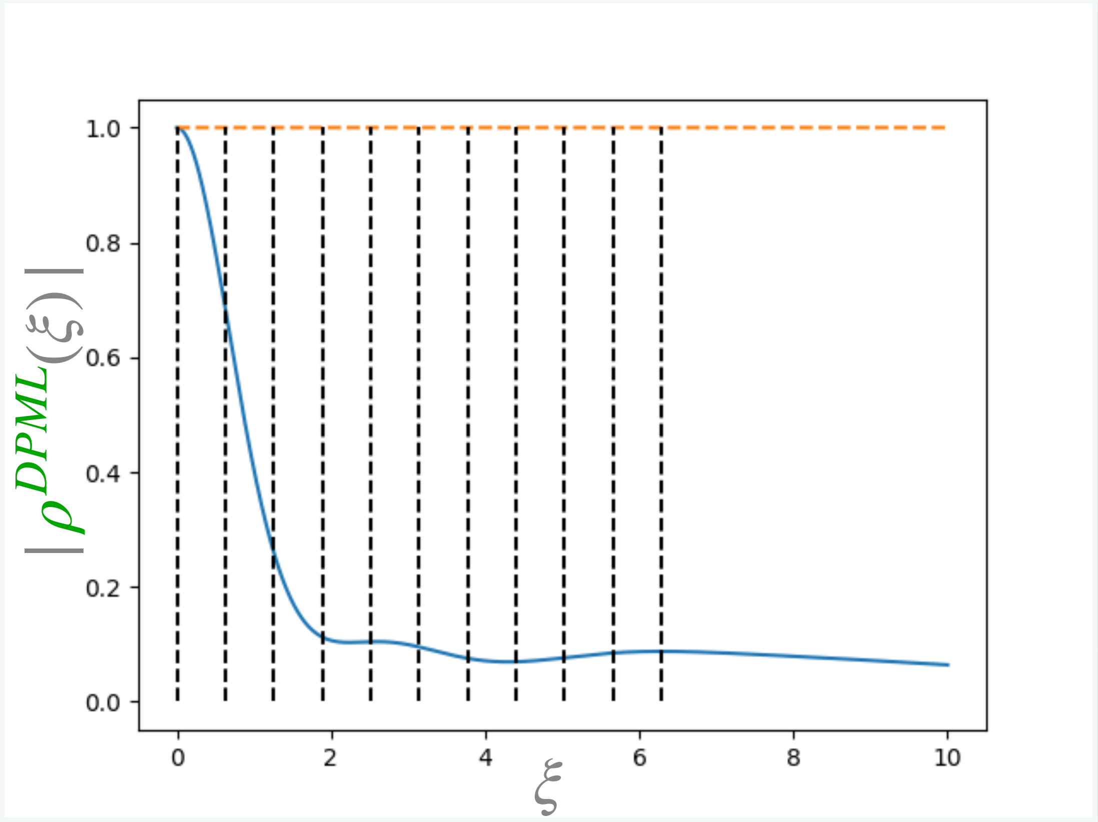
Convergence analysis on a toy problem

Convergence analysis (D-PML case)

$$\left|\begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)



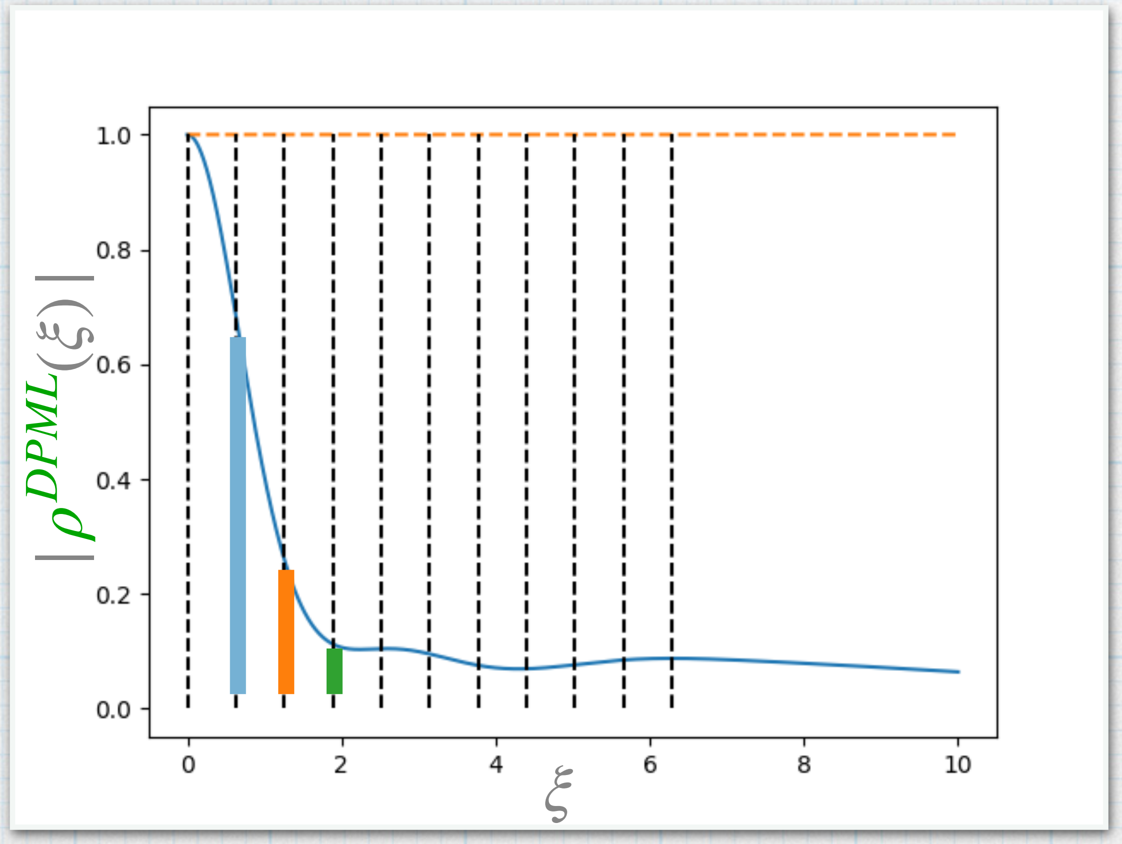
Convergence analysis on a toy problem

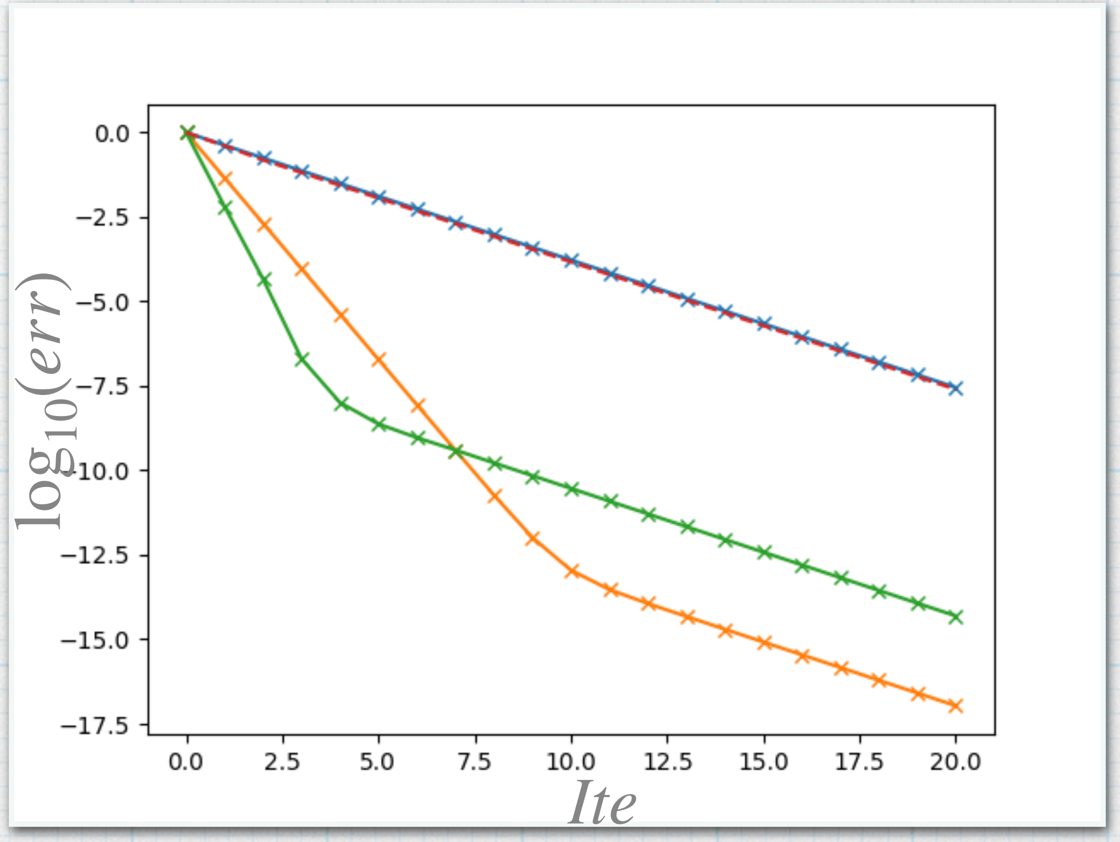
Convergence analysis (D-PML case)

$$\left|\begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{1,n} = 0 & \text{in} & \widetilde{\Omega}_1 \\ \mathbf{v}^{1,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{1,n} = (\partial_x + \widetilde{p}_{1,2})\mathbf{v}^{2,n-1} & \text{on} & \widetilde{\Gamma}_{12} \end{array}\right.$$

$$\left|\begin{array}{lll} \mathscr{L}_H^{PML}\mathbf{v}^{2,n} = 0 & \text{in} & \widetilde{\Omega}_2 \\ \mathbf{v}^{2,n} = 0 & \text{on} & \partial\widetilde{\Omega} \\ (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{2,n} = (\partial_x + \widetilde{p}_{2,1})\mathbf{v}^{1,n-1} & \text{on} & \widetilde{\Gamma}_{21} \end{array}\right.$$

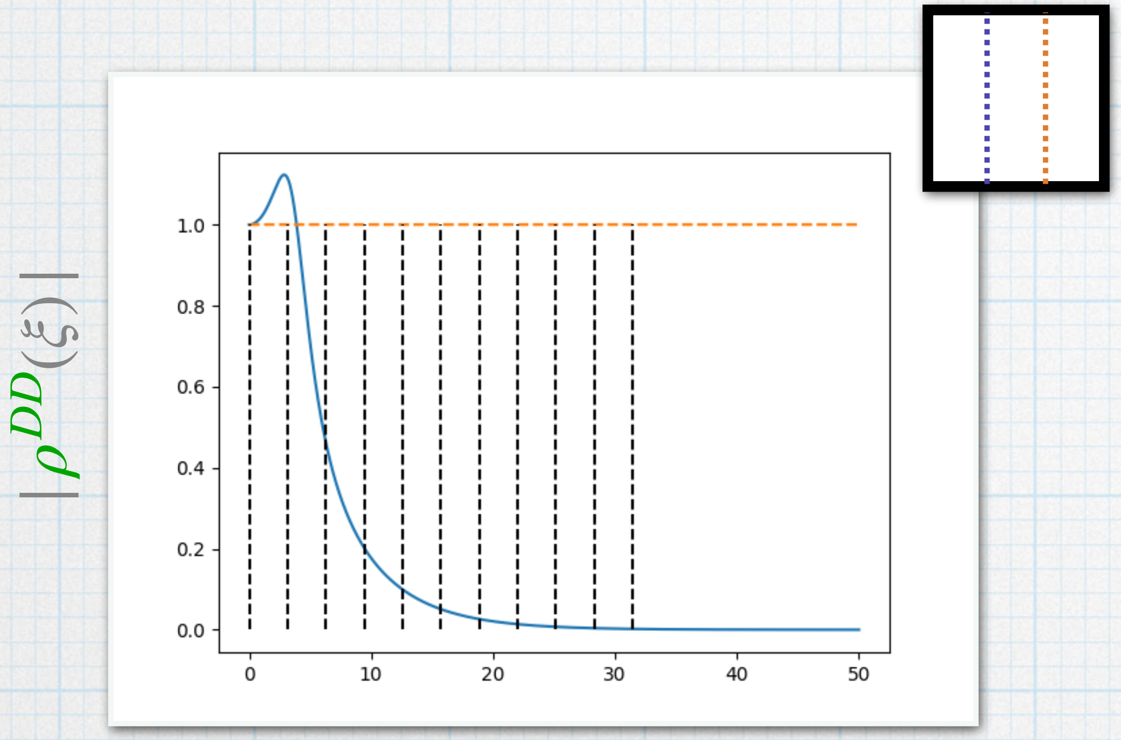
Illustration on an example (Wave Ray: $-\Delta\mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\widetilde{p}_{1,2} = \widetilde{p}_{2,1} = i\omega$)

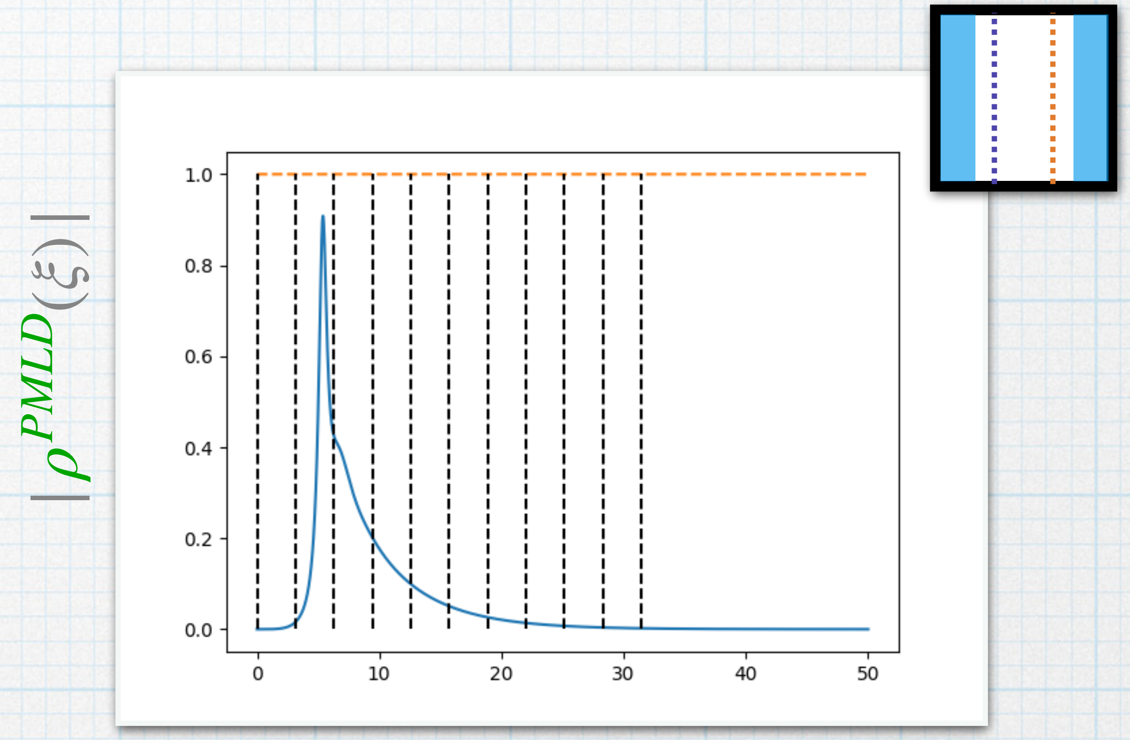


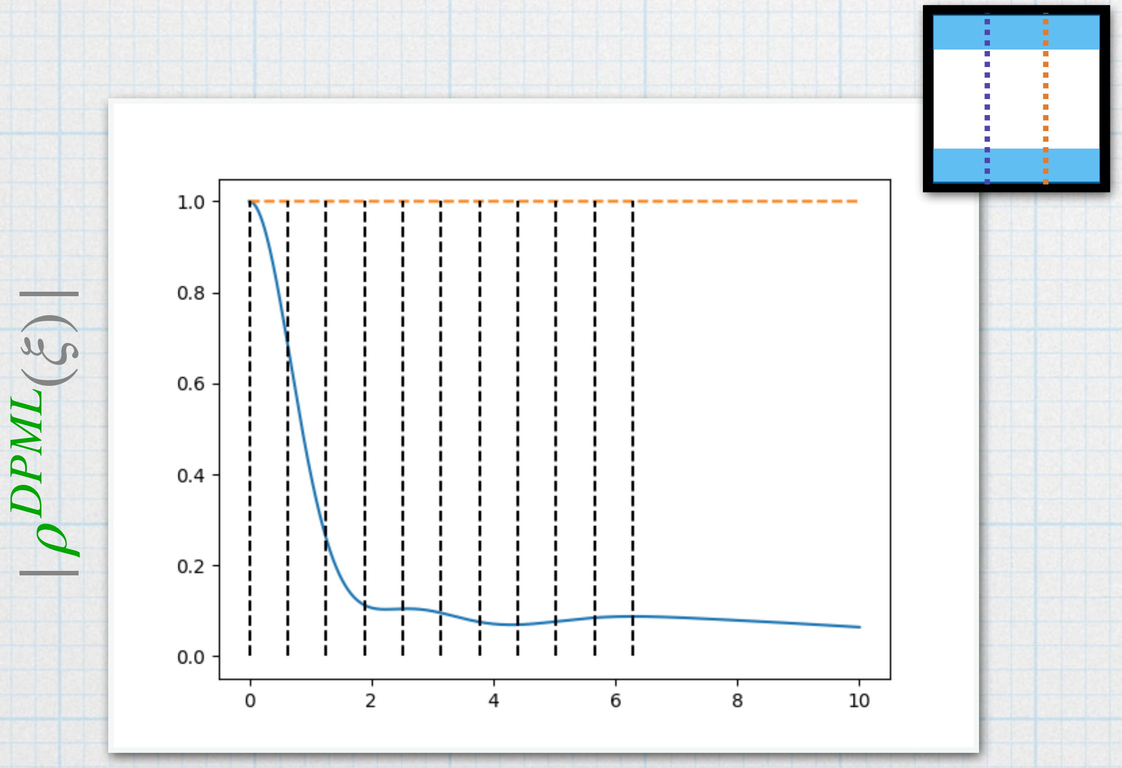


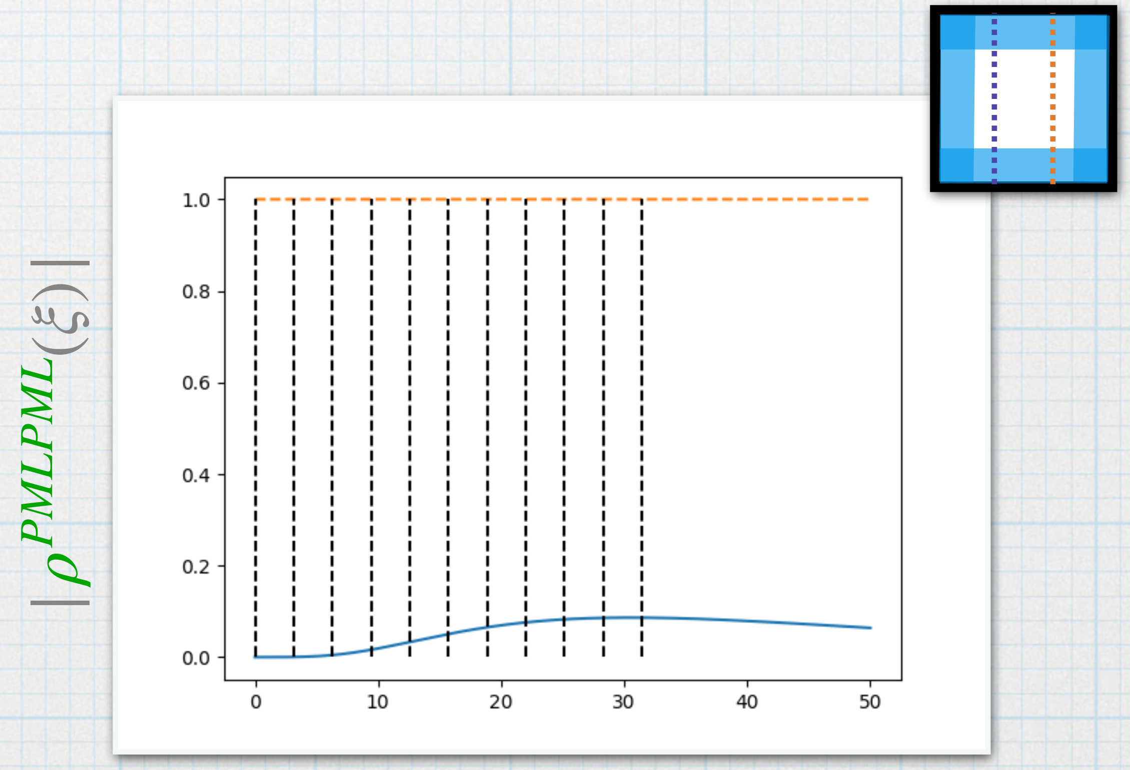
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 10$)



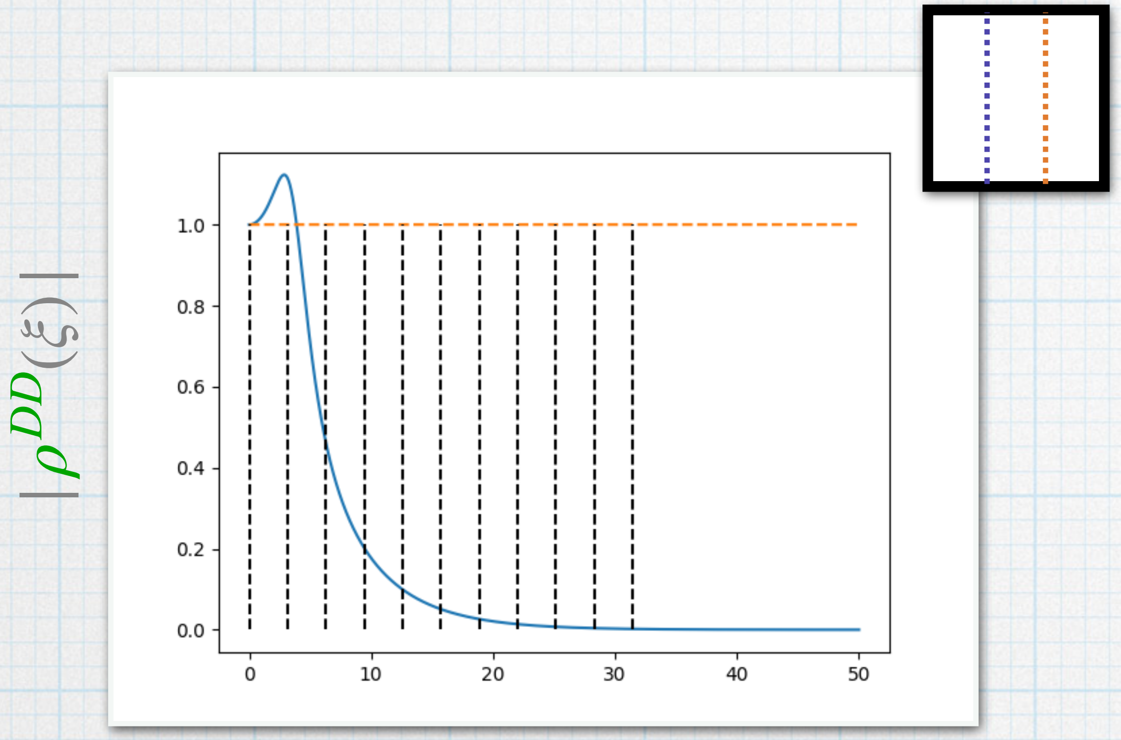


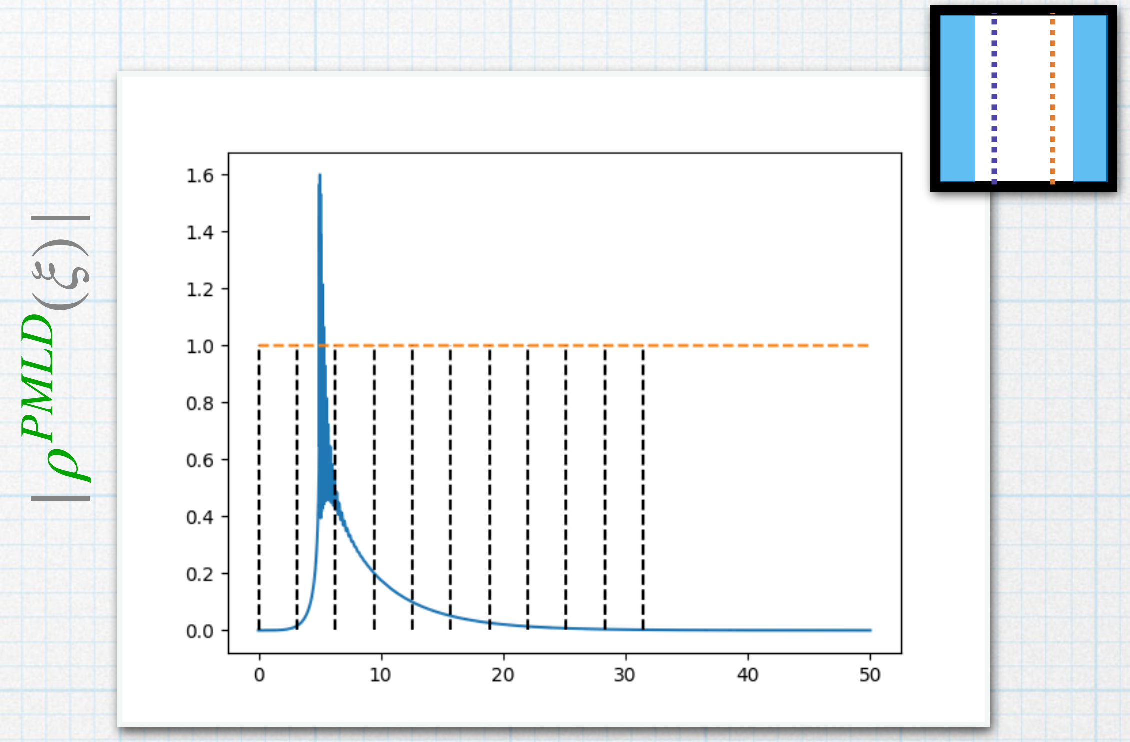


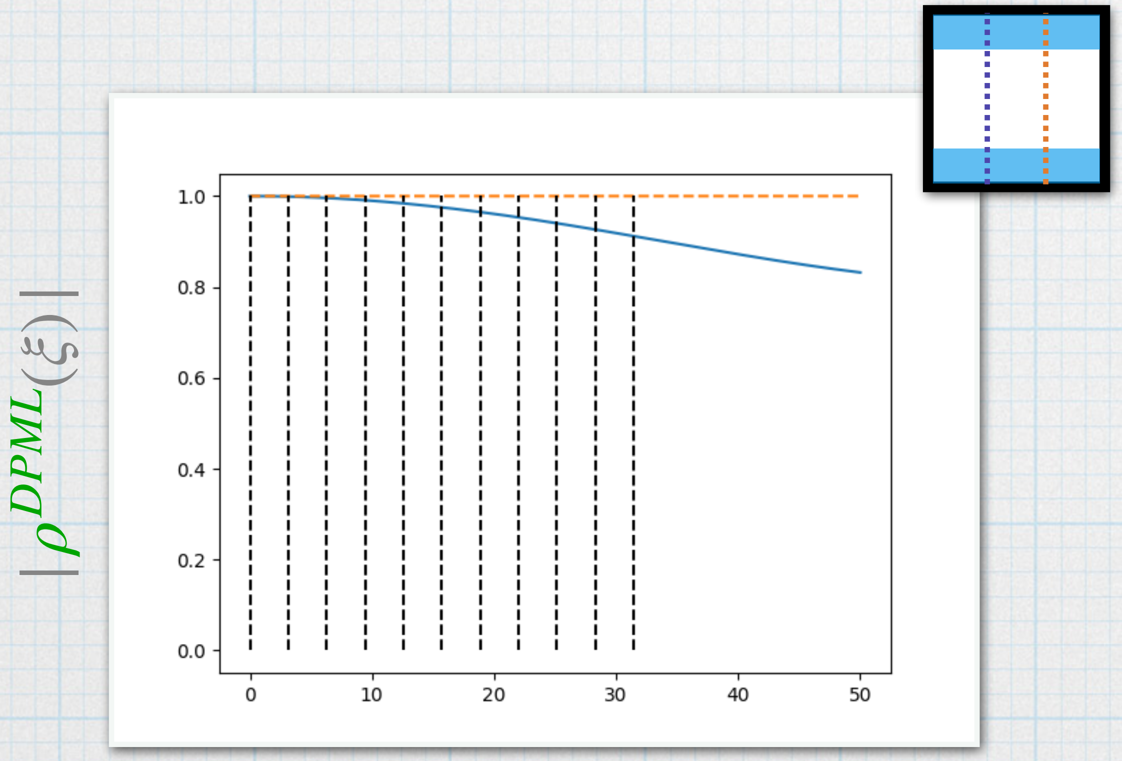


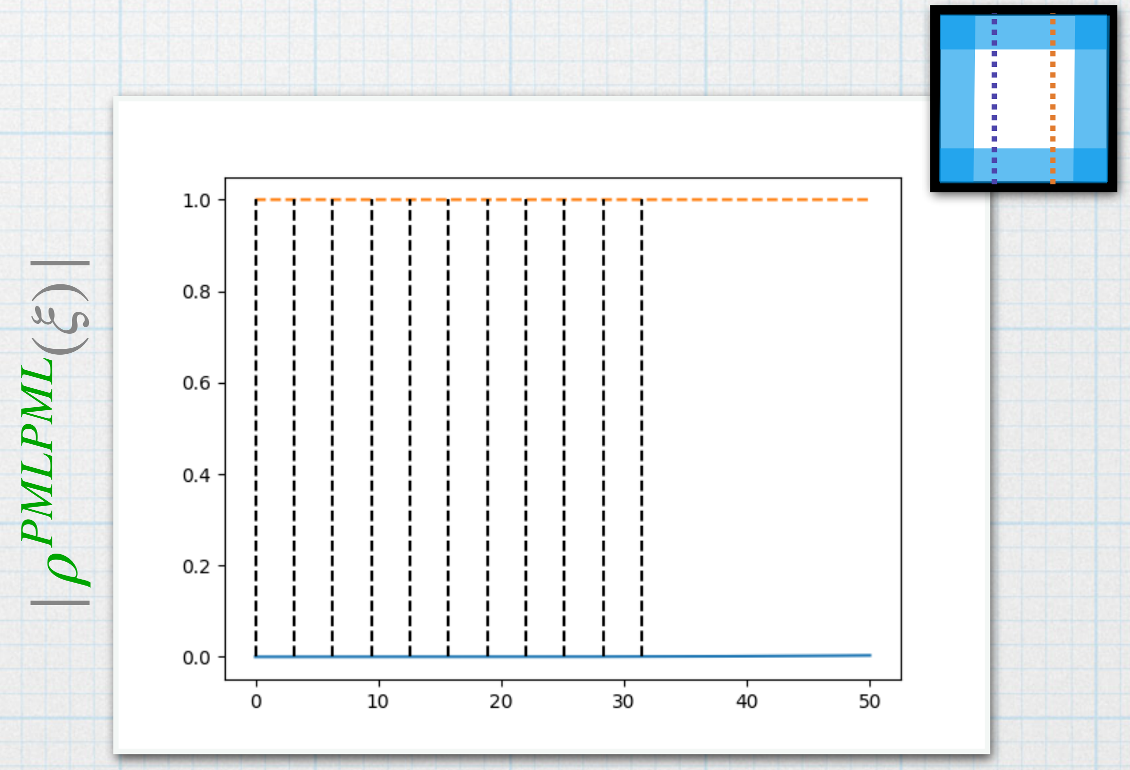
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 50$)





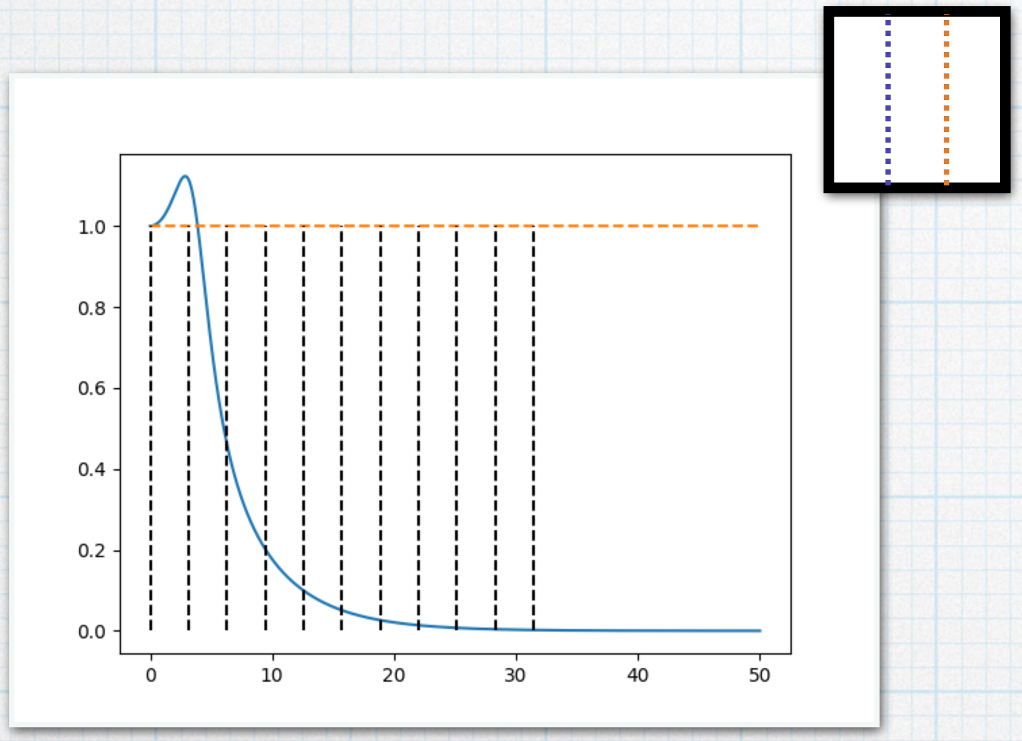




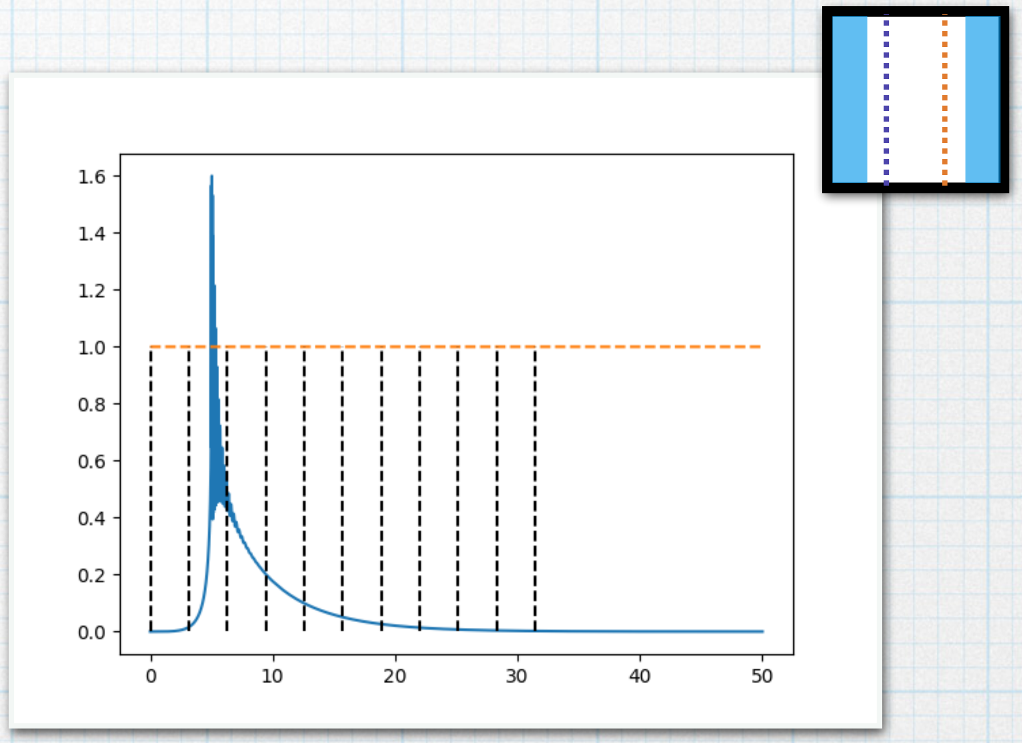
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 50$)

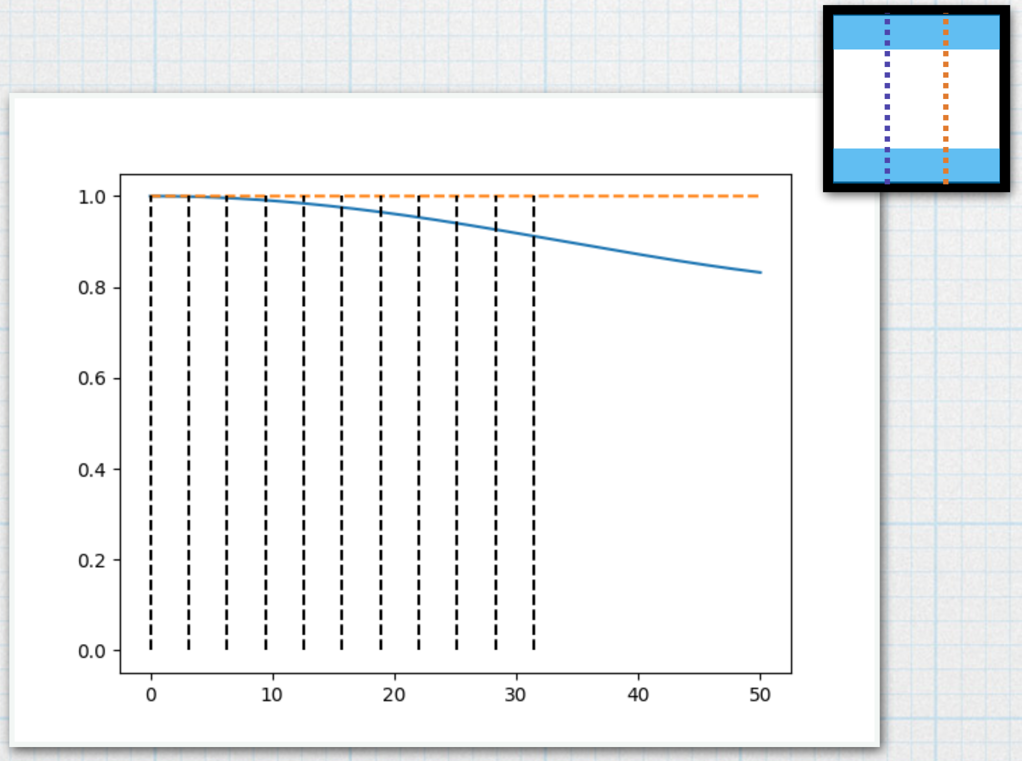
$|\rho^{DD}(\xi)|$



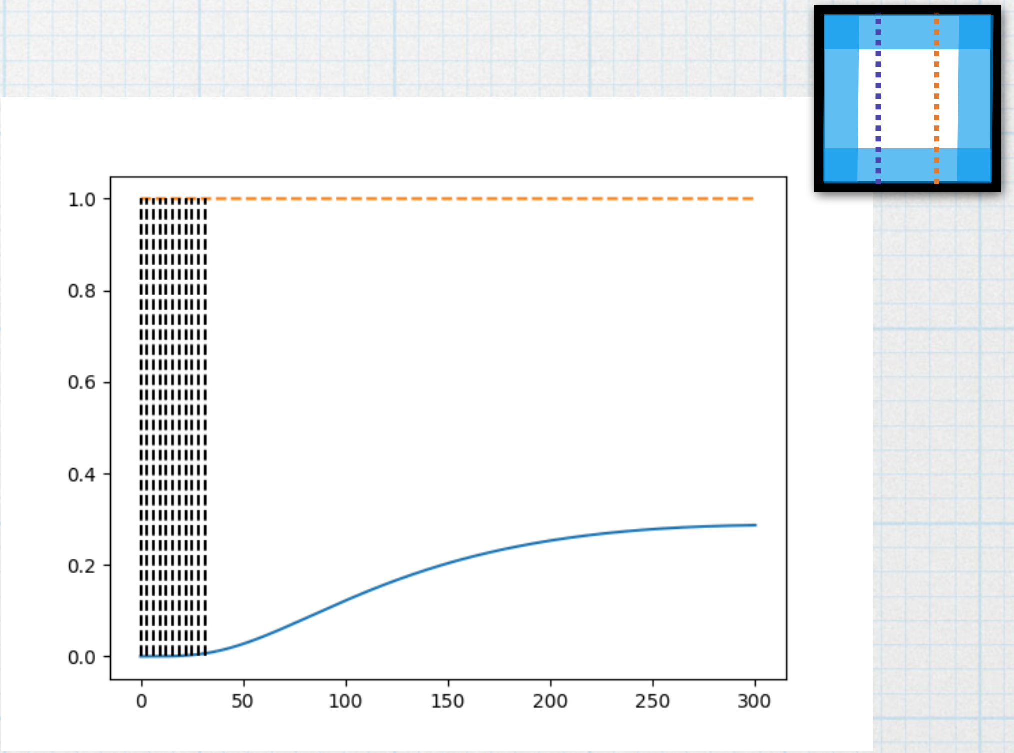
$|\rho^{PMLD}(\xi)|$



$|\rho^{DPML}(\xi)|$

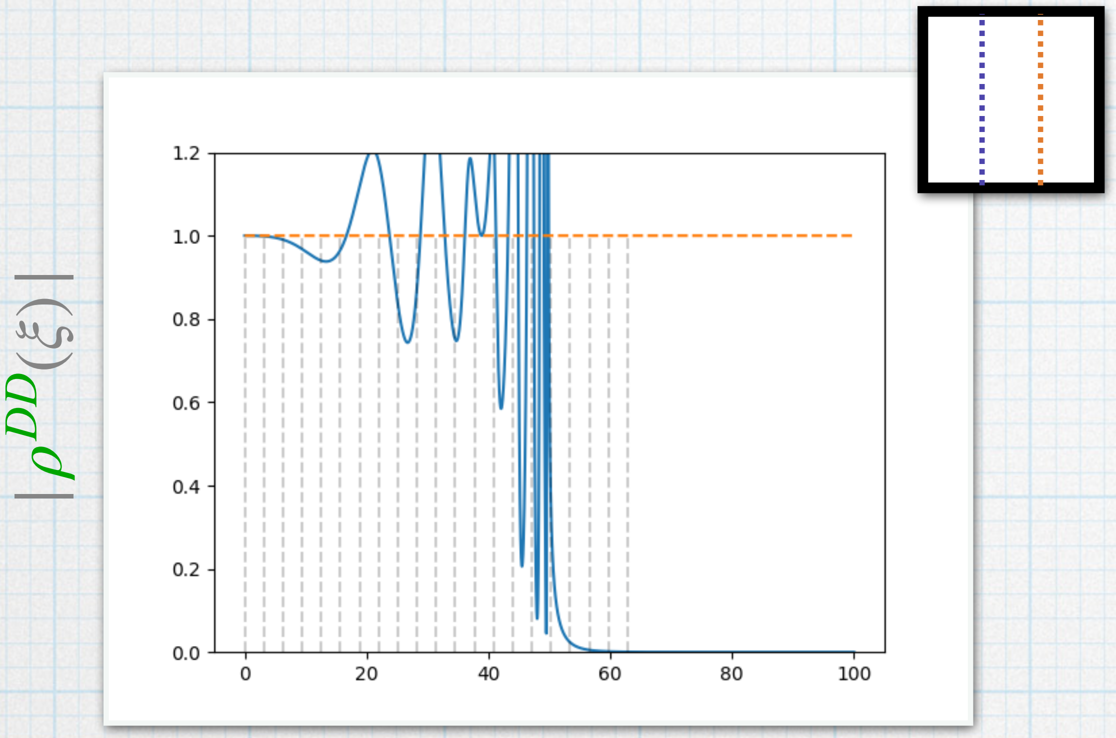


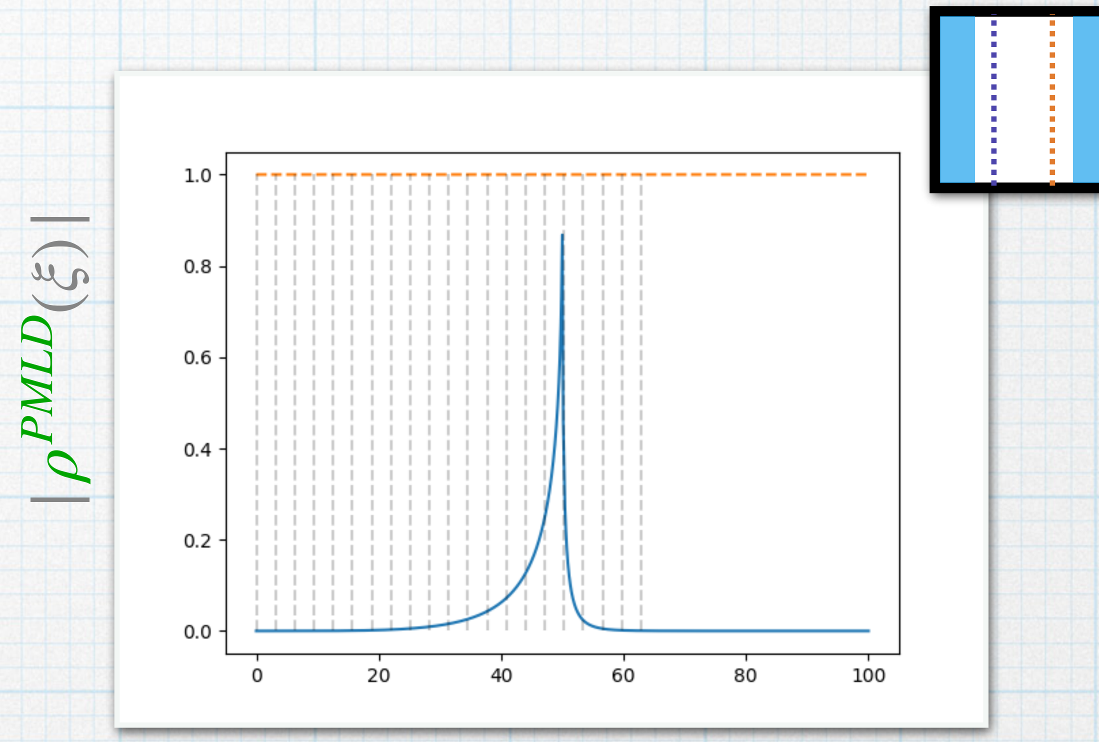
$|\rho^{PMLPML}(\xi)|$

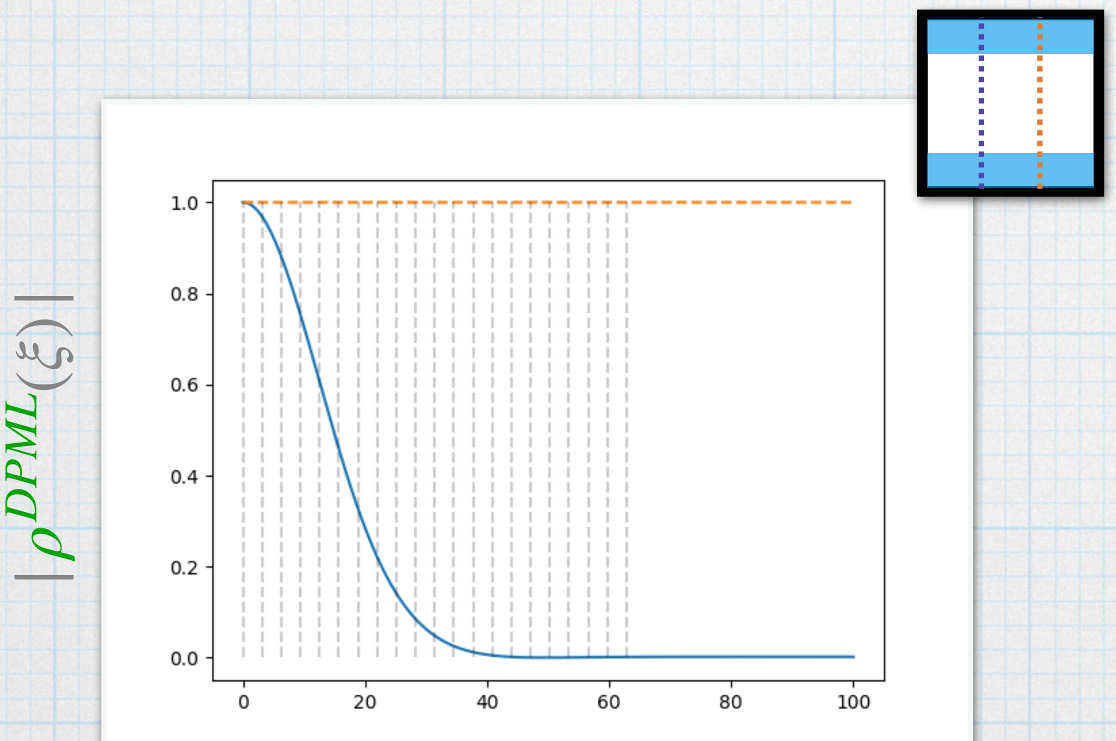


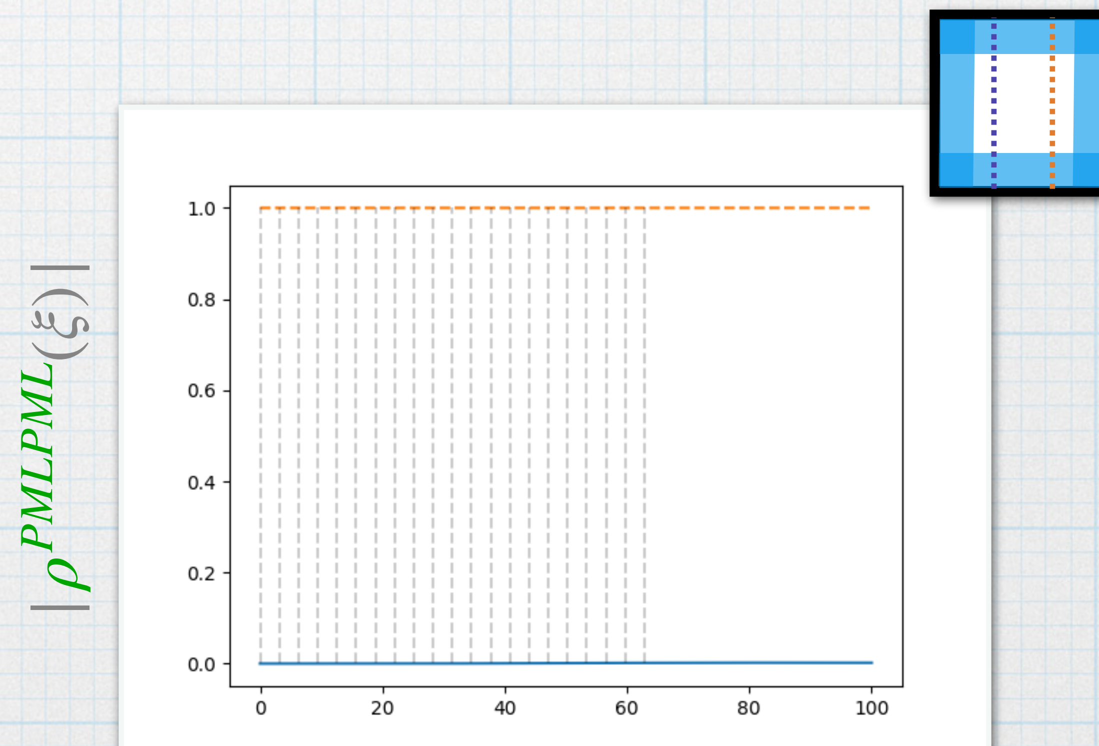
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=0$, $\widetilde{\omega}=50$, $\sigma_{PML}=10$)



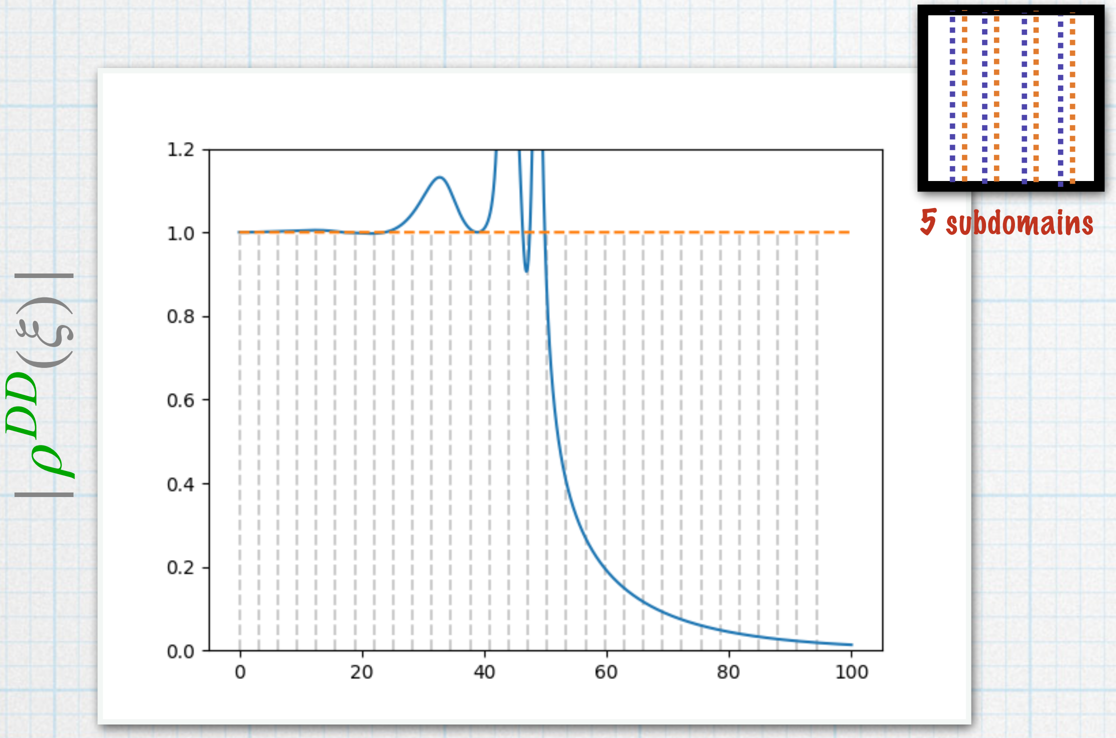


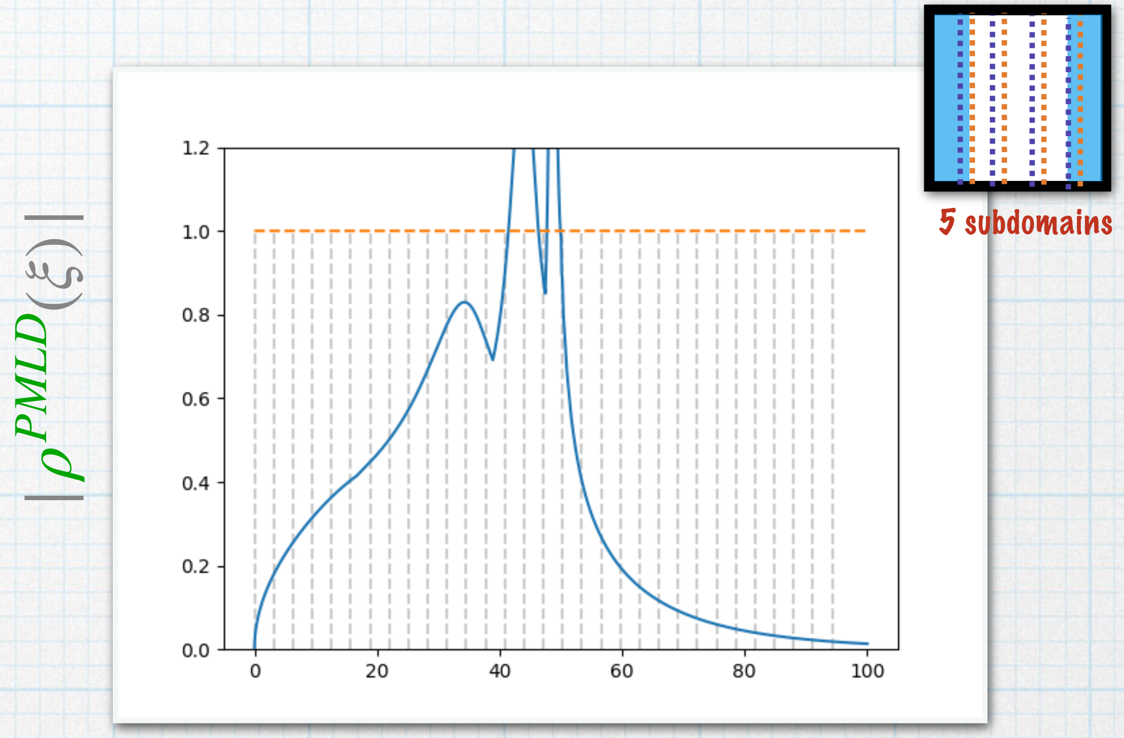


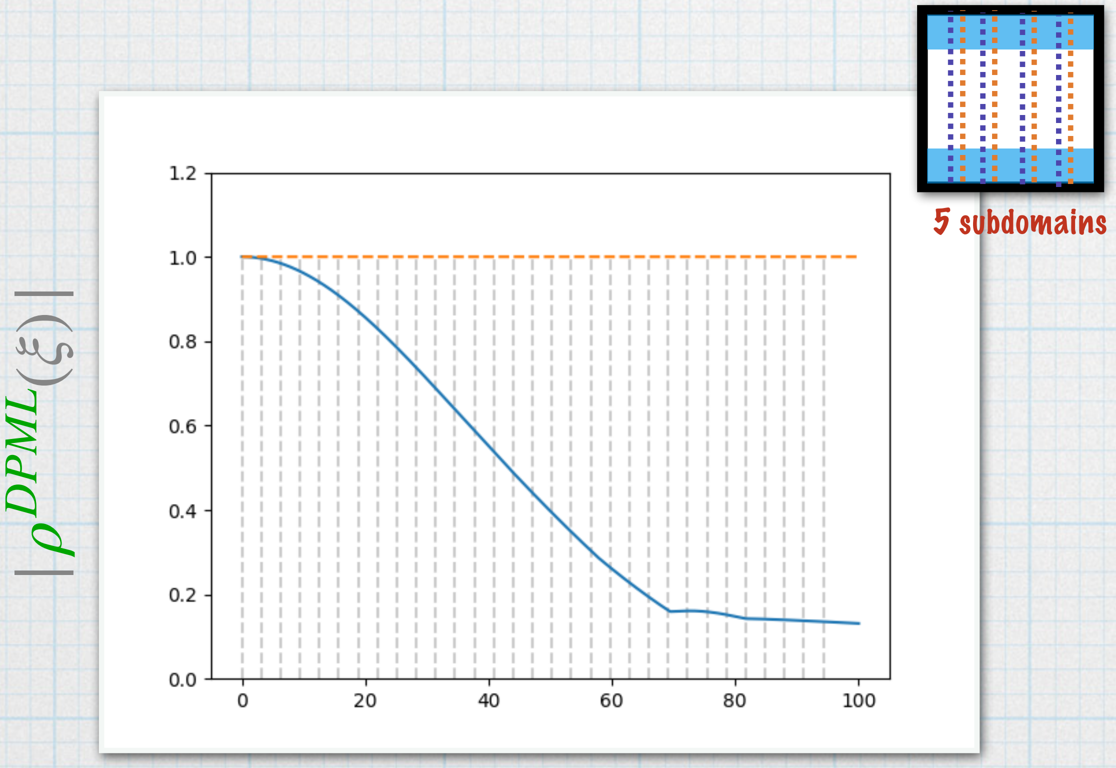


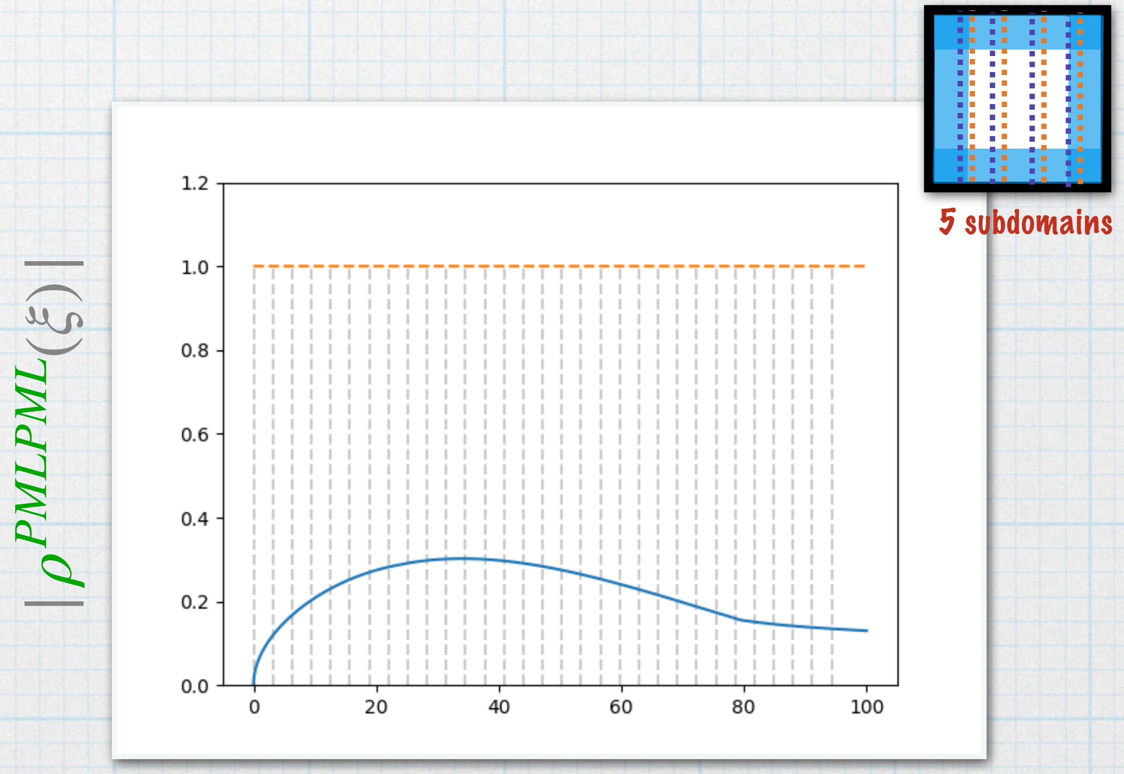
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$, $\sigma_{PML} = 10$)



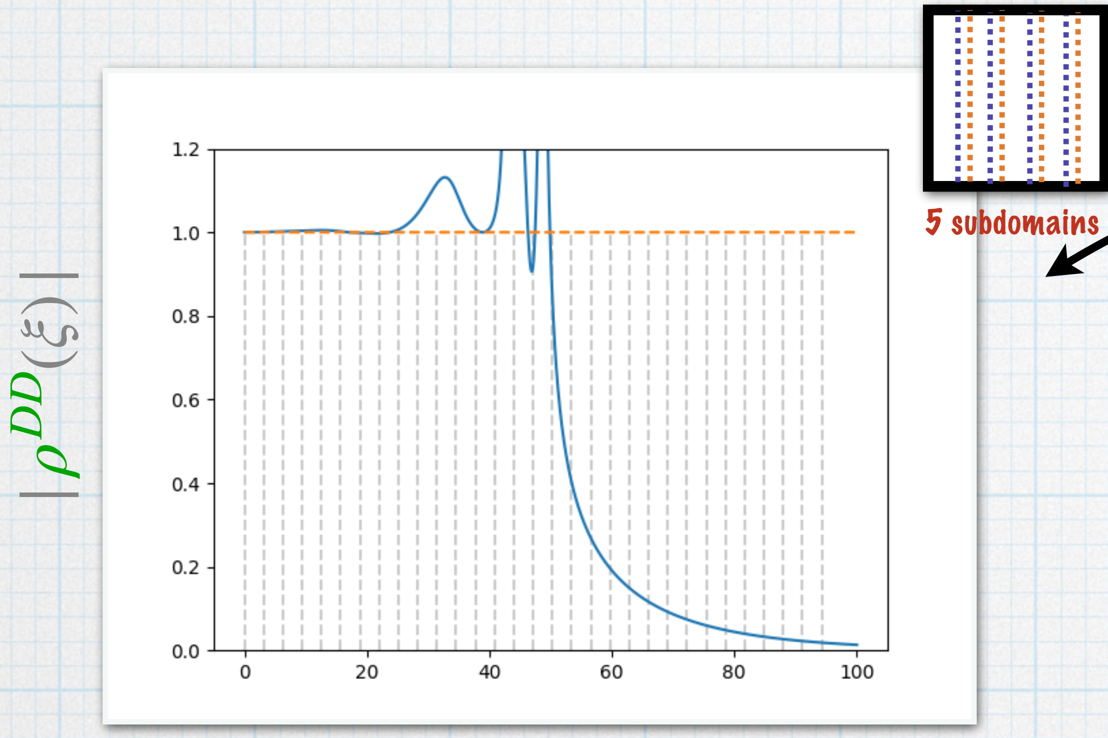




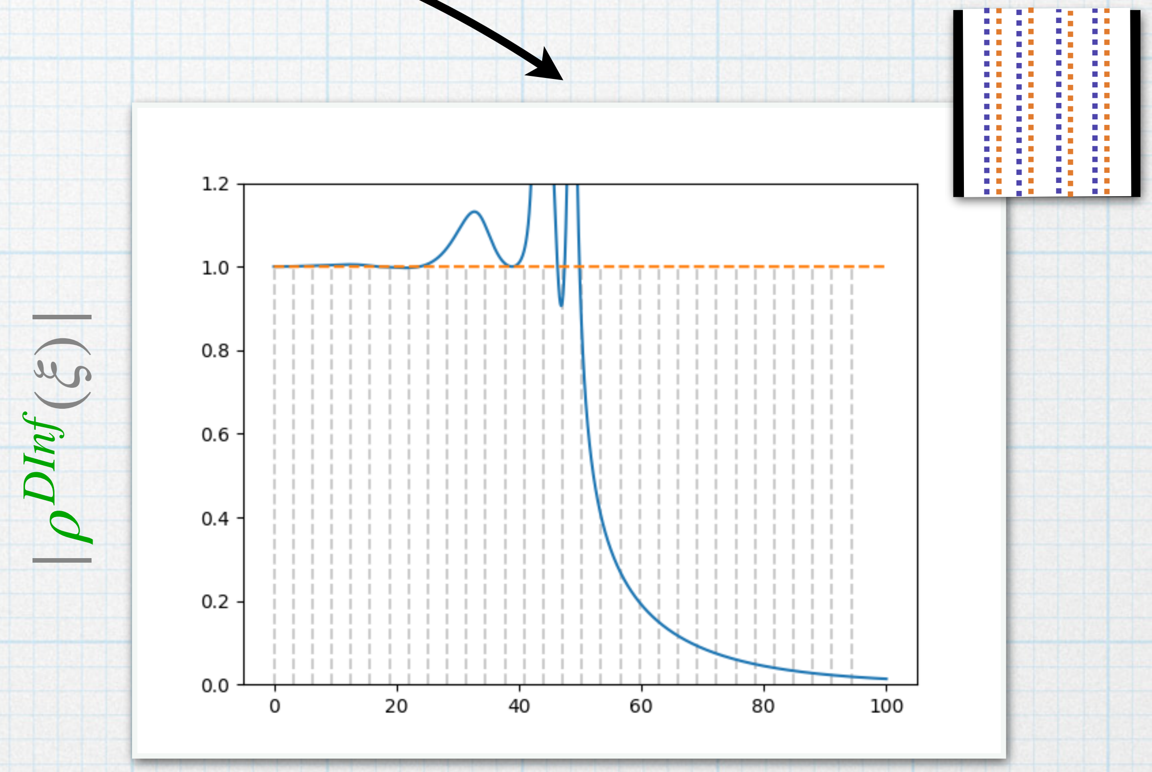


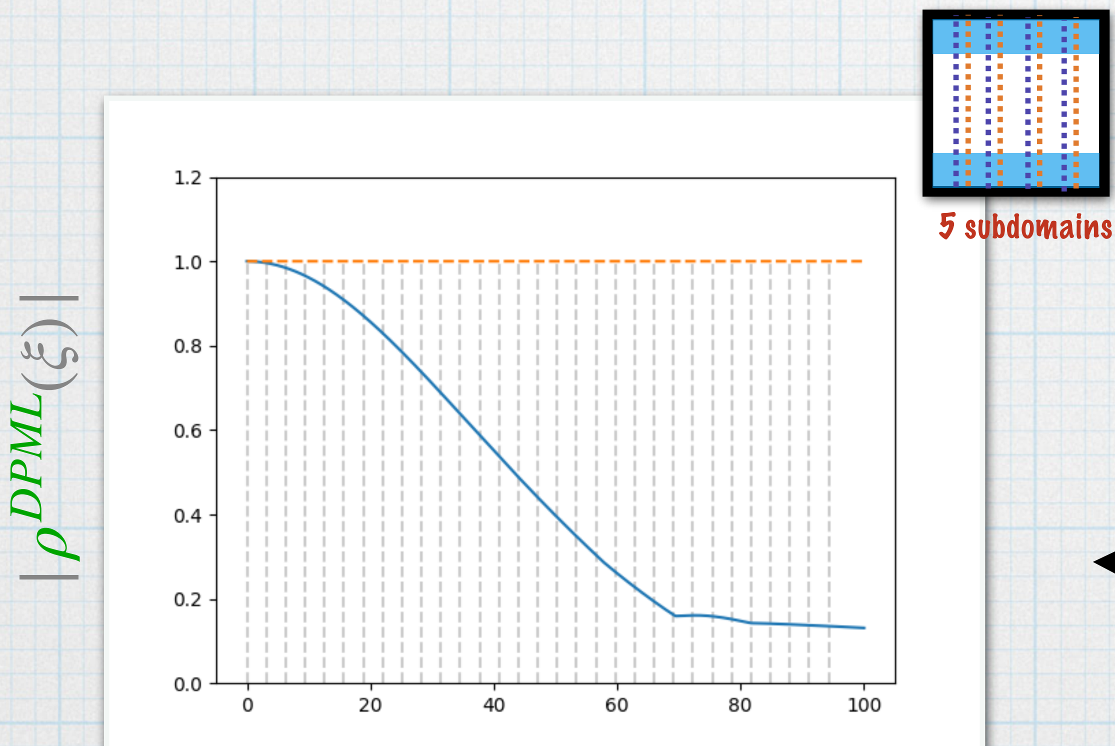
Convergence analysis on a toy problem

Comparison of the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 50$, $\sigma_{PML} = 10$)



Same convergence factor
but $\neq$ solution !





« Same solution »
but $\neq$ convergence factor

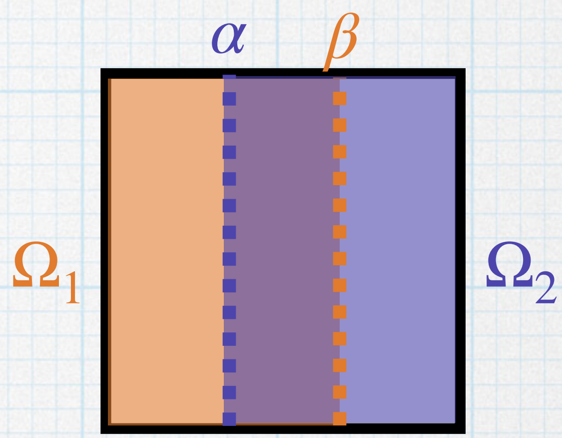
In the next...

1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

An alternated iterative algorithm

Step 1 $i \in \{1,2\}$

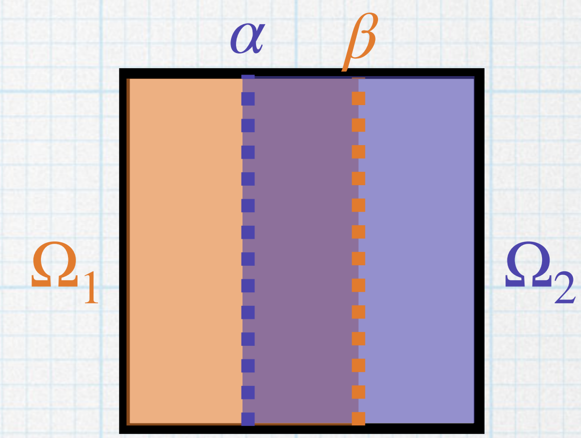
$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathbf{u}^{i,n+1/2} = \mathrm{f}_i & \text{in} & \Omega_i \\ \mathbf{u}^{i,n+1/2} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{i,i\pm1})\mathbf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1})\mathbf{u}^{i\pm1,n} & \text{on} & \Gamma_{ii\pm1} \end{array}\right.$$



An alternated iterative algorithm

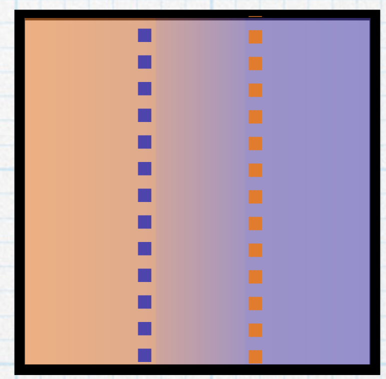
Step 1 $\quad i \in \{1,2\}$

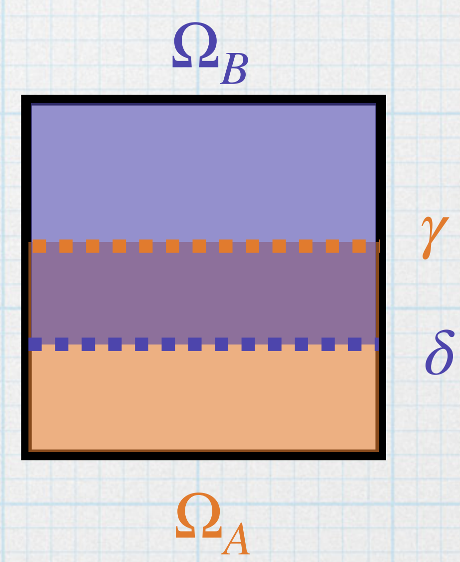
$$
\left|
\begin{array}{lll}
\mathscr{L}_{CH}\mathbf{U}^{i,n+1/2} = \mathbf{f}_i & \text{in} & \Omega_i \\
\mathbf{U}^{i,n+1/2} = 0 & \text{on} & \partial\Omega \\
(\partial_x + p_{i,i\pm 1})\mathbf{U}^{i,n+1/2} = (\partial_x + p_{i,i\pm 1})\mathbf{U}^{i\pm 1,n} & \text{on} & \Gamma_{ii\pm 1}
\end{array}
\right.
$$



Step 2 $\quad i \in \{1,2\},\ j \in \{A,B\}$

$$
\left|
\begin{array}{l}
\mathbf{U}^n_{Glo} = \mathscr{P}_{12\to Glo}((\mathbf{U}^{i,n+1/2})_i) \\
\mathbf{U}^{j,n} = \mathbf{U}^n_{Glo}|_{\Omega_j}
\end{array}
\right.
$$

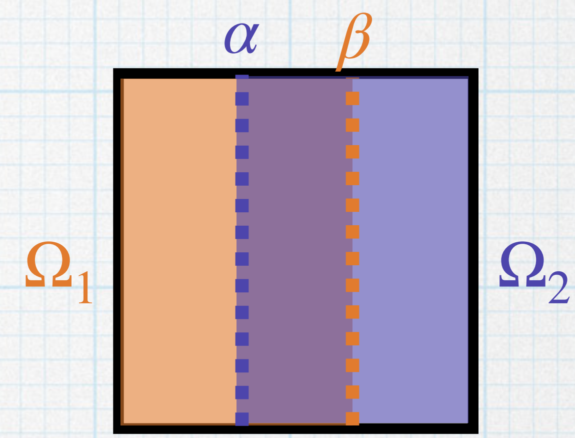




An alternated iterative algorithm

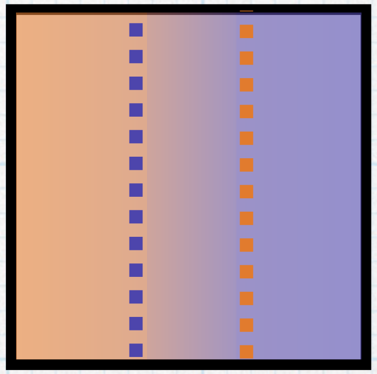
Step 1 $i \in \{1,2\}$

$$\begin{cases} \mathscr{L}_{CH} \mathbf{u}^{i,n+1/2} = \mathbf{f}_i & \text{in} \quad \Omega_i \\ \mathbf{u}^{i,n+1/2} = 0 & \text{on} \quad \partial\Omega \\ (\partial_x + p_{i,i\pm1}) \mathbf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1}) \mathbf{u}^{i\pm1,n} & \text{on} \quad \Gamma_{ii\pm1} \end{cases}$$



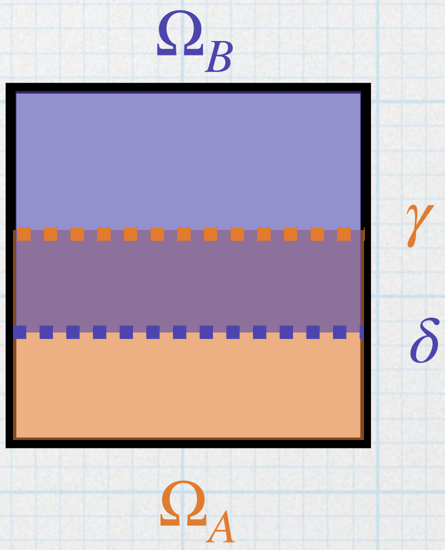
Step 2 $i \in \{1,2\}, \ j \in \{A,B\}$

$$\begin{cases} \mathbf{u}^n_{Glo} = \mathscr{P}_{12\rightarrow Glo}((\mathbf{u}^{i,n+1/2})_i) \\ \mathbf{u}^{j,n} = \mathbf{u}^n_{Glo}|_{\Omega_j} \end{cases}$$



Step 3 $j \in \{A,B\}$

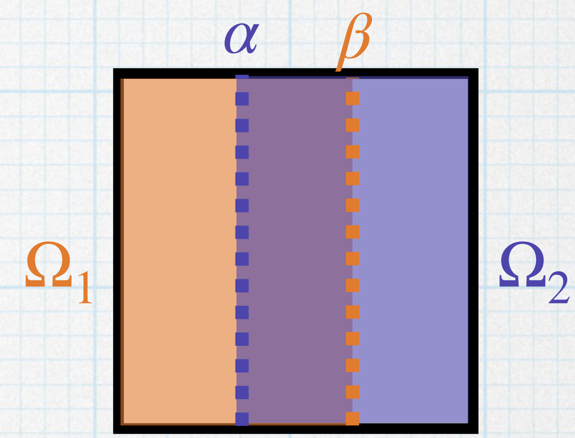
$$\begin{cases} \mathscr{L}_{CH} \mathbf{u}^{j,n+1/2} = \mathbf{f}_j & \text{in} \quad \Omega_j \\ \mathbf{u}^{j,n+1/2} = 0 & \text{on} \quad \partial\Omega \\ (\partial_x + p_{j,j\pm1}) \mathbf{u}^{j,n+1/2} = (\partial_x + p_{j,j\pm1}) \mathbf{u}^{j\pm1,n} & \text{on} \quad \Gamma_{jj\pm1} \end{cases}$$



An alternated iterative algorithm

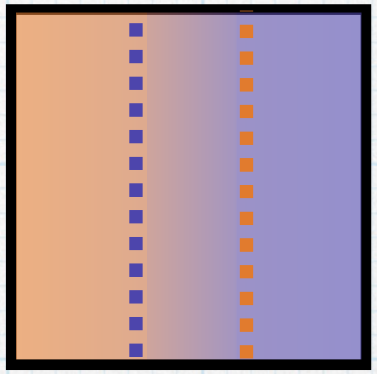
Step 1 $i \in \{1,2\}$

$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathbf{u}^{i,n+1/2} = \mathrm{f}_i & \text{in} & \Omega_i \\ \mathbf{u}^{i,n+1/2} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{i,i\pm1})\mathbf{u}^{i,n+1/2} = (\partial_x + p_{i,i\pm1})\mathbf{u}^{i\pm1,n} & \text{on} & \Gamma_{ii\pm1} \end{array}\right.$$



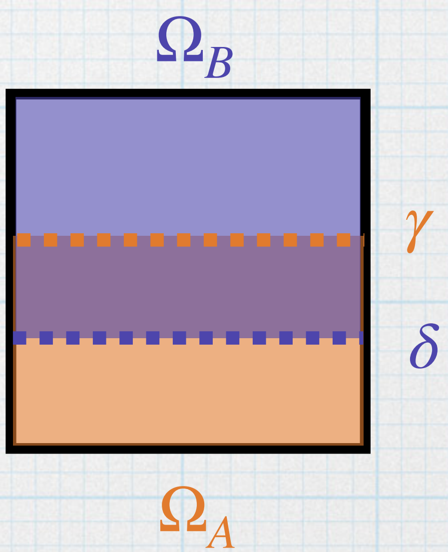
Step 2 $i \in \{1,2\},\ j \in \{A,B\}$

$$\left|\begin{array}{l} \mathbf{u}^n_{Glo} = \mathscr{P}_{12\to Glo}((\mathbf{u}^{i,n+1/2})_i) \\ \mathbf{u}^{j,n} = \mathbf{u}^n_{Glo}|_{\Omega_j} \end{array}\right.$$



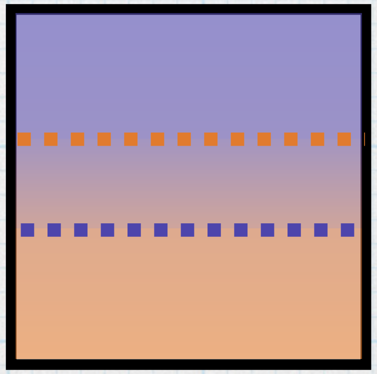
Step 3 $j \in \{A,B\}$

$$\left|\begin{array}{lll} \mathscr{L}_{CH}\mathbf{u}^{j,n+1/2} = \mathrm{f}_j & \text{in} & \Omega_j \\ \mathbf{u}^{j,n+1/2} = 0 & \text{on} & \partial\Omega \\ (\partial_x + p_{j,j\pm1})\mathbf{u}^{j,n+1/2} = (\partial_x + p_{j,j\pm1})\mathbf{u}^{j\pm1,n} & \text{on} & \Gamma_{jj\pm1} \end{array}\right.$$



Step 4 $i \in \{1,2\},\ j \in \{A,B\}$

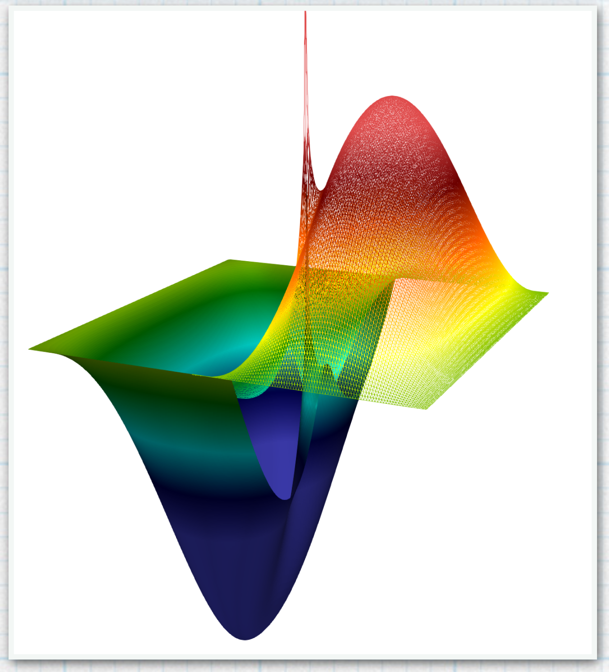
$$\left|\begin{array}{l} \mathbf{u}^{n+1}_{Glo} = \mathscr{P}_{AB\to Glo}((\mathbf{u}^{j,n+1/2})_j) \\ \mathbf{u}^{i,n+1} = \mathbf{u}^n_{Glo}|_{\Omega_i} \end{array}\right.$$



An alternated iterative algorithm

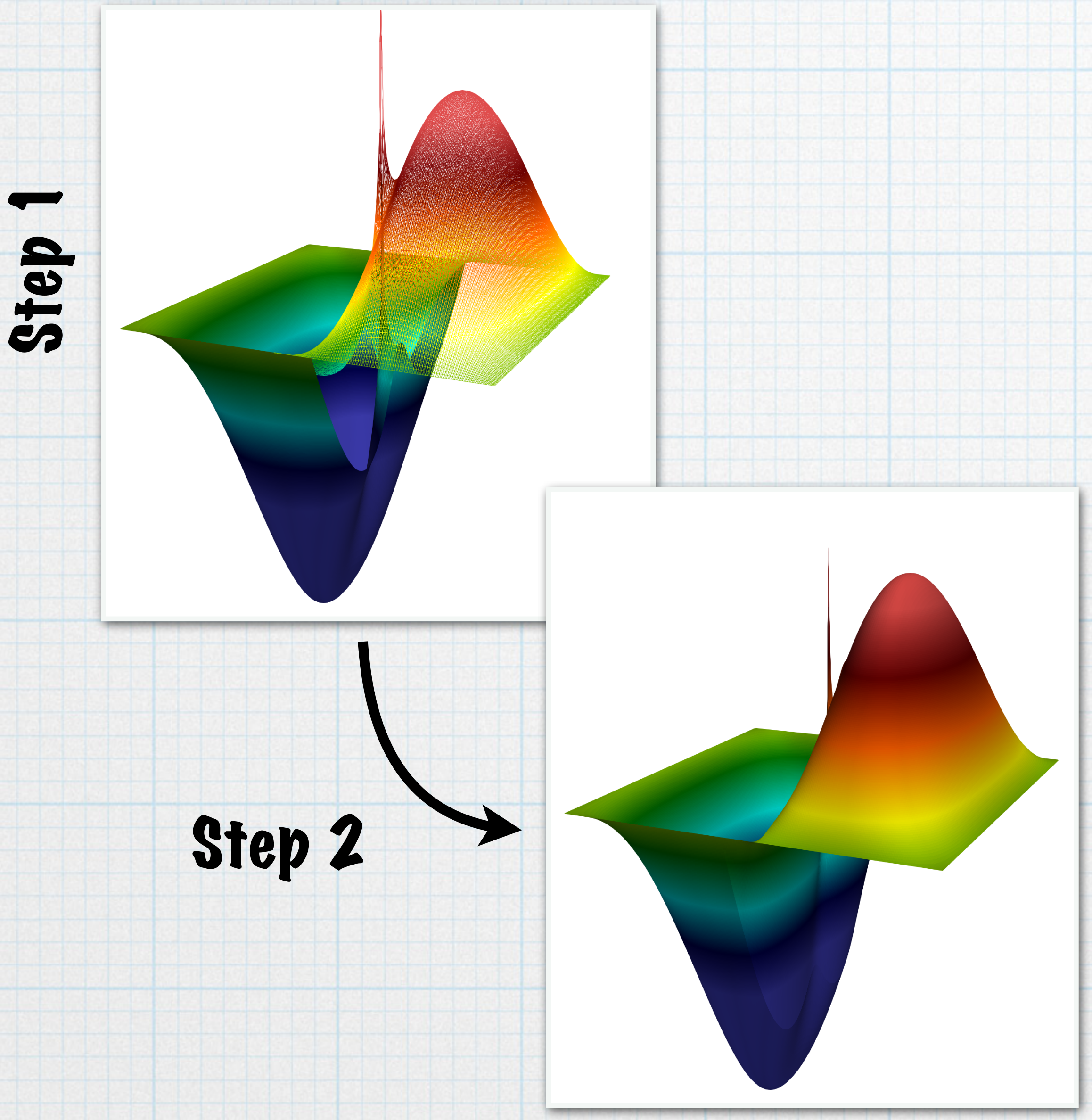
Illustration of the algorithm (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = \delta$, $\widetilde{\omega} = 5$)

Step 1



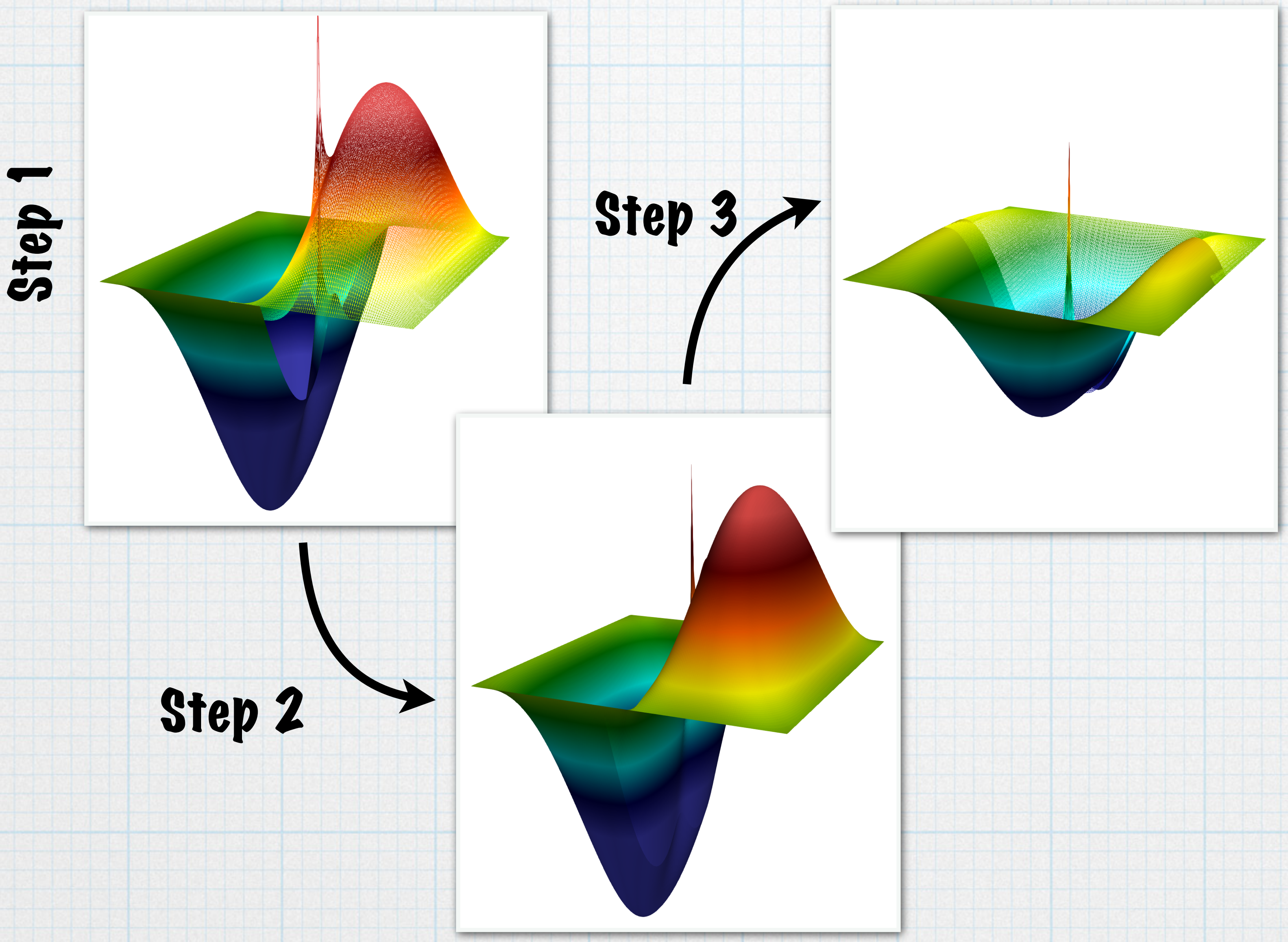
An alternated iterative algorithm

Illustration of the algorithm (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=\delta,\ \widetilde{\omega}=5$)



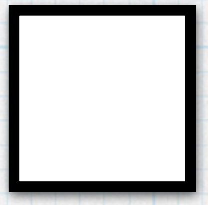
An alternated iterative algorithm

Illustration of the algorithm (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=\delta$, $\widetilde{\omega}=5$)

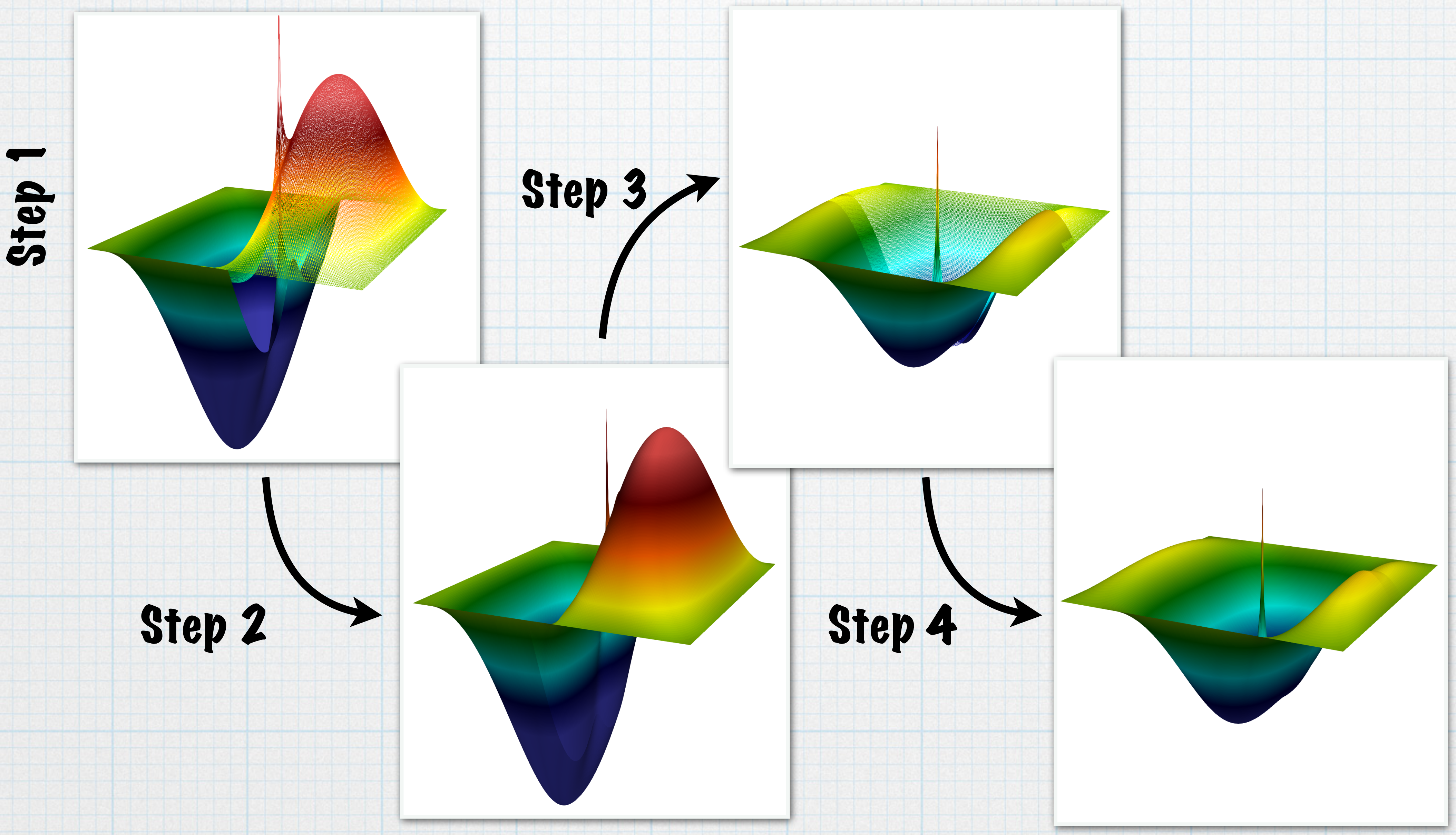


An alternated iterative algorithm

Illustration of the algorithm (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=\delta,\ \widetilde{\omega}=5$)

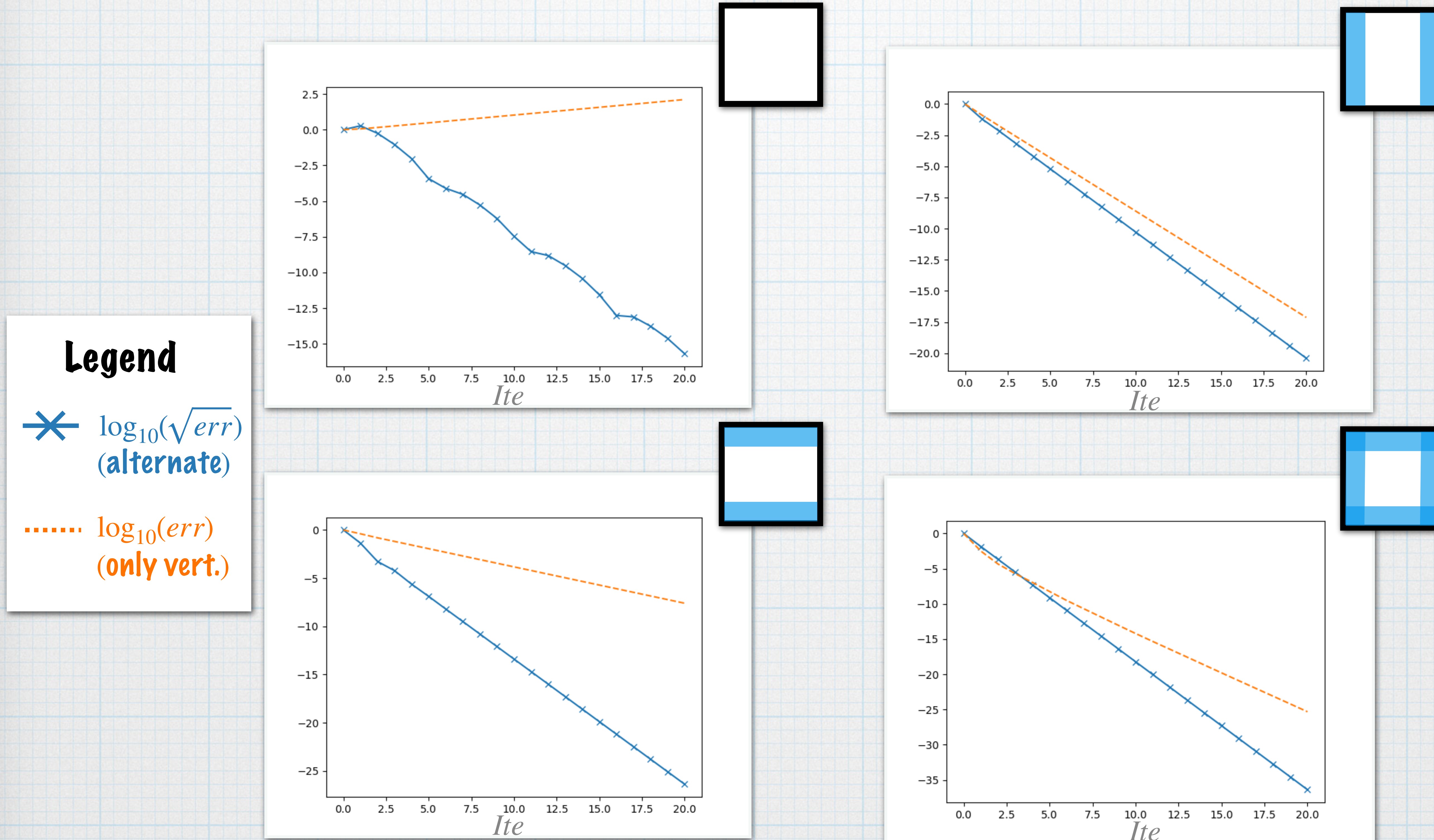


D-D



An alternate iterative algorithm

Convergence in the four situations (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a} \cdot \nabla \mathbf{u} = 0$, $\widetilde{\omega} = 5$, $\sigma_{PML} = 10$)



An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$\mathbf{A}^{i,n}(\xi_k)$

An alternated iterative algorithm

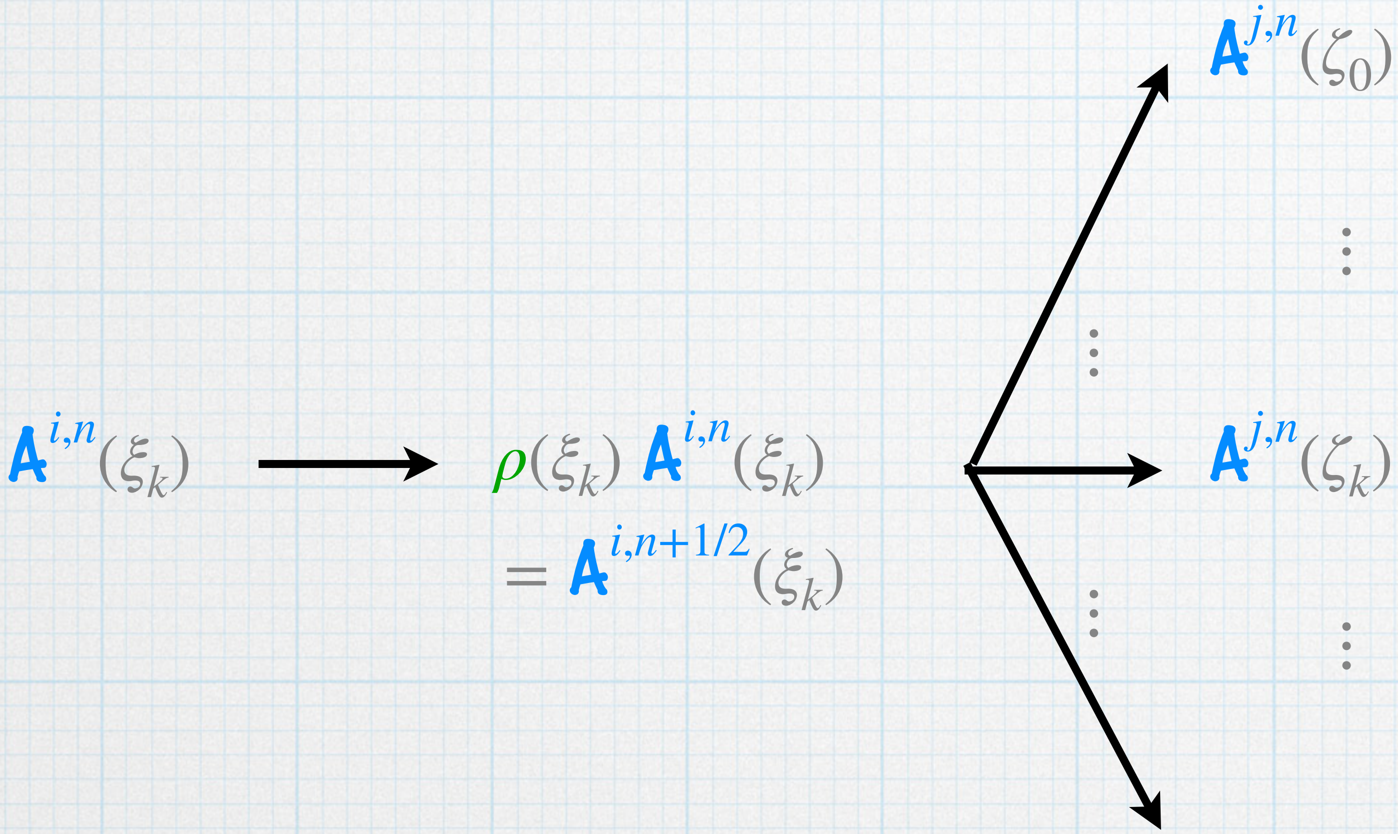
Some ideas on the convergence analysis of the alternated algorithm:

$$\mathsf{A}^{i,n}(\xi_k) \longrightarrow \rho(\xi_k)\,\mathsf{A}^{i,n}(\xi_k) = \mathsf{A}^{i,n+1/2}(\xi_k)$$

Step 1

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:



Step 1 **Step 2**

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$$
\begin{tikzcd}
 & & \mathbf{A}^{j,n}(\zeta_0) \arrow[r] & \rho'(\zeta_0)\,\mathbf{A}^{j,n}(\zeta_0) = \mathbf{A}^{j,n+1/2}(\zeta_0) \\
 & \vdots & \vdots & \vdots \\
\mathbf{A}^{i,n}(\xi_k) \arrow[r] & \rho(\xi_k)\,\mathbf{A}^{i,n}(\xi_k) = \mathbf{A}^{i,n+1/2}(\xi_k) \arrow[uur] \arrow[r] \arrow[dr] & \mathbf{A}^{j,n}(\zeta_k) \arrow[r] & \rho'(\zeta_k)\,\mathbf{A}^{j,n}(\zeta_k) = \mathbf{A}^{j,n+1/2}(\zeta_l) \\
 & \vdots & \vdots & \vdots \\
 & & &
\end{tikzcd}
$$

Step 1 **Step 2** **Step 3**

An alternated iterative algorithm

Some ideas on the convergence analysis of the alternated algorithm:

$$\mathbf{A}^{i,n}(\xi_k) \longrightarrow \rho(\xi_k)\,\mathbf{A}^{i,n}(\xi_k) = \mathbf{A}^{i,n+1/2}(\xi_k)$$

$$\mathbf{A}^{j,n}(\zeta_0) \longrightarrow \rho'(\zeta_0)\,\mathbf{A}^{j,n}(\zeta_0) = \mathbf{A}^{j,n+1/2}(\zeta_0)$$

$$\vdots \qquad \vdots$$

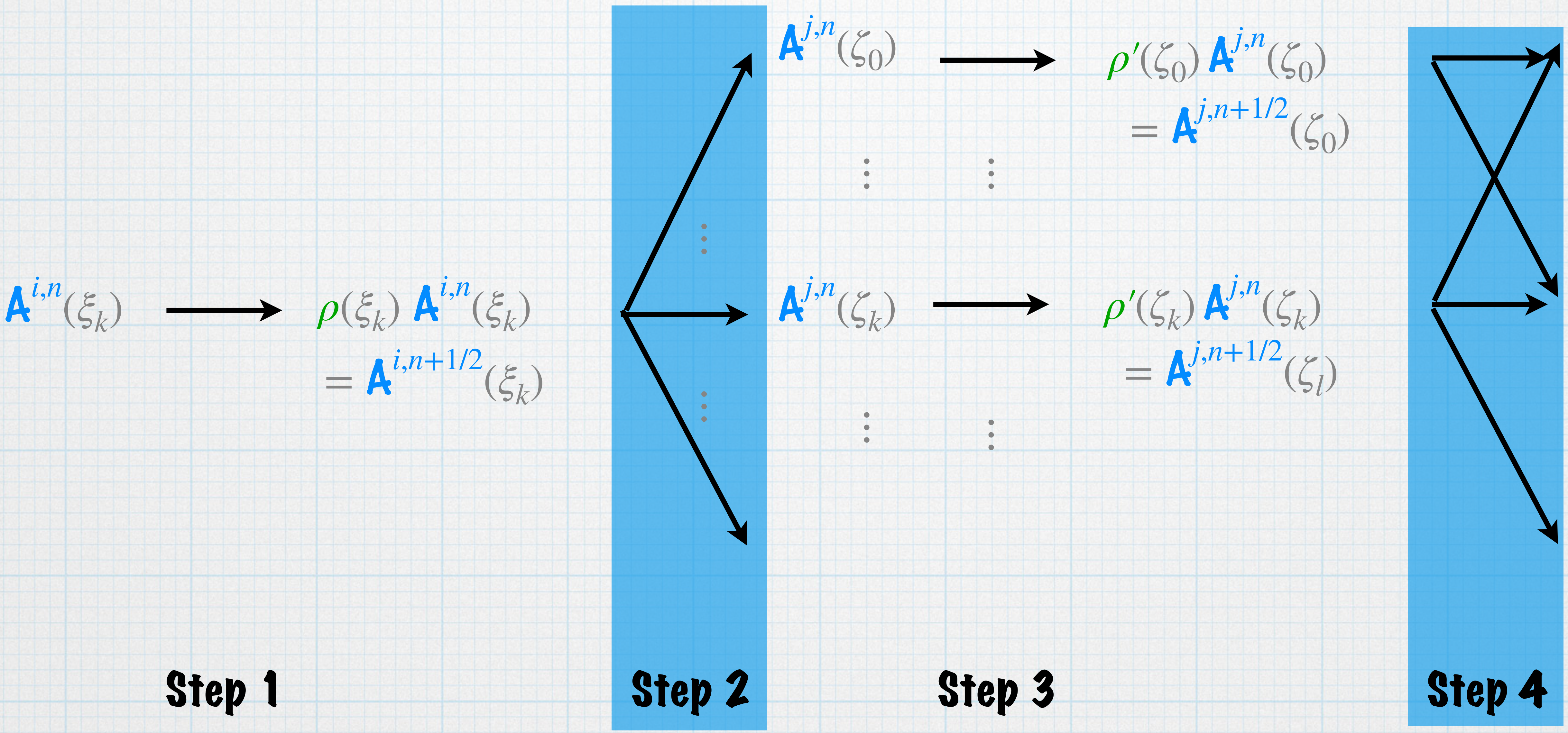
$$\mathbf{A}^{j,n}(\zeta_k) \longrightarrow \rho'(\zeta_k)\,\mathbf{A}^{j,n}(\zeta_k) = \mathbf{A}^{j,n+1/2}(\zeta_l)$$

$$\vdots \qquad \vdots$$

Step 1 Step 2 Step 3 Step 4

An alternated iterative algorithm

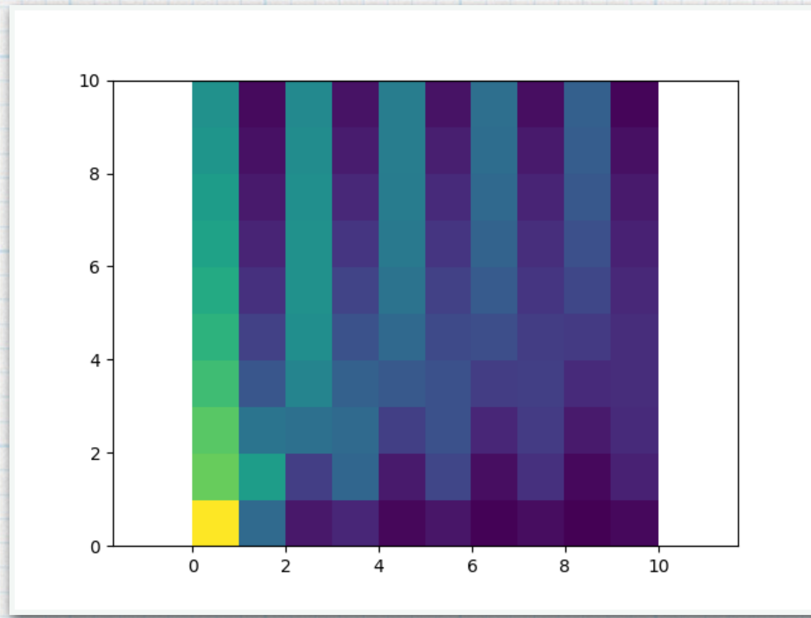
Some ideas on the convergence analysis of the alternated algorithm:



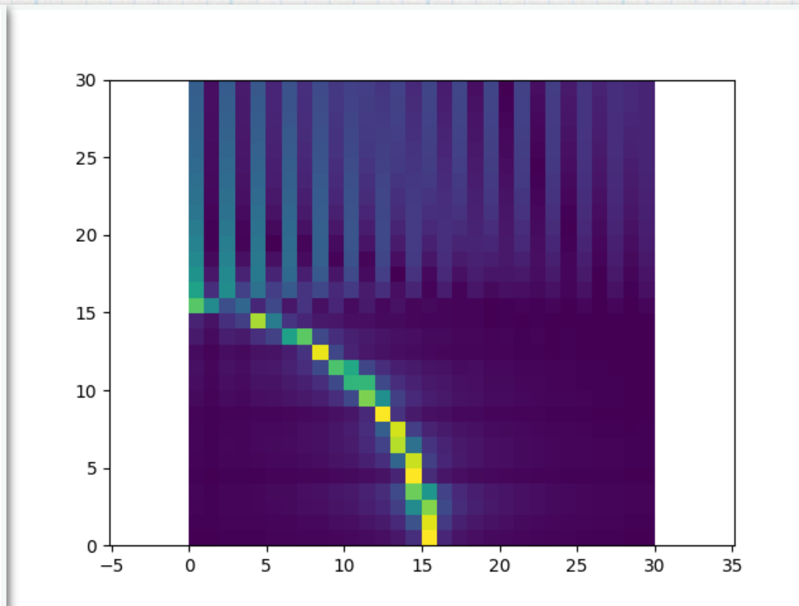
Let us denote by $\left[\mathscr{P}_{12\to AB}\right]_{kl}$ the coefficients that maps $\mathbf{A}^{i,n+1/2}(\xi_k)$ to $\mathbf{A}^{j,n}(\zeta_l)$.

An alternated iterative algorithm

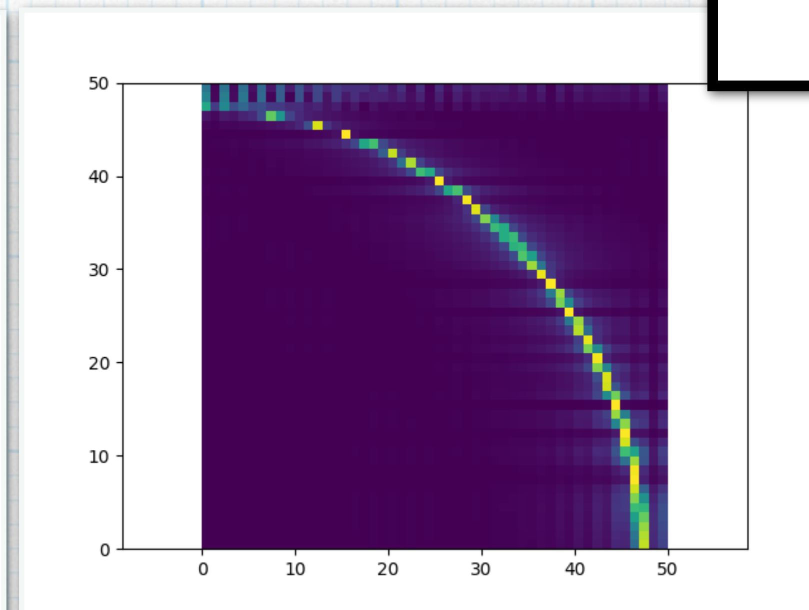
Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=0$)



$\widetilde{\omega}=5$



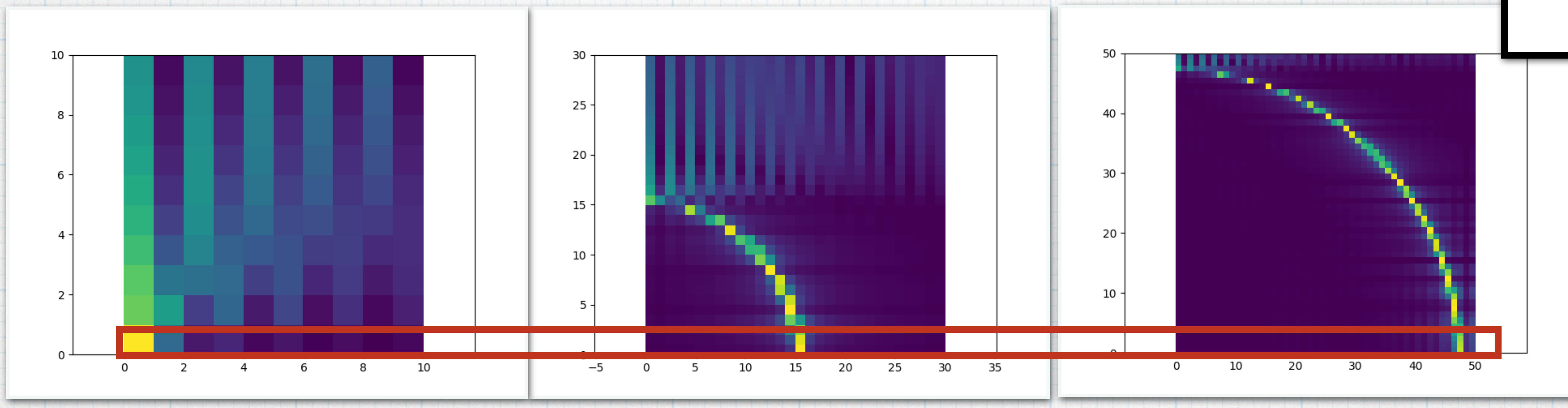
$\widetilde{\omega}=50$



$\widetilde{\omega}=150$

An alternated iterative algorithm

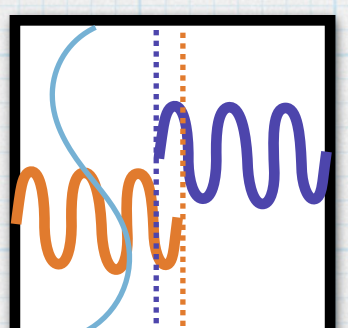
Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$)



$\widetilde{\omega} = 5$ $\widetilde{\omega} = 50$ $\widetilde{\omega} = 150$

Interpretation: $\mathbf{v}^{i,n}(\xi) = \sin(\xi\pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x} + \mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right)$ **where** $\mathcal{S}(\xi) = \sqrt{\widetilde{\mu} - (\xi\pi)^2}$.

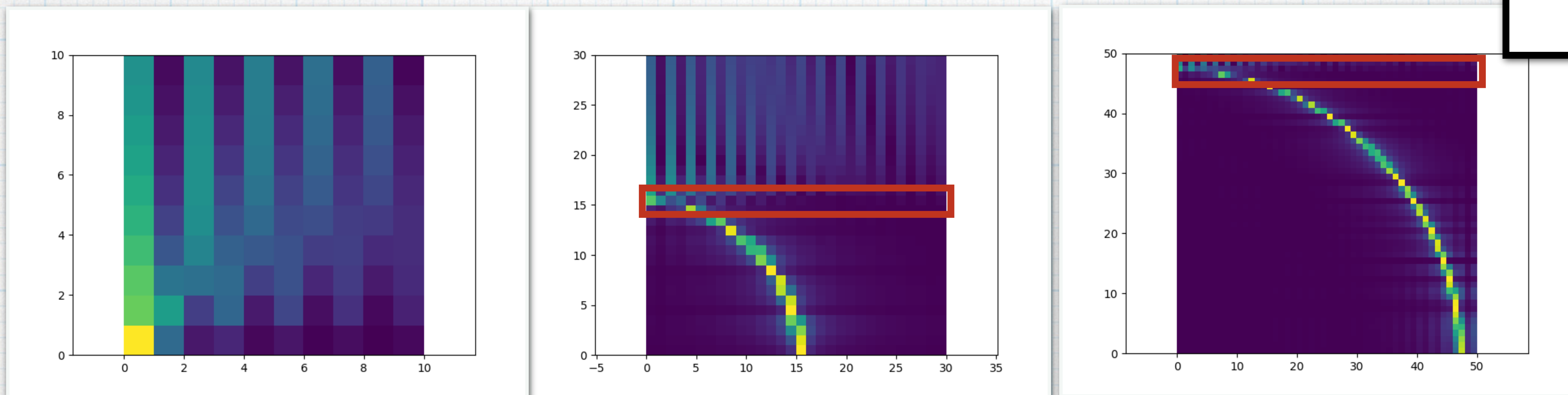
Small ξ



High oscillation

An alternated iterative algorithm

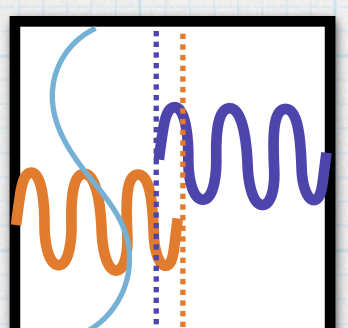
Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta\mathbf{u}+i\mathbf{a}\cdot\nabla\mathbf{u}=0$)



$\widetilde{\omega}=5$ $\widetilde{\omega}=50$ $\widetilde{\omega}=150$

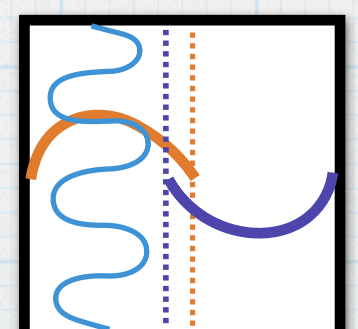
Interpretation: $\mathbf{v}^{i,n}(\xi)=\sin(\xi\pi y)\left(\mathbf{A}^{i,n}(\xi)e^{i\mathcal{S}(\xi)x}+\mathbf{B}^{i,n}(\xi)e^{-i\mathcal{S}(\xi)x}\right)$ where $\mathcal{S}(\xi)=\sqrt{\widetilde{\mu}-(\xi\pi)^2}$.

Small ξ



High oscillation

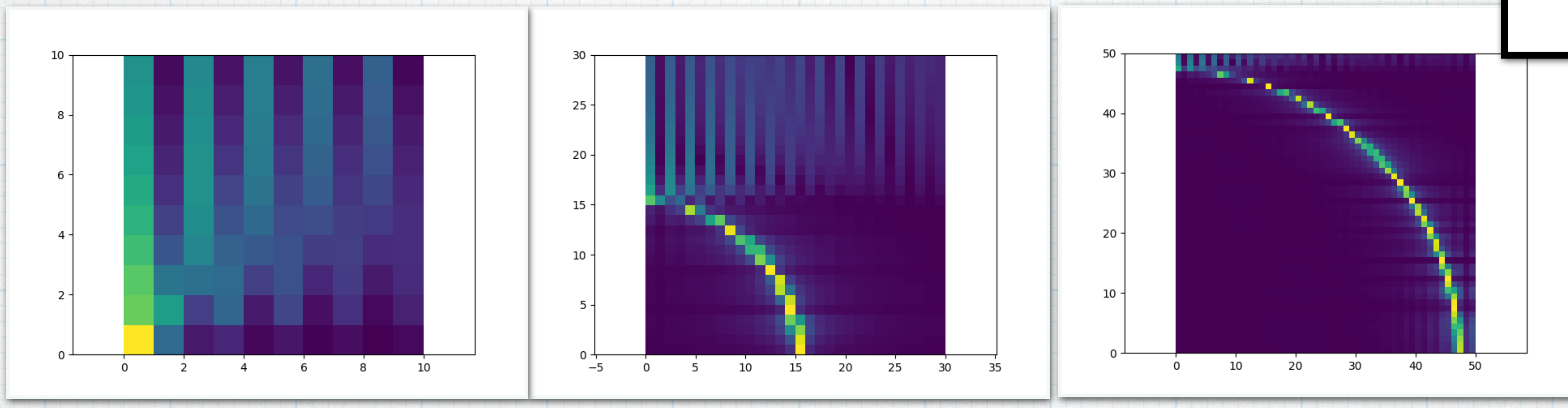
Large ξ



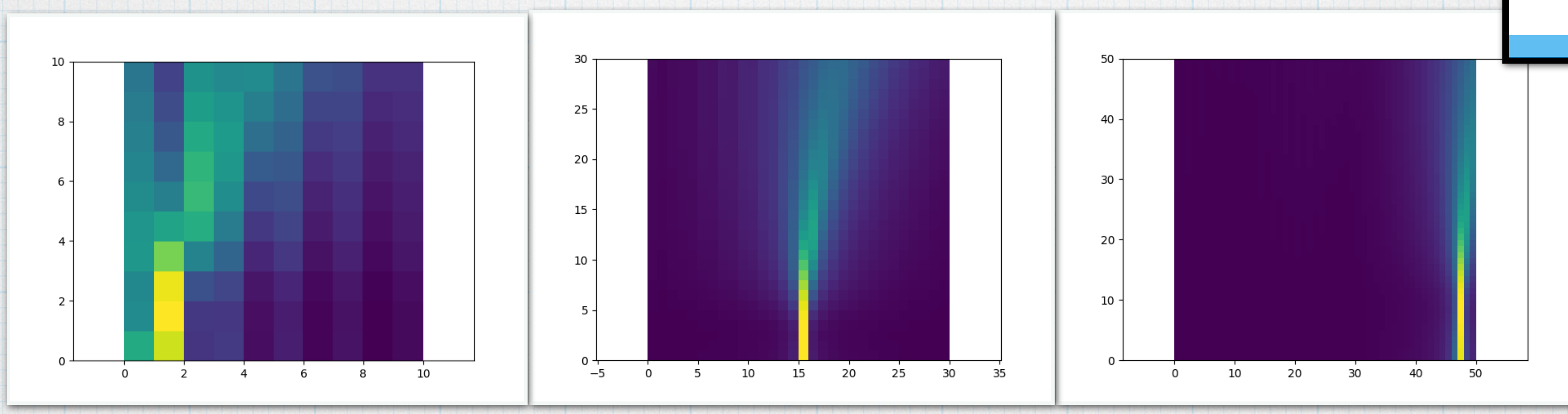
Slow oscillation

An alternated iterative algorithm

Illustration of the matrix $\mathscr{P}_{12\to AB}$ (Wave Ray: $-\Delta \mathbf{u} + i\mathbf{a}\cdot\nabla\mathbf{u} = 0$)



$\widetilde{\omega} = 5$ $\widetilde{\omega} = 50$ $\widetilde{\omega} = 150$

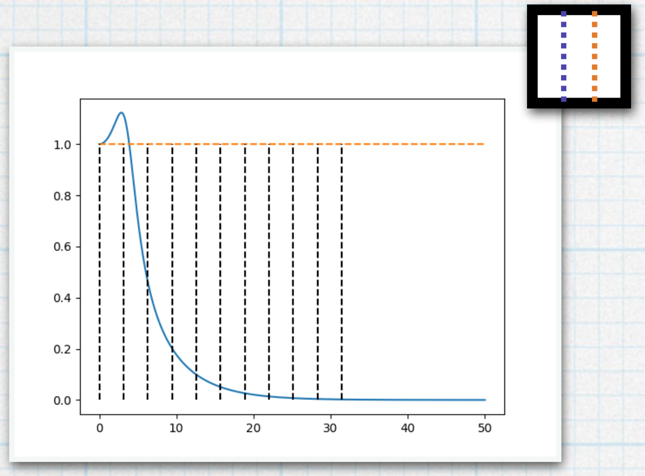
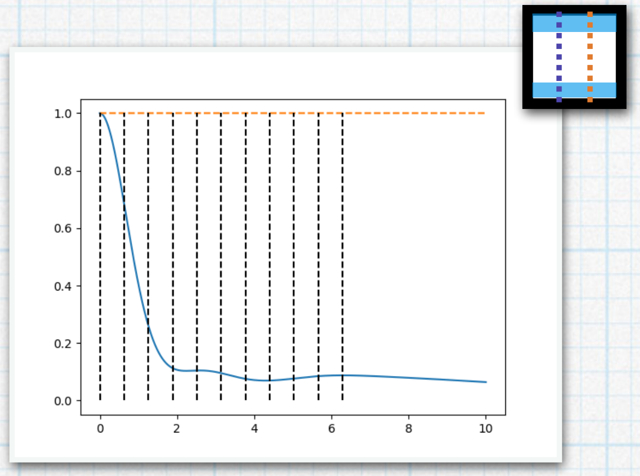
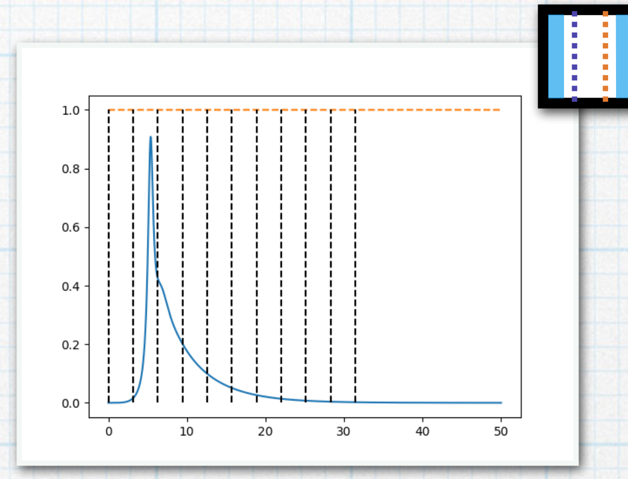
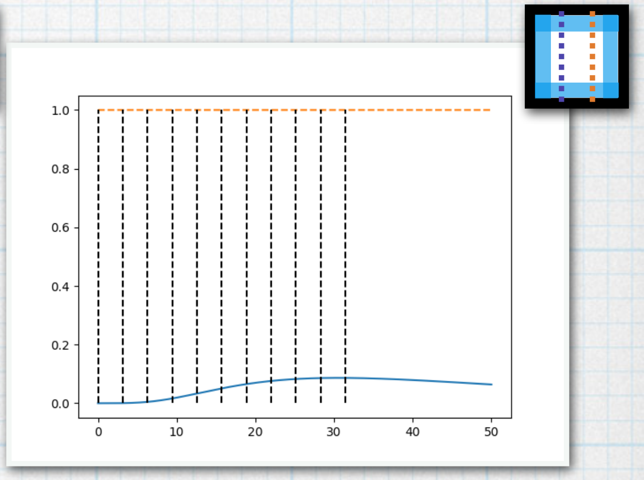


In the next...

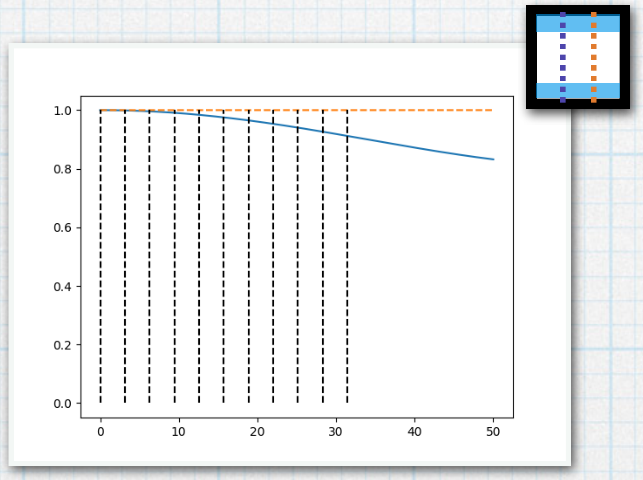
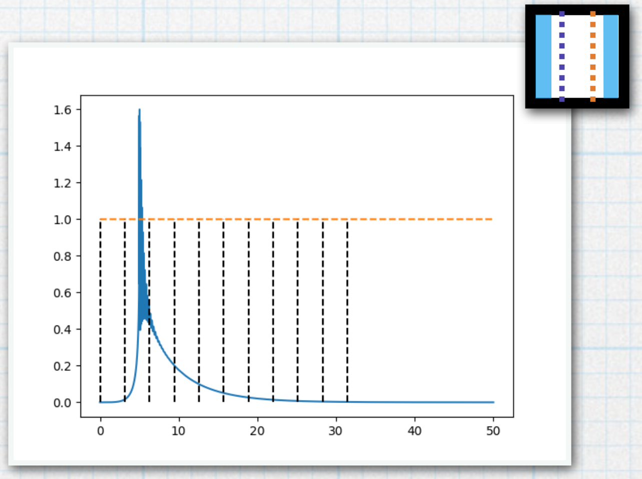
1. Motivation
2. Link with Helmholtz equation
3. Convergence analysis on a toy problem
4. An alternated iterative algorithm
5. Conclusion

Conclusion

Using **PML has a strong impact on the convergence** of classical iterative algorithm.

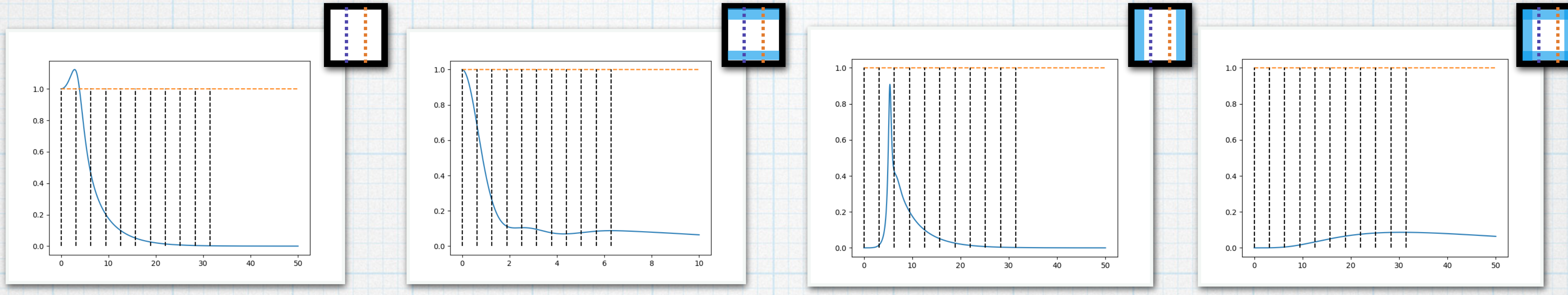





Also, the **PML coefficients must be chosen carefully !**

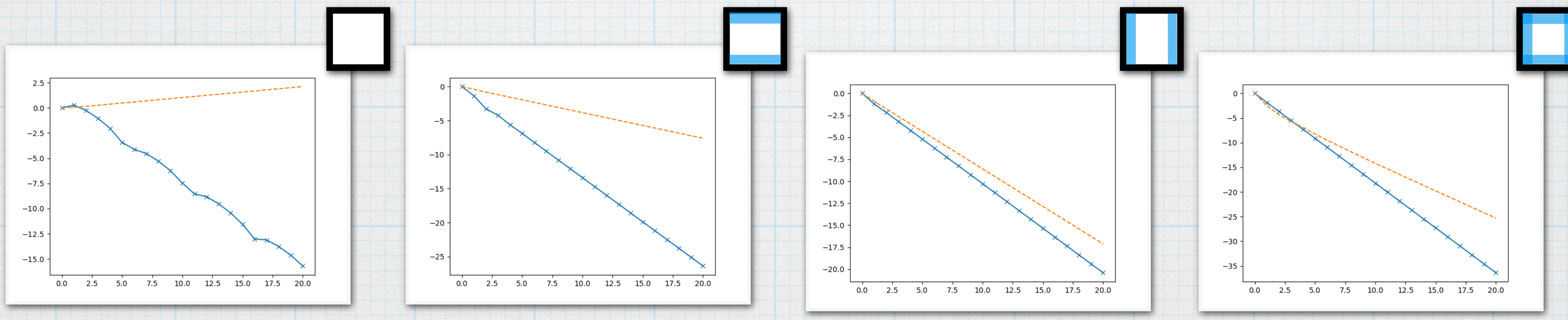
Conclusion

Using **PML has a strong impact on the convergence** of classical iterative algorithm.



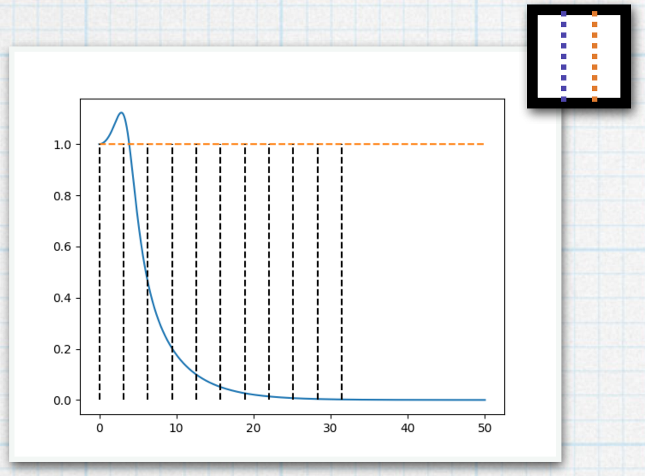
We propose an **alternated algorithm** based on splitting once vertically and once horizontally the domain. This algorithm :

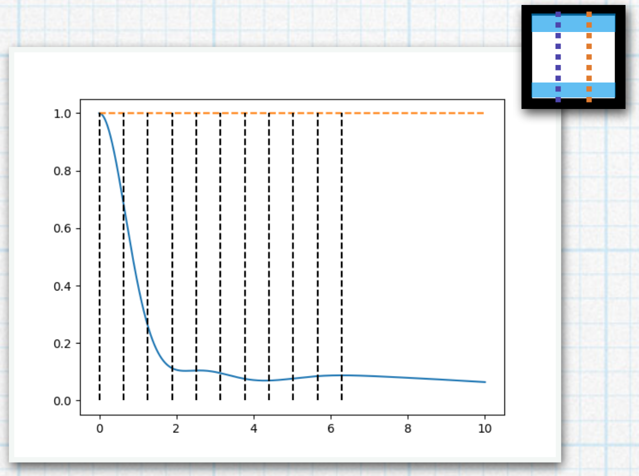
* **improve the convergence factor** in every case

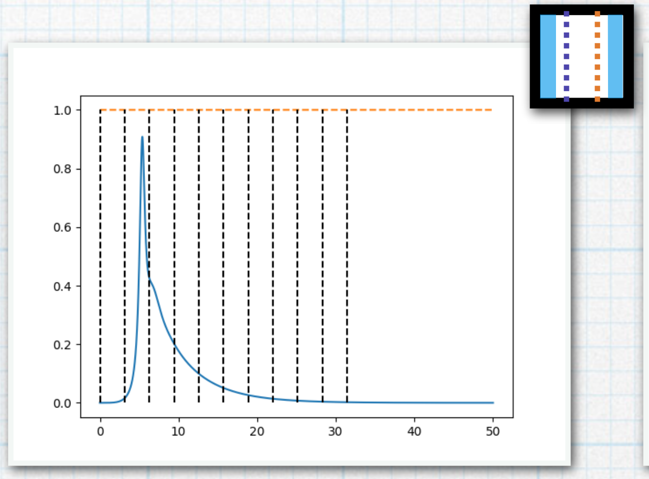


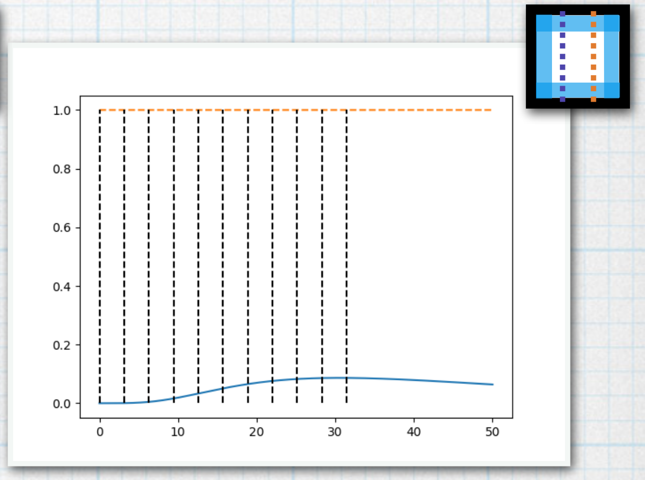
Conclusion

Using **PML has a strong impact on the convergence** of classical iterative algorithm.



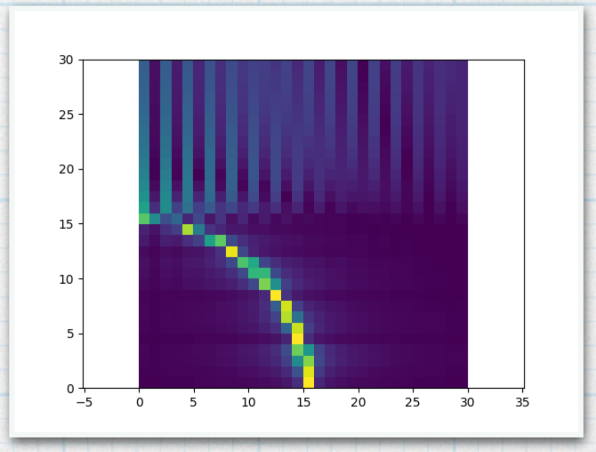


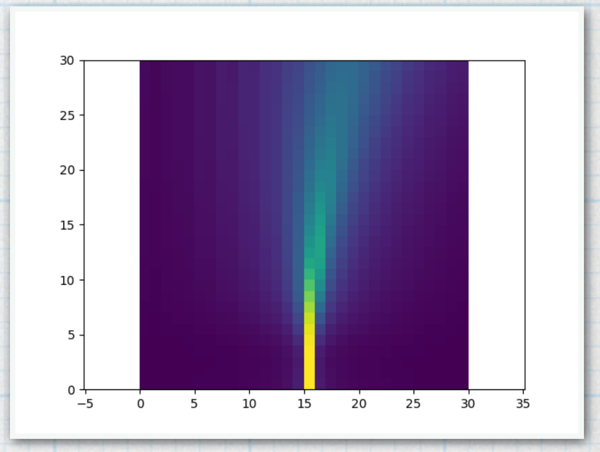




We propose an **alternated algorithm** based on splitting once vertically and once horizontally the domain. This algorithm :

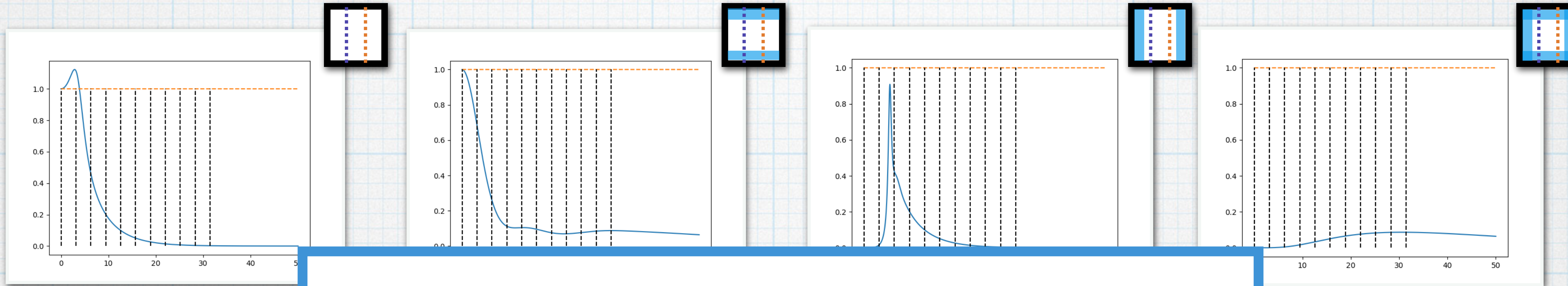
* **improve the convergence factor** in every case
* and have a **different behaviour depending on the PML BCs,**





Questions ?

Using **PML has a strong impact on the convergence** of classical iterative algorithm.



Thank you for your attention !!

We propose an ... cally and once horizontally the domain. This algorithm :

* improve the convergence factor in every case
* and have a different behaviour depending on the PML BCs,