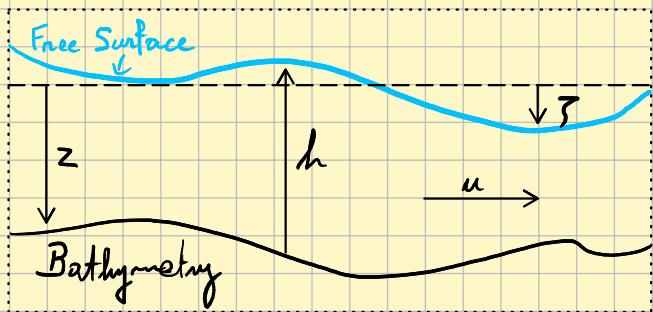


Fully implicit kinetic scheme for the 1D Saint-Venant system

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The 1D Saint-Venant system: notations & properties



- $h(t, x)$ → Water height
- $z(x)$ → Bathymetry
- $\zeta = h + z$ → Free surface
- $u(t, x)$ → Horizontal velocity

Hyperbolic system of conservation laws:

$$\begin{cases} \partial_t h + \partial_x hu = 0 \\ \partial_t hu + \partial_x (hu^2 + \frac{g}{2}h^2) = -gh\partial_x z \end{cases} \quad (\text{SV}) \longrightarrow$$

$$\partial_t u + \partial_x F(u) = S(u, z)$$

↳ Eigenvalues of $D\mathbf{F}$ given by
 $\lambda = u \pm \sqrt{2}c$, $c = \sqrt{\frac{g}{2}h}$ sound speed

Positivity: $h > 0$, stationary state: $h+z=K$, $u=0$

$$\text{Energy } E(t, x) = \frac{hu^2}{2} + \frac{gh^2}{2} + gLz \text{ satisfies } \partial_t E + \partial_x \left[(E + \frac{gh^2}{2})u \right] \leq 0 \quad (1)$$

Kinetic representation for the 1D Saint-Venant system

Idea: $f(t, x, \xi) \geq 0$ particle distribution, new variable $\xi \rightarrow$ velocity

↳ Recover macroscopic quantities by integrating: $M_f(t, x) = \int_{\mathbb{R}} \left(\frac{1}{\xi}\right) f(t, x, \xi) d\xi$

$$\text{Boltzmann kinetic equation: } \underbrace{\partial_t f + \xi \partial_x f}_{\text{Linear transport}} = \frac{1}{\varepsilon} \underbrace{Q[f](t, x, \xi)}_{\text{Collision operator with } \int_{\mathbb{R}} \left(\frac{1}{\xi}\right) Q d\xi = 0} \quad (2)$$

$$\text{Gibbs equilibrium: } f \in \text{Ker } Q \iff f = M(M_f, \xi) := \frac{1}{g^{\frac{1}{\varepsilon}}} \sqrt{(2gh_f - (\xi - u_f)^2)_+}$$

Maxwellian $M(M, \xi)$ satisfies the moment relations:

$$\forall M \in \mathbb{R}_+ \times \mathbb{R}, \int_{\mathbb{R}} \left(\frac{1}{\xi}\right) M(M, \xi) d\xi = M, \int_{\mathbb{R}} \xi \left(\frac{1}{\xi}\right) M(M, \xi) d\xi = F(M) \quad (3)$$

Lemma 1: $M = (M, mu)$ is a weak solution of (SV) iff $M(M, \xi)$ satisfies

$$\partial_t M + \xi \partial_x M - g(\partial_x \xi) \partial_\xi M = \mu(t, x, \xi) \quad (\text{KR})$$

with $\int_{\mathbb{R}} \left(\frac{1}{\xi}\right) \mu(t, x, \xi) d\xi = 0$ for a.e. (t, x) .

Explicit kinetic scheme with a flat bottom

Kinetic representation (KR) obtained as the limit of (2) when $\varepsilon \rightarrow 0$.

BGK operator $Q[f] = M(u_f, \xi) - f \rightarrow$ replace (KR) with BGK-splitting:

$$\begin{cases} \partial_t f = \frac{1}{\varepsilon} (M(u_f, \xi) - f) \\ \partial_t f + \xi \partial_x f = 0 \end{cases} \xrightarrow{\varepsilon \rightarrow 0} \left| \begin{array}{l} \text{Solve } \partial_t f + \xi \partial_x f = 0 \text{ with initial data} \\ f^0(x, \xi) = M(u_f^0(x), \xi). \end{array} \right.$$

Discretize: $t^n = n \Delta t$, $x_j = j \Delta x$, $C_j = (x_{j-1/2}, x_{j+1/2})$, $f_j^n(\xi) = \frac{1}{\Delta x} \int_{C_j} f(t^n, x, \xi) dx$

1st order upwind scheme: $\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{\xi}{\Delta x} \left(1_{\xi < 0} (f_{j+1/2}^n - f_j^n) + 1_{\xi > 0} (f_j^n - f_{j-1/2}^n) \right) = 0 \quad (4)$
 where $f_j^n = M(u_j^n, \xi)$.

Integrate (4) against $(\frac{1}{\xi})$: $\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{\Delta x} \left(F(u_j^n, u_{j+1/2}^n) - F(u_{j-1/2}^n, u_j^n) \right) = 0$

Macroscopic kinetic flux $F(u_L, u_R) = \int_{\xi < 0} \xi \left(\frac{1}{\xi} \right) M(u_R, \xi) d\xi + \int_{\xi > 0} \xi \left(\frac{1}{\xi} \right) M(u_L, \xi) d\xi$

Implicit version of the kinetic scheme with flat bottom

Consider the scheme $\frac{f_j^{n+1} - f_j^n}{\Delta t} + \frac{3}{\Delta x} \left(\mathbb{1}_{z < 0} (f_{j+1}^{n+1} - f_j^{n+1}) + \mathbb{1}_{z > 0} (f_j^{n+1} - f_{j-1}^{n+1}) \right) = 0 \quad (5)$

N = number of cells, define $f^n = (f_1^n, f_2^n, \dots, f_N^n)^T \in (\mathbb{R}_+)^N$

(5) becomes $(I + \sigma L) f^{n+1} = f^n + \sigma B^{n+1}$ with $\sigma = \frac{\Delta t}{\Delta x}$, B^{n+1} accounts for limit conditions
 ↳ Ghost cells

$$L = \begin{pmatrix} 1 & 3\mathbb{1}_{z < 0} & & & \\ 3\mathbb{1}_{z > 0} & 1 & 3\mathbb{1}_{z < 0} & & \\ & & 1 & 3\mathbb{1}_{z < 0} & \\ & & & 3\mathbb{1}_{z > 0} & 1 \\ & & & & 1 \end{pmatrix}, \quad B^{n+1} = \begin{pmatrix} 3M_0^{n+1} \mathbb{1}_{z > 0} \\ 0 \\ \vdots \\ 0 \\ -3M_{N+1}^{n+1} \mathbb{1}_{z < 0} \end{pmatrix} \in \mathbb{R}^N$$

Remark 2: Matrix $(I + \sigma L)^{-1}$ can be computed by hand, and we have analytic expressions for

$$\int_R (I + \sigma L)^{-1} M d\zeta \text{ and } \int_R \mathbb{1} (I + \sigma L)^{-1} M d\zeta.$$

Since B^{n+1} unknown, proceed in two steps: $\begin{cases} 1/ (h, h\mu)^{n+1/2} = \int_R \left(\frac{1}{\zeta}\right) \otimes (I + \sigma L)^{-1} M^n d\zeta \\ 2/ (h, h\mu)^{n+1} = \int_R \left(\frac{1}{\zeta}\right) \otimes (I + \sigma L)^{-1} B^{n+1/2} d\zeta \end{cases}$

Discrete entropy inequality and positivity

Energy inequality $\partial_t E + \partial_x \left[(E + g\frac{h^2}{2})u \right] < 0 \rightarrow$ discrete counterpart?

Consider kinetic entropy $H(f, \xi) = \frac{\xi^2}{2} f + \frac{g^2 \pi^2}{6} f^3 + g z f$ as an energy distribution

Lemma 3: $\forall u \in \mathbb{R}_+ \times \mathbb{R}$, $\int_{\mathbb{R}} H(M(u, \xi), \xi) d\xi = E$, $\int_{\mathbb{R}} \xi H(M(u, \xi), \xi) d\xi = (E + g\frac{h^2}{2})u$.

$M(u, \cdot)$ minimizes $f \mapsto \int_{\mathbb{R}} H(f(\xi), \xi) d\xi$ under the constraint $\int_{\mathbb{R}} (\frac{1}{f}) f(\xi) d\xi = u$

Prop 4: Under CFL $\sigma|\xi| \leq 1$, explicit scheme (4) satisfies $h_j^{n+1} \geq 0$ and $\bar{E}_j^{n+1} \leq E_j^n - \sigma [G_{j+\frac{1}{2}}^n - G_{j-\frac{1}{2}}^n]$

Proof: $f_j^{n+1} = (1-\sigma/\xi)M_j^n + \sigma/\xi M_{j\pm 1}^n \xrightarrow[\text{Convex combination}]{\text{CFL}} f_j^{n+1} \geq 0$

Using lemma 3: $\bar{E}_j^{n+1} = \int_{\mathbb{R}} H(M_j^{n+1}(\xi), \xi) d\xi \leq \int_{\mathbb{R}} H(f_j^{n+1}(\xi), \xi) d\xi \leq \int_{\mathbb{R}} (1-\sigma/\xi)H_j^n + \sigma/\xi H_{j\pm 1}^n d\xi$
 \hookrightarrow Convexity of H

Prop 5: $\forall \Delta t > 0$, implicit scheme (5) satisfies $h_j^{n+1} \geq 0$ and $E_j^{n+1} \leq E_j^n - \sigma [\tilde{G}_{j+\frac{1}{2}}^{n+1} - \tilde{G}_{j-\frac{1}{2}}^{n+1}] + D_j$, $D_j \leq 0$

Proof: $I + \sigma L$ is monotone: $(I + \sigma L)f \geq 0 \Rightarrow f \geq 0$.

Define D_j such that $H(f_j^{n+1}) = H(M_j^n) - \underbrace{\sigma \partial_p H(f_j^{n+1})}_{\left[f_j^{n+1} - M_j^n \right]} + D_j \rightarrow$ we find $D_j \leq 0$.

Bounded from above by conservative term.

Accounting for manflat bottom: hydrostatic reconstruction

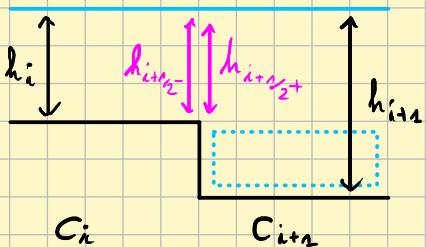
Discretize source term $S(U, z)$ in (SV):

$$\begin{cases} \partial_t h + \partial_x h u = 0 \\ \partial_t h u + \partial_x (h u^2 + \frac{\rho}{2} h^2) = -g h \partial_x z \end{cases}$$

Constraint: preserve lakes at rest $h+z=K$, $u=0$.

Problem: 1/ Upwinding introduces diffusion on $h \Rightarrow F(U_i, U_{i+1}) - F(U_{i-1}, U_i) \neq 0$

2/ Need to balance pressure variation with source term: $\partial_x (\frac{\rho}{2} h^2) = -g h \partial_x z$



Hydrostatic reconstruction: $Z_{i+1/2} = \max(z_i, z_{i+1})$

$$h_{i+1/2+} = (h_{i+1} + z_{i+1} - z_{i+1/2})_+, \quad h_{i+1/2-} = (h_i + z_i - z_{i+1/2})_+$$

$$\tilde{U}_{i+1/2+} = \begin{pmatrix} h_{i+1/2+} \\ h_{i+1/2+} \cdot u_{i+1} \end{pmatrix}, \quad \tilde{U}_{i+1/2-} = \begin{pmatrix} h_{i+1/2-} \\ h_{i+1/2-} \cdot u_i \end{pmatrix}$$

Modify numerical flux F as

$$\begin{cases} F_{i+1/2-} = F(\tilde{U}_{i+1/2-}, \tilde{U}_{i+1/2+}) + \frac{\rho}{2} (h_i^2 - h_{i+1/2-}^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ F_{i-1/2+} = F(\tilde{U}_{i-1/2+}, \tilde{U}_{i-1/2-}) + \frac{\rho}{2} (h_i^2 - h_{i-1/2+}^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Over lakes at rest, $U_{i+1/2-} = U_{i+1/2+} \Rightarrow$ No more diffusion on h .

Kinetic interpretation of the hydrostatic reconstruction

We remark that :

$$\int_{\mathbb{R}} \left(\frac{1}{\tau} \right) (\tau - \mu_i) (M_i - M_{i+\tau_2^-}) d\tau = \begin{pmatrix} 0 \\ \delta_2 (h_i^2 - h_{i+\tau_2^-}^2) \end{pmatrix}, \quad M_{i+\tau_2^-} = M(\tilde{M}_{i+\tau_2^-}, \tau)$$

$$\int_{\mathbb{R}} \left(\frac{1}{\tau} \right) (\tau - \mu_i) (M_i - M_{i-\tau_2^+}) d\tau = \begin{pmatrix} 0 \\ \delta_2 (h_i^2 - h_{i-\tau_2^+}^2) \end{pmatrix}, \quad M_{i-\tau_2^+} = M(\tilde{M}_{i-\tau_2^+}, \tau)$$

Define the interfacial Maxwellian $M_{i+\tau_2} = M(\tilde{M}_{i+\tau_2}, \tau) \mathbf{1}_{\tau < 0} + M(\tilde{M}_{i+\tau_2^-}, \tau) \mathbf{1}_{\tau > 0}$.

The previous explicit kinetic scheme (4) with hydrostatic reconstruction writes :

$$\frac{f_i^{m+1} - f_i^m}{\Delta t} + \frac{\tau}{\Delta x} (M_{i+\tau_2}^m - M_{i-\tau_2}^m) + \sigma (\tau - \mu_i^m) [n_{i-\tau_2^+}^m - M_{i+\tau_2^-}] = 0 \quad (6)$$

Prop 6: Scheme (6) admits a discrete entropy inequality with an error term :

$$\tilde{E}_i^{m+1} \leq E_i^m - \sigma [\tilde{G}_{i+\tau_2} - \tilde{G}_{i-\tau_2}] + D_i, \quad D_i \geq 0 \quad \rightarrow \text{Total energy can increase.}$$

See E. Audusse, M.-O. Bristeau, B. Perthame, Kinetic entropy inequality and hydrostatic reconstruction scheme for the Saint-Venant system, 2000

Implicit kinetic scheme with hydrostatic reconstruction

Replace explicit kinetic scheme with hydrostatic reconstruction (6) by:

$$\frac{f_i^{m+1} - f_i^m}{\Delta t} + \frac{\gamma}{\Delta x} (M_{i+1/2}^{m+1} - M_{i-1/2}^{m+1}) + \sigma (\gamma - M_i^{m+1}) [n_{i-1/2+}^{m+1} - M_{i+1/2-}^{m+1}] = 0 \quad (7)$$

↳ Nonlinear system

We propose an iterative approximation for (7):

$$(1+\alpha) f_i^{m+1,k+1} = f_i^m + \alpha f_i^{m+1,k} - \sigma \gamma (M_{i+1/2}^{m+1,k} - M_{i-1/2}^{m+1,k}) + \sigma (\gamma - M_i^{m+1,k}) [M_{i+1/2-}^{m+1,k} - M_{i-1/2+}^{m+1,k}] \quad (8)$$

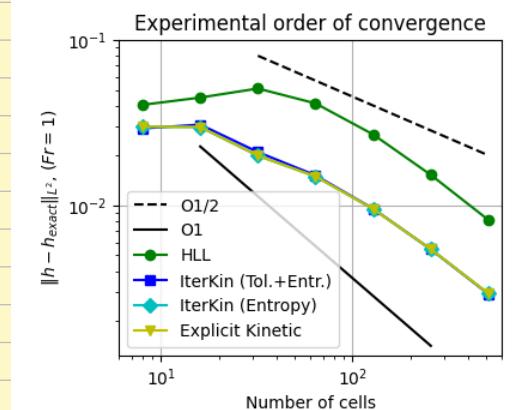
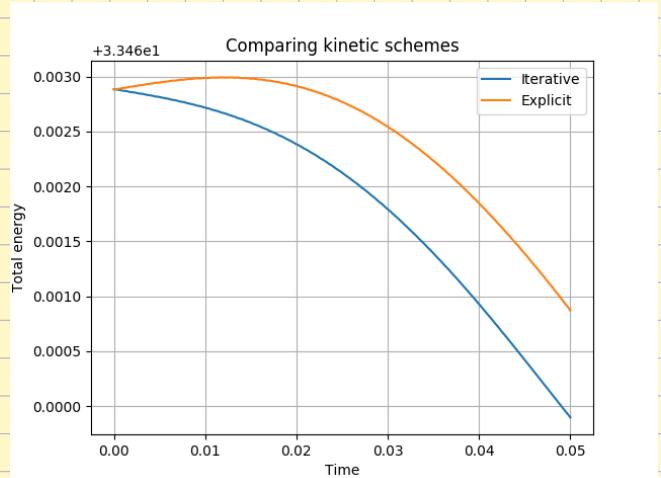
$\alpha > 0$ is a relaxation parameter.

Prop 7: Let δ, K_1, K_2 be non-negative parameters. Assume that each iteration (8) keeps

$$M_i^{m+1,k} = \int_R \left(\frac{1}{\gamma} f_i^{m+1,k} \right) d\zeta \text{ in the net } \Theta = \{(h, h_m) \mid 0 \leq h \leq K_1, 1 \leq m \leq K_2\} \text{ for all } k \in \mathbb{N}.$$

Then under CFL condition $\sigma \leq C(K_2, K_1, \gamma/\delta)$, sequence $(f_i^{m+1,k})_k$ converges to f_i^{m+1} satisfying (7).

Comparing explicit and iterative kinetic schemes with hydrostatic reconstruction



Domain: $[0,1]$ discretized with 100 cells

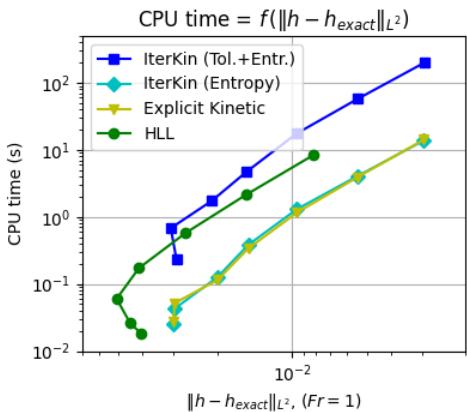
Boundary conditions: periodic

Initial data: $h+z = K$, $\mu = K' > 0$



We plot the total energy

$$E_{\text{tot}}(t) = \int_{[0,1]} E(t, x) dx$$



Perspectives:

- Proof of energy dissipation in the iterative case
- Weaker assumption to get the convergence of iterative scheme
- 2D over cartesian mesh ?

References:

Hydrostatic reconstruction:

- * E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, B. Perthame, A fast and stable well-balanced scheme with hydrostatic reconstruction for Shallow Water flows, 2004

1D Explicit kinetic schemes

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2D Explicit kinetic schemes

- * E. Audusse, M.-O. Bristeau, B. Perthame, Kinetic scheme for Saint-Venant equations with source terms on unstructured grids, 2000