The discrete divergence-free condition in lattice Boltzmann magnetohydrodynamics: an approach inspired by data

Paul DELLAR, OCIAM, Mathematical Institute - University of Oxford

Magnetohydrodynamics (MHD) combines the Navier–Stokes and Maxwell equations to describe the flow of electrically conducting fluids in magnetic fields. The magnetic field $\mathbf{B}$ evolves according to Maxwell's equation $\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0$. This evolution preserves the divergence-free condition $\nabla \cdot \mathbf{B} = 0$ that expresses the absence of magnetic monopoles. It is natural to ask what discrete forms of the $\nabla \cdot \mathbf{B} = 0$ condition are preserved by numerical algorithms.
The lattice Boltzmann approach embeds the target system of partial differential equations to be simulated into a larger linear constant-coefficient hyperbolic system. All nonlinearity is confined to algebraic source terms, implemented locally at individual lattice points in the discretised system. The author's embedding of the MHD equations leads to the magnetic evolution equation [1, 2]

$$\partial_t \mathbf{B} + \nabla \cdot \Lambda = 0.$$

The curl of $\mathbf{E}$ is replaced by the divergence of a general rank-2 tensor Λ that evolves according to

$$\partial_t \Lambda + \nabla \cdot \mathsf{M} = -\frac{1}{\tau}\left(\Lambda - \Lambda^{(0)}\right) \tag{$\star$}$$

in the simplest case. Slowly varying solutions satisfy $\operatorname{Tr}\Lambda = -\tau\Theta\,\nabla \cdot \mathbf{B} + \mathcal{O}(\tau^3)$ for a constant Θ, so it is natural to adopt $\operatorname{Tr}\Lambda \approx 0$ as a proxy for $\nabla \cdot \mathbf{B} = 0$. Numerical simulations typically maintain $\operatorname{Tr}\Lambda$ at close to round-off error in 64-bit arithmetic.
However, one can still ask what divergence-free condition is best satisfied by the components of the magnetic field $\mathbf{B}$ on the lattice. In two dimensions, there are two natural finite difference approximations Δ^+ and $\Delta^\times$ for $\nabla \cdot \mathbf{B}$, using four-point stencils along the axes and the diagonals of a square lattice respectively. Any convex linear combination $\Delta^\phi = (1-\phi)\,\Delta^+ + \phi\,\Delta^\times$ is also a second-order accurate finite difference approximation for $\nabla \cdot \mathbf{B}$. Minimising the sum of squares

$$\sum_{\text{lattice}} \Big((1-\phi)\,\Delta^+ + \phi\,\Delta^\times\Big)^2$$

for the outputs of simulations with different values of τ shows a very distinct minimum at

$$\phi = 2\left(1/4 - (\tau/\Delta t)^2\right).$$

We derive this optimal ϕ analytically by applying operator algebra techniques to the discrete evolution equations for the diagonal components of the tensors Λ and M on the lattice.
Finally, we show that adjusting the right-hand side of ($\star$) to replace τ with a different relation time τ_ψ for the trace $\operatorname{Tr}\Lambda$ implements a set of extended MHD equations with divergence cleaning, for which

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} + \nabla\psi = 0.$$

The extra scalar field ψ is related to $\operatorname{Tr}\Lambda$. It acts like a hydrodynamic pressure to maintain the divergence-free condition. The joint evolution of ψ and Δ^ϕ, the optimal finite difference approximation for $\nabla \cdot \mathbf{B}$ on the lattice, in simulations converges to analytical solutions of the extended magnetohydrodynamics equations with hyperbolic divergence cleaning.

[1] P. J. Dellar. *Lattice kinetic schemes for magnetohydrodynamics.* J. Comput. Phys., **179**, 95–126, 2002.

[2] P. J. Dellar. *Lattice Boltzmann magnetohydrodynamics with current-dependent resistivity.* J. Comput. Phys., **237**, 115–131, 2013.

Contact : `paul.dellar@maths.ox.ac.uk`