Origin of Fourier Series II

Zigzags from Bernoulli to Combinatorics

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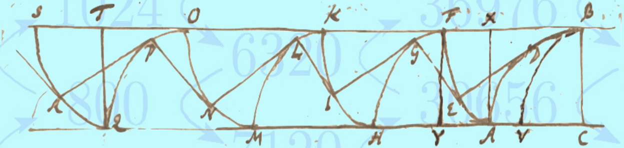
CANUM, Évian, June 2022



$E_{1,1} = 1$

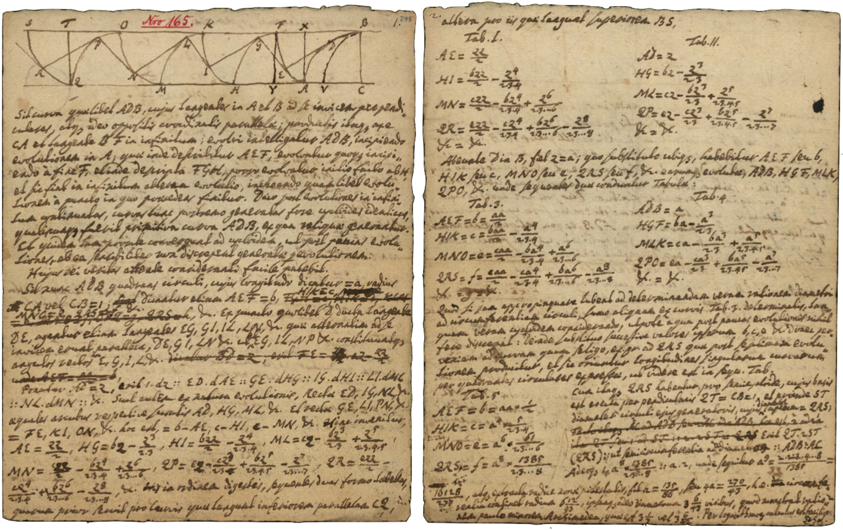
$E_{2,1} = 0$, $E_{2,2} = 1$

$E_{3,3} = 1$, $E_{3,2} = 1$, $E_{3,1} = 0$

$E_{4,1} = 0$, $E_{4,2} = 1$, $E_{4,3} = 2$, $E_{4,4} = 2$

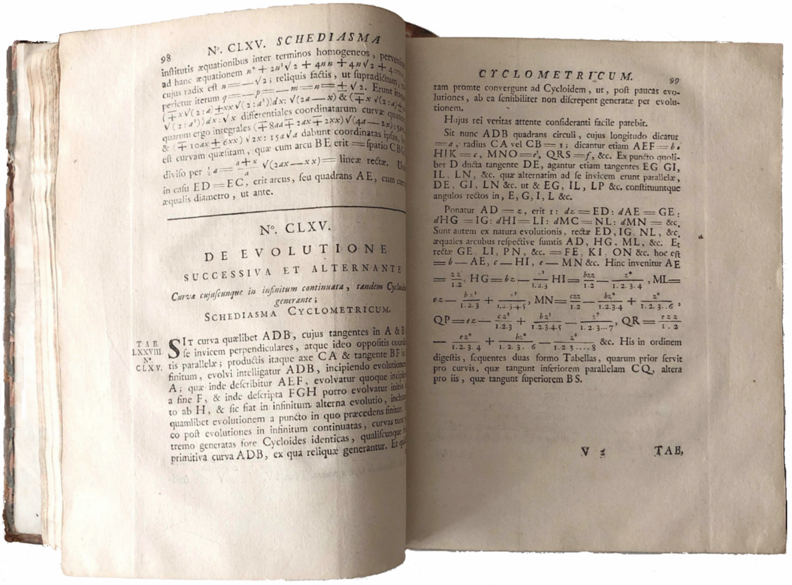
$E_{5,5} = 5$, $E_{5,4} = 5$, $E_{5,3} = 4$, $E_{5,2} = 2$, $E_{5,1} = 0$

$E_{6,1} = 0$, $E_{6,2} = 5$, $E_{6,3} = 10$, $E_{6,4} = 14$, $E_{6,5} = 16$, $E_{6,6} = 16$



UB Basel, L Ia 12, 4.1, fol. 245–247.

Johann's Manuscript Edited by G. Cramer



98 N°. CLXV. SCHEDIASMA

institutis æquationibus inter terminos homogeneos, perven... ad hanc æquationem $n^4 + 2n^3\sqrt{2} + 4nn + 4n\sqrt{2} + 4 = \ldots$ cujus radix est $n = -\sqrt{2}$; reliquis factis, ut supradictum, ... pericitur iterum $q = -p = -m = n = \pm\sqrt{2}$. Erunt itaq... $(\mp x\sqrt{(2:a)} \pm xx\sqrt{(2:a^3)})dx : \sqrt{(2a - x)}$ & $(\mp x\sqrt{(2:a)} + \ldots$ $\sqrt{(2:a^3)})dx : \sqrt{x}$ differentiales coordinatarum curvæ quæ... quarum ergo integrales $(\mp 8aa \mp 2ax \mp 2xx)\sqrt{(4a - 2x)} : 5a\ldots$ & $(\mp 10ax \pm 6xx)\sqrt{2x} : 15a\sqrt{a}$ dabunt coordinatas ipsas, ... est curvam quæsitam, quæ cum arcu BE erit = spatio CBG... diviso per $\frac{1}{2}a = \frac{a+x}{a}\sqrt{(2ax - xx)} =$ lineæ rectæ. Un... in casu ED = EC, erit arcus, seu quadrans AE, cum ... æqualis diametro, ut ante.

N°. CLXV.

DE EVOLUTIONE

SUCCESSIVA ET ALTERNANTE

Curvæ cujuscunque in infinitum continuata, tandem Cycloidem generante;

SCHEDIASMA CYCLOMETRICUM.

TAB. LXXVIII. N°. CLXV.

Sit curva quælibet ADB, cujus tangentes in A & B se invicem perpendiculares, atque ideo oppositis coord... tis parallelæ; productis itaque axe CA & tangente BF in... finitum, evolvi intelligatur ADB, incipiendo evolutione... A; quæ inde describitur AEF, evolvatur quoque incip... a fine F, & inde descripta FGH porro evolvatur initio... to ab H, & sic fiat in infinitum alterna evolutio, inchoa... quamlibet evolutionem a puncto in quo præcedens finitur. ... co post evolutiones in infinitum continuatas, curvas tunc ... tremo generatas fore Cycloides identicas, qualiscunque f... primitiva curva ADB, ex qua reliquæ generantur. Et qu...

CYCLOMETRICUM. 99

tam promte convergunt ad Cycloidem, ut, post paucas evolutiones, ab ea sensibiliter non discrepent generatæ per evolutionem.

Hujus rei veritas attente consideranti facile patebit.

Sit nunc ADB quadrans circuli, cujus longitudo dicatur $= a$, radius CA vel CB $= 1$; dicantur etiam AEF $= b$, HIK $= c$, MNO $= e$, QRS $= f$, &c. Ex puncto quolibet D ducta tangente DE, agantur etiam tangentes EG GI, IL, LN, &c. quæ alternatim ad se invicem erunt parallelæ, DE, GI, LN &c. ut & EG, IL, LP &c. constituuntque angulos rectos in, E, G, I, L &c.

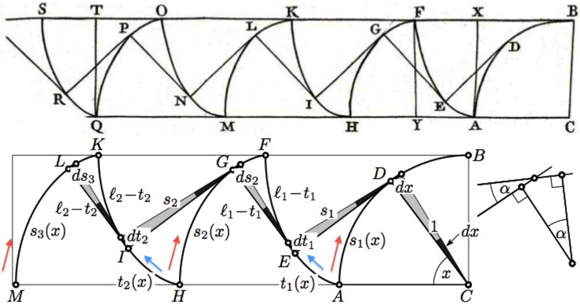
Ponatur AD $= z$, erit $1 : dz =$ ED $: d$AE $=$ GE $: d$HG $=$ IG $: d$HI $=$ LI $: d$MC $=$ NL $: d$MN $=$ &c. Sunt autem ex natura evolutionis, rectæ ED, IG, NL, &c. æquales arcubus respective sumtis AD, HG, ML, &c. Et rectæ GE, LI, PN, &c. $=$ FE, KI, ON &c. hoc est $= b -$ AE, $c -$ HI, $e -$ MN &c. Hinc invenitur AE $= \frac{zz}{1.2}$, HG $= bz - \frac{z^3}{1.2.3}$, HI $= \frac{bzz}{1.2} - \frac{z^4}{1.2.3.4}$, ML $= ez - \frac{bz^3}{1.2.3} + \frac{z^5}{1.2.3.4.5}$, MN $= \frac{czz}{1.2} - \frac{bz^4}{1.2.3.4} + \frac{z^6}{1.2.3..6}$, QP $= ez - \frac{cz^3}{1.2.3} + \frac{bz^5}{1.2.3.4.5} - \frac{z^7}{1.2.3...7}$, QR $= \frac{ezz}{1.2} - \frac{ez^4}{1.2.3.4} + \frac{bz^6}{1.2.3..6} - \frac{z^8}{1.2.3....8}$ &c. His in ordinem digestis, sequentes duas formo Tabellas, quarum prior servit pro curvis, quæ tangunt inferiorem parallelam CQ, altera pro iis, quæ tangunt superiorem BS.

V 2 TAB.

Opera Omnia, Tomus quartus (quo continentur ANEKΔOTA), 1742, pp. 98–108.

"Ce théorème remarquable est dû à Jean Bernoulli (...)." (Poisson, 1820)

Start with a quarter circle and consider the successive involutes.



Johann's claim: **the successive involutes converge to a cycloid.**

"Hujus rei veritas attente consideranti facile patebit."

Notations: $s_1(x) = x, \quad s_i(0) = t_i(0) = 0, \quad t_i(\pi/2) = \ell_i$

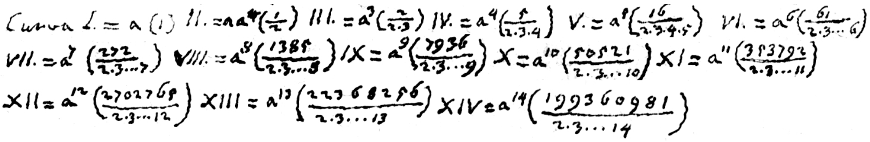
$$t_i(x) = \int_0^x s_i(\xi)\, d\xi, \qquad s_i(x) = \int_0^x (\ell_{i-1} - t_{i-1}(\xi))\, d\xi$$

Computation of Arc Lengths

$$
\begin{aligned}
t_1(x) &= \tfrac{x^2}{2!} \\
t_2(x) &= \ell_1 \tfrac{x^2}{2!} - \tfrac{x^4}{4!} \\
t_3(x) &= \ell_2 \tfrac{x^2}{2!} - \ell_1 \tfrac{x^4}{4!} + \tfrac{x^6}{6!}
\end{aligned}
\qquad
\begin{aligned}
s_1(x) &= x \\
s_2(x) &= \ell_1 x - \tfrac{x^3}{3!} \\
s_3(x) &= \ell_2 x - \ell_1 \tfrac{x^3}{3!} + \tfrac{x^5}{5!} \\
s_4(x) &= \ell_3 x - \ell_2 \tfrac{x^3}{3!} + \ell_1 \tfrac{x^5}{5!} - \tfrac{x^7}{7!}
\end{aligned}
$$

$$
a = \tfrac{\pi}{2},\ 0 = \ell_i - t_i(a) \Rightarrow
\begin{cases}
0 = \ell_1 - \dfrac{a^2}{2!} & \Rightarrow \ell_1 = \dfrac{a^2}{2!} \\
0 = \ell_2 - \ell_1 \dfrac{a^2}{2!} + \dfrac{a^4}{4!} & \Rightarrow \ell_2 = \dfrac{5a^4}{4!} \\
0 = \ell_3 - \ell_2 \dfrac{a^2}{2!} + \ell_1 \dfrac{a^4}{4!} - \dfrac{a^6}{6!} & \Rightarrow \ell_3 = \dfrac{61a^6}{6!} \quad \text{etc.}
\end{cases}
$$

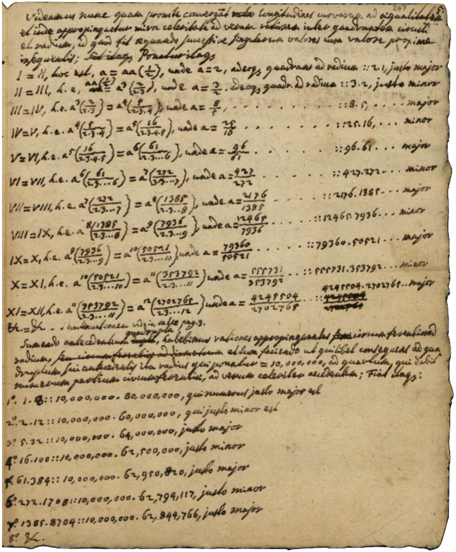
Johann finds the sequence:


Curva I. = a (1) II. = aa $(\frac{1}{2})$ III. = $a^3(\frac{2}{2.3})$ IV. = $a^4(\frac{5}{2.3.4})$ V. = $a^5(\frac{16}{2.3.4.5})$ VI. = $a^6(\frac{61}{2.3\ldots6})$
VII. = $a^7(\frac{272}{2.3\ldots7})$ VIII. = $a^8(\frac{1385}{2.3\ldots8})$ IX = $a^9(\frac{7936}{2.3\ldots9})$ X = $a^{10}(\frac{50521}{2.3\ldots10})$ XI = $a^{11}(\frac{353792}{2.3\ldots11})$
XII = $a^{12}(\frac{2702765}{2.3\ldots12})$ XIII = $a^{13}(\frac{22368256}{2.3\ldots13})$ XIV = $a^{14}(\frac{199360981}{2.3\ldots14})$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E_i	1	1	2	5	16	61	272	1385	7936	50521	353792	2702765	22368256	199360981

Approximation of π

$$\text{VII} = \text{VIII} \Rightarrow \frac{272a^7}{7!} = \frac{1385a^8}{8!} \Rightarrow a = \frac{272 \cdot 8}{1385} \Rightarrow 2a = \pi = \frac{4352}{1385}$$

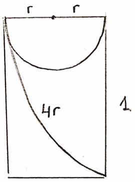

Videamus nunc quam promte convergat longitudines curvarum ad aequalitatem et inde appropinquetur mira celeritate ad veram rationem inter quadrantem circuli et radium, id quod fit aequando successive singularum valores cum valore proxime insequenti; Ponatur itaque
I = II, hoc est, a = aa(1/2), unde a = 2, adeoque quadrans ad radium :: 2.1, justo major
II = III, h.e. aa(1/2) = a³(2/2.3), unde a = 3/2. adeoque quadr. ad radium :: 3.2, justo minor
III = IV, h.e. a³(2/2.3) = a⁴(5/2.3.4), unde a = 8/5, . . . :: 8.5, . . . major
IV = V, h.e. a⁴(5/2.3.4) = a⁵(16/2.3.4.5), unde a = 25/16 . . . :: 25.16, . . . minor
V = VI, h.e. a⁵(16/2.3.4.5) = a⁶(61/2.3…6), unde a = 96/61 . . . :: 96.61 . . . major
VI = VII, h.e. a⁶(61/2.3…6) = a⁷(272/2.3…7), unde a = 427/272 . . . :: 427.272 . . . minor
VII = VIII, h.e. a⁷(272/2.3…7) = a⁸(1385/2.3…8), unde a = 2176/1385 . . . :: 2176.1385 . . . major
VIII = IX, h.e. a⁸(1385/2.3…8) = a⁹(7936/2.3…9), unde a = 12465/7936 . . . :: 12465.7936 . . . minor
IX = X, h.e. a⁹(7936/2.3…9) = a¹⁰(50521/2.3…10), unde a = 79360/50521 . . . :: 79360.50521 . . . major
X = XI, h.e. a¹⁰(50521/2.3…10) = a¹¹(353792/2.3…11), unde a = 555731/353792 . . . :: 555731.353792 . . . minor
XI = XII, h.e. a¹¹(353792/2.3…11) = a¹²(2702765/2.3…12), unde a = 4245504/2702765 . . . :: 4245504.2702765 . . major
&c. = &c. . . continuationem vide in calce pag. 3.
[illegible]
1°. 1.8 :: 10,000,000 . 80,000,000, qui numerus justo major est
2°. 2.12 :: 10,000,000 . 60,000,000, qui justo minor est
3° 5.32 :: 10,000,000 . 64,000,000, justo major
4° 16.100 :: 10,000,000 . 62,500,000, justo minor
5° 61.384 :: 10,000,000 . 62,950,820, justo major
6° 272.1708 :: 10,000,000 . 62,794,117, justo minor
7° 1385.8704 :: 10,000,000 . 62,844,766, justo major
8° &c.

	$2a$	error
I = II	4	−0.85840734
II = III	3	0.14159265
III = IV	$\frac{16}{5}$	−0.05840734
IV = V	$\frac{50}{16}$	0.01659265
V = VI	$\frac{192}{61}$	−0.00594833
VI = VII	$\frac{854}{272}$	0.00188677
VII = VIII	$\frac{4352}{1385}$	−0.00064561
VIII = IX	$\frac{24930}{7936}$	0.00021160
IX = X	$\frac{158720}{50521}$	−0.00007120
X = XI	$\frac{1111462}{353792}$	0.00002359
XI = XII	$\frac{8491008}{2702765}$	−0.00000789
XII = XIII	$\frac{70271890}{22368256}$	0.00000262
XIII = XIV	$\frac{626311168}{199360981}$	−0.00000087

Other Approximation of π

Suppose VIII = semi-cycloid

$BC = 1 \Rightarrow r = \frac{1}{\pi}$



$$a^8 \frac{1385}{8!} = 4 \cdot \frac{1}{\pi} = \frac{2}{a}$$

$$\Rightarrow a^9 = \frac{2 \cdot 8!}{1385}$$

Cum itaque QRS habeatur pro ſemi-Cycloide, cujus baſis eſt erecta perpendicularis QT = CB = 1, & proinde ST diameter circuli ejus generatoris, cujus diametri duplum = QRS; erit QT : 2ST [QRS] = ut ſemicircumferentia ad 2 diametros = ADB : 2AC. Adeoque $1 : a^8 \times \frac{1385}{1.2.3 \ldots 8} = a : 2$; unde ſequitur $a^9 = \frac{2 \times 1.2.3.4 \ldots 8}{1385} = \frac{16128}{277}$; atque extracta radice nonæ poteſtatis, fit $a = \frac{175}{56}$, ſeu $4a = \frac{175}{14}$, hoc est, circumferentia continet radium $6\frac{2}{45}$, ipſumque adeo diametrum $3\frac{6}{55}$ vicibus, quod monſtrat rationem paulo minorem *Archimedea*, quæ eſt $3\frac{1}{7}$, vel $3\frac{6}{45}$. Per logarithmos calculus eſt facilior.

“Curva”	$2a$	error =
I	$2\sqrt{2}$	0.3131655288
II	$2\sqrt[3]{4}$	−0.0332094503
III	$2\sqrt[4]{6}$	0.0114234934
IV	$2\sqrt[5]{\frac{2\cdot4!}{5}}$	−0.0024196887
V	$2\sqrt[6]{\frac{2\cdot5!}{16}}$	0.0007570486
VI	$2\sqrt[7]{\frac{2\cdot6!}{61}}$	−0.0001999873
VII	$2\sqrt[8]{\frac{2\cdot7!}{272}}$	0.0000609333
VIII	$2\sqrt[9]{\frac{2\cdot8!}{1385}}$	−0.0000175640
IX	$2\sqrt[10]{\frac{2\cdot9!}{7936}}$	0.0000053536
X	$2\sqrt[11]{\frac{2\cdot10!}{50521}}$	−0.0000016065
XI	$2\sqrt[12]{\frac{2\cdot11!}{353792}}$	0.0000004937
XII	$2\sqrt[13]{\frac{2\cdot12!}{2702765}}$	−0.0000001513
XIII	$2\sqrt[14]{\frac{2\cdot13!}{22368256}}$	0.0000000469
XIV	$2\sqrt[15]{\frac{2\cdot14!}{199360981}}$	−0.0000000145

Euler's Proof (*Demonstratio Theorematis Bernoulliani*, E300, 1766)

"Insigne igitur hoc Theorema soli quasi observationi innixum (...)."

First part: "ausserordentliche Kühnheit" (Speiser, 1954)

- $y_1(x) = AD$, $y_2(x) = AE$, ..., $y_i(\frac{\pi}{2}) = L_i$ and $z = y_\infty$ the "curva infinitissima".
- "nihil enim impedit" $z = A\,\sin.\alpha v + B\,\sin.\beta v + C\,\sin.\gamma v + D\,\sin.\delta v + E\,\sin\,\varepsilon v + \text{etc.}$
- by analogy

$$x=0: \quad y_7=0 \quad y_7'=L_6 \quad y_7''=0 \quad y_7'''=-L_4 \quad y_7^{(4)}=0 \quad y_7^{(5)}=L_2 \quad y_7^{(6)}=0$$

$$x=\tfrac{\pi}{2}: \quad y_7=L_7 \quad y_7'=0 \quad y_7''=-L_5 \quad y_7'''=0 \quad y_7^{(4)}=L_3 \quad y_7^{(5)}=0 \quad y_7^{(6)}=-L_1$$

$$\begin{matrix}\text{si } v=0\\ \text{si } v=\rho\end{matrix}\Bigg\} ;\; z=\begin{cases}0\\+f\end{cases};\; \frac{dz}{dv}=\begin{cases}+g\\0\end{cases};\; \frac{ddz}{dv^2}=\begin{cases}0\\-f'\end{cases};\; \frac{d^3z}{dv^3}=\begin{cases}-g'\\0\end{cases};\; \frac{d^4z}{dv^4}=\begin{cases}0\\+f''\end{cases}$$

$$\begin{matrix}\text{si } v=0\\ \text{si } v=\rho\end{matrix}\Bigg\} ;\; \frac{d^5z}{dv^5}=\begin{cases}+g''\\0\end{cases};\; \frac{d^6z}{dv^6}=\begin{cases}0\\-f'''\end{cases};\; \frac{d^7z}{dv^7}=\begin{cases}-g'''\\0\end{cases} \text{ etc.}$$

$$z = \frac{gv}{1} - \frac{g'v^3}{1.2.3} + \frac{g''v^5}{1.2.3.4.5} - \frac{g'''v^7}{1.2\ldots 7} + \text{etc.}$$

"*perspicitur*" and $y_\infty^{(2k+1)}(\frac{\pi}{2}) = 0 \Rightarrow \alpha = 1, \beta = 3, \gamma = 5, \delta = 7, \varepsilon = 9, \ldots$

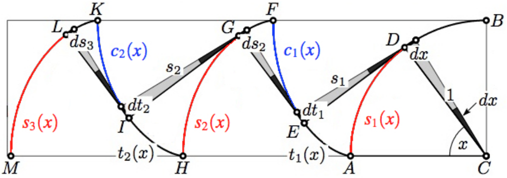
$$\begin{aligned} g &= A + 3\,B + 5\,C + 7\,D + 9\,E + \text{etc.} & g''' &= A + 3^7B + 5^7C + 7^7D + 9^7E + \text{etc.}\\ g' &= A + 3^3B + 5^3C + 7^3D + 9^3E + \text{etc.} & &\text{etc.}\\ g'' &= A + 3^5B + 5^5C + 7^5D + 9^5E + \text{etc.} & g^{(\infty)} &= A + 3^\infty B + 5^\infty C + 7^\infty D + 9^\infty E + \text{etc.}\end{aligned}$$

$\Rightarrow y_\infty(x) = A\sin(x)$ *"(...) sicque habemus perfectam demonstrationem theorematis* BERNOULLIANI."

Second part: $\Phi > \frac{\pi}{2}$ (resp. $<$) epicycloid (resp. hypo-), $\Phi = \frac{\pi}{2}$ cycloid GeoGebra !

Euler's Proof Rearranged ($\simeq$ Puiseux, 1844)

"(...) il m'a semblé qu'on la simplifiait beaucoup en la présentant de la manière suivante (...)" (Puiseux)



$$c_i(x) := \ell_i - t_i(x) = \int_x^{\pi/2} s_i(\xi)\,d\xi$$

$$s_i(x) = \int_0^x c_{i-1}(\xi)\,d\xi$$

$$c_0(x) := 1 = \tfrac{4}{\pi}\left(+\cos(x) - \tfrac{1}{3}\cos(3x) + \tfrac{1}{5}\cos(5x) - \dots\right)$$

$$\int_x^{\pi/2} \sin(k\xi)\,d\xi = \tfrac{1}{k}\cos(kx) \quad k = 1, 3, 5, \dots$$

$$\int_0^x \cos(k\xi)\,d\xi = \tfrac{1}{k}\sin(kx)$$

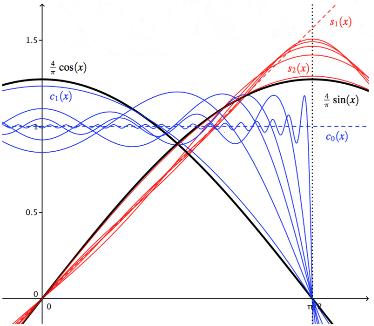
$$s_1(x) = x = \tfrac{4}{\pi}\left(+\sin(x) - \tfrac{1}{3^2}\sin(3x) + \tfrac{1}{5^2}\sin(5x) - \dots\right)$$

$$c_1(x) = \tfrac{4}{\pi}\left(+\cos(x) - \tfrac{1}{3^3}\cos(3x) + \tfrac{1}{5^3}\cos(5x) - \dots\right)$$

$$s_2(x) = \tfrac{4}{\pi}\left(+\sin(x) - \tfrac{1}{3^4}\sin(3x) + \tfrac{1}{5^4}\sin(5x) - \dots\right)$$

$\dots$

$$s_i(x) \to \tfrac{4}{\pi}\sin(x) \qquad c_i(x) \to \tfrac{4}{\pi}\cos(x)$$



Euler's Formulas (E130, 1740)

$x = \frac{\pi}{2}$ for $s_1, s_2, s_3, \ldots$, $x = 0$ for $c_0, c_1, c_2, \ldots$ and $L_i = E_i \frac{(\frac{\pi}{2})^i}{i!}$

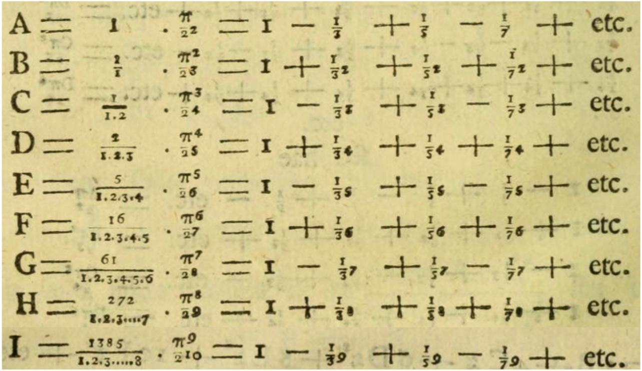
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{1\pi}{0! \cdot 2^2} \qquad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{1\pi^2}{1! \cdot 2^3}$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots = \frac{1\pi^3}{2! \cdot 2^4} \qquad 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \frac{2\pi^4}{3! \cdot 2^5}$$

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \cdots = \frac{5\pi^5}{4! \cdot 2^6} \qquad 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \cdots = \frac{16\pi^6}{5! \cdot 2^7}$$

$$1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \cdots = \frac{61\pi^7}{6! \cdot 2^8} \qquad 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \cdots = \frac{272\pi^8}{7! \cdot 2^9}$$

$$1 - \frac{1}{3^{2\ell+1}} + \frac{1}{5^{2\ell+1}} - \frac{1}{7^{2\ell+1}} + \frac{1}{9^{2\ell+1}} - \cdots = \frac{\pi^{2\ell+1}}{(2\ell)! \cdot 2^{2\ell+2}} E_{2\ell} \qquad 1 + \frac{1}{3^{2\ell}} + \frac{1}{5^{2\ell}} + \frac{1}{7^{2\ell}} + \frac{1}{9^{2\ell}} + \cdots = \frac{\pi^{2\ell}}{(2\ell-1)! \cdot 2^{2\ell+1}} E_{2\ell-1}$$



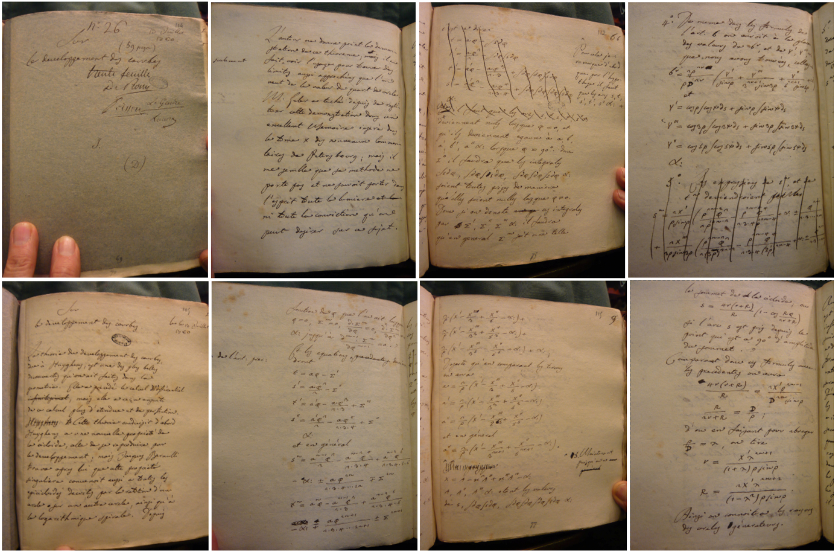
$$A = 1 \cdot \frac{\pi}{2^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.}$$
$$B = \frac{2}{1} \cdot \frac{\pi^2}{2^3} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}$$
$$C = \frac{1}{1.2} \cdot \frac{\pi^3}{2^4} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \text{etc.}$$
$$D = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}$$
$$E = \frac{5}{1.2.3.4} \cdot \frac{\pi^5}{2^6} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \text{etc.}$$
$$F = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.}$$
$$G = \frac{61}{1.2.3.4.5.6} \cdot \frac{\pi^7}{2^8} = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \text{etc.}$$
$$H = \frac{272}{1.2.3 \ldots 7} \cdot \frac{\pi^8}{2^9} = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.}$$
$$I = \frac{1385}{1.2.3 \ldots 8} \cdot \frac{\pi^9}{2^{10}} = 1 - \frac{1}{3^9} + \frac{1}{5^9} - \frac{1}{7^9} + \text{etc.}$$

Institut de France Library.

Sur le developpement des courbes (Lagrange, 1780)

"(...) il me semble que sa methode ne porte pas et ne sauroit porter dans l'esprit toute la lumiere ni toute la conviction qu'on peut desirer sur ce sujet."

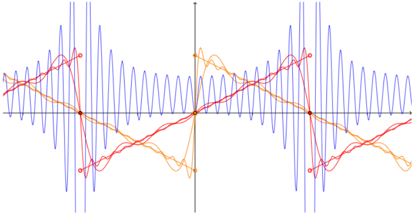
Long correct proof. No trigonometric series in Lagrange's manuscript (59 pages)...



Controversery between Lagrange and Fourier ($\simeq$ 1807)

Lagrange: Calculus applies only to functions satisfying the “loi de continuité”.

He asserts that (b) is false if $x > \frac{\pi}{2}$:



(a) $\sin(y) + \frac{1}{2}\sin(2y) + \frac{1}{3}\sin(3y) + \frac{1}{4}\sin(4y) + \ldots = -\frac{y}{2} + \frac{\pi}{2}$

$\overset{y=\pi-x}{\Leftrightarrow}$

(b) $\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) \ldots = \frac{x}{2}$

(c) $\cos(x) - \cos(2x) + \cos(3x) - \cos(4x) + \ldots = \frac{1}{2}$

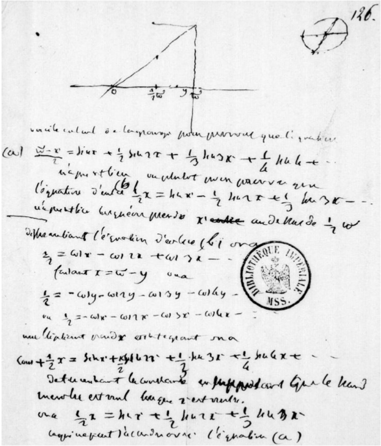
$\overset{y=\pi-x}{\Leftrightarrow}$

(d) $-\cos(y) - \cos(2y) - \cos(3y) - \cos(4y) - \ldots = \frac{1}{2}$

(e) $\sin(y) + \frac{1}{2}\sin(2y) + \frac{1}{3}\sin(3y) + \frac{1}{4}\sin(4y) + \ldots = \mathrm{Cons} - \frac{1}{2}y$

(f) $\sin(y) + \frac{1}{2}\sin(2y) + \frac{1}{3}\sin(3y) + \frac{1}{4}\sin(4y) + \ldots = -\frac{1}{2}y$

(f) is a contradiction with (a) !



BNF, Ms. Fr. 22529

Fourier: (b) converges everywhere !

“Je vous prie instamment de jeter les yeux sur cette note (...) si de pareilles démonstrations peuvent être douteuses, il faut renoncer à écrire quelque chose d’exact en mathématiques.”

Alternating Permutations (Désiré André, 1879,1881)

Sur les permutations alternées;

PAR M. DÉSIRÉ ANDRÉ.

Le présent travail a pour point de départ la notion, probablement toute nouvelle, des permutations alternées de n lettres distinctes, et pour objet l'étude du nombre $2A_n$ de ces permutations.

Nous définissons les permutations alternées de n éléments distincts; nous donnons le moyen de calculer, de proche en proche, la moitié A_n de leur nombre; nous déterminons la fonction génératrice de la fraction $\frac{A_n}{n!}$; nous en déduisons les développements de $\operatorname{tang} x$ et de $\operatorname{séc} x$ suivant les puissances croissantes de x; nous appliquons les résultats obtenus à différents développements ou séries; nous en tirons plusieurs conséquences touchant les développements des fonctions elliptiques; enfin nous donnons de nouvelles et plus simples formules pour calculer les nombres A_n.

Il se trouve que ces nombres A_n ne sont autres que les coefficients de $\frac{x^n}{n!}$ dans le développement soit de $\operatorname{tang} x$, soit de $\operatorname{séc} x$. Ces coefficients avaient été considérés déjà. Grâce aux permutations alternées, nous en pouvons donner une définition combinatoire très simple, très nette, et indépendante de tout développement.

I. — *Définition des permutations alternées.*

1. Considérons n éléments distincts $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ et formons-en toutes les permutations. Si, dans l'une quelconque d'entre elles, nous retranchons chaque indice du suivant, nous obtenons une suite de $n-1$ différences, dont aucune n'est égale à zéro, et qui, dans toutes les permutations, sauf deux, sont les unes positives, les autres négatives. Lorsque, tout le long de cette suite, ces différences sont *alternativement* positives et négatives, la permutation correspondante est *alternée*. Lorsque, au contraire, cette continuelle alternance des signes ne se présente pas, la permutation n'est pas alternée.

Par exemple, dans le cas particulier où $n = 4$, les permutations

1324 3241

$$\alpha_1\alpha_3\alpha_2\alpha_4, \quad \alpha_3\alpha_2\alpha_4\alpha_1$$

sont alternées, et les permutations

2341 3214

$$\alpha_2\alpha_3\alpha_4\alpha_1, \quad \alpha_3\alpha_2\alpha_1\alpha_4$$

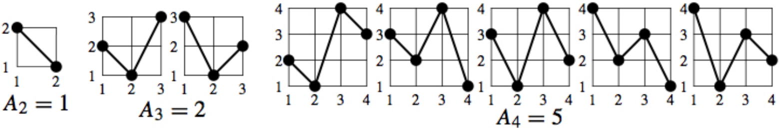
ne le sont pas.

2. Le nombre des permutations alternées de n éléments distincts dépend évidemment, et uniquement, de n. Par sa nature même, il est forcément positif et entier; et l'on peut démontrer, d'une manière très simple, qu'il est toujours pair. Il suffit effectivement de faire observer que les permutations alternées de n éléments distincts se correspondent deux à deux. L'un des moyens les plus commodes d'établir cette correspondance consiste à associer les permutations qui présentent les mêmes indices dans un ordre exactement inverse.

En associant de cette façon les permutations alternées de quatre éléments, lesquelles sont au nombre de dix, on obtient les cinq couples suivants :

1324	$\alpha_1\alpha_3\alpha_2\alpha_4$	et	$\alpha_4\alpha_2\alpha_3\alpha_1$,	4231
1423	$\alpha_1\alpha_4\alpha_2\alpha_3$	et	$\alpha_3\alpha_2\alpha_4\alpha_1$,	3241
up-down 2314	$\alpha_2\alpha_3\alpha_1\alpha_4$	et	$\alpha_4\alpha_1\alpha_3\alpha_2$,	4132 down-up
2413	$\alpha_2\alpha_4\alpha_1\alpha_3$	et	$\alpha_3\alpha_1\alpha_4\alpha_2$,	3142
3412	$\alpha_3\alpha_4\alpha_1\alpha_2$	et	$\alpha_2\alpha_1\alpha_4\alpha_3$.	2143

Alternating Permutations (Désiré André, 1879,1881)



$$E_0 = 1,\ E_1 = 1,\ E_2 = 1,\ E_3 = 2,\ E_4 = 5,\ E_5 = 16,\ E_6 = 61,\ E_7 = 272,\ E_8 = 1385,\ \ldots$$

Same numbers as Johann's numbers for arc lengths !

"Je ferai (...) avec un grand plaisir la connaissance de M. D. André, dont je me rappelle bien les Notes dans les *Comptes rendus*. L'ingénieuse définition de certains nombres entiers qui donnent aussitôt les coefficients dans les développements de tang x, séc x, comme nombres de permutations, jouissant de certaines propriétés, s'est gravée dans mon esprit." (Stieltjes to Hermite, Paris, February 1886)

$$t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8}\ \&c.$$

$$s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7}\ \&c.$$

Gregory, 1671

Theorem.

$$E(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec(x) + \tan(x)$$

Corollary. ("définition très simple, très nette")

$$\sum_{k=0}^{\infty} E_{2k} \frac{x^{2k}}{(2k)!} = \sec(x)$$

$$\sum_{k=0}^{\infty} E_{2k+1} \frac{x^{2k+1}}{(2k+1)!} = \tan(x)$$

André's Recurrence

"C'est un très joli exercice." (Arnol'd, 2000)

"[Ceci] s'établit par les raisonnements combinatoires les plus simples." (André, 1879)

Proof.

We count alternating permutations of $[n+1]$ according to the position of $n+1$:

$$\underbrace{\square \ldots \square > \square}_{k} < \overset{\sigma_{k+1}=n+1}{\underset{k+1}{\square}} > \underbrace{\square < \square \ldots \square}_{n-k}$$

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \quad (n \neq 0)$$

$$\begin{aligned} \Rightarrow E(x)^2 &= E_0^2 + \left(\tfrac{E_0}{0!}\tfrac{E_1}{1!} + \tfrac{E_1}{1!}\tfrac{E_0}{0!}\right)x + \left(\tfrac{E_0}{0!}\tfrac{E_2}{2!} + \tfrac{E_1}{1!}\tfrac{E_1}{1!} + \tfrac{E_2}{2!}\tfrac{E_0}{0!}\right)x^2 + \ldots \\ &= E_1 + 2\tfrac{E_2}{1!}x + 2\tfrac{E_3}{2!}x^2 + \ldots \\ &= 2E'(X) - 1 \end{aligned}$$

$$\overset{E(0)=1}{\Rightarrow} E(x) = \sec(x) + \tan(x)$$

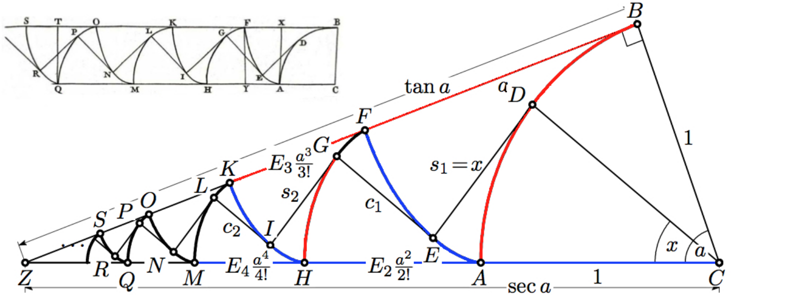
■

Geometric Interpretation of E (H. & Wanner, 2019)

Johann's arc lengths:

$$CA = \boxed{\frac{1a^0}{0!}},\ BA = \boxed{\frac{1a^1}{1!}},\ AF = \boxed{\frac{1a^2}{2!}},\ FH = \boxed{\frac{2a^3}{3!}},\ HK = \boxed{\frac{5a^4}{4!}},\ KM = \boxed{\frac{16a^5}{5!}},\ \ldots$$

Nowhere assumed that $a = \frac{\pi}{2}$. Case $a < \frac{\pi}{2}$:



$$\sum_{k=0}^{\infty} E_{2k} \frac{x^{2k}}{(2k)!} = \sec(x) \qquad \sum_{k=0}^{\infty} E_{2k+1} \frac{x^{2k+1}}{(2k+1)!} = \tan(x)$$

$$\sec. x = \alpha + \frac{\beta}{1.2} xx + \frac{\gamma}{1.2.3.4} x^4 + \frac{\delta}{1.2\ldots6} x^6 + \frac{\varepsilon}{1.2\ldots8} x^8 + \&c. \qquad tg\,x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots\ldots6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots\ldots8} + \&c.$$

Euler, 1755

Seidel-**Entringer**-Arnol'd Triangle (Entringer, 1966)

André is not cited !

A COMBINATORIAL INTERPRETATION
OF THE EULER AND BERNOULLI NUMBERS

BY

R. C. ENTRINGER

The following problem is mentioned by A. J. KEMPNER in [1]. "In how many ways can the numbers $1, 2, \ldots, n-1$ be arranged so that the numbers are alternately larger and smaller (the first number at the left to have a smaller number at its right)?" There are, for example, five such permutations for $n = 5$ since we count just the following: $(2, 1, 4, 3)$, $(3, 1, 4, 2)$, $(3, 2, 4, 1)$, $(4, 1, 3, 2)$, $(4, 2, 3, 1)$. Since a solution of the problem was not essential to the further development of his topic, KEMPNER merely stated a recurrence relation without proof and with this relation calculated the number of such permutations for small n.

In this paper, for convenience, we will consider the equivalent complementary problem, i.e., the first number at the left shall have a *larger* number at its right. We obtain a recurrence relation involving the number of such permutations and show that the number of such permutations is closely related to an Euler or Bernoulli number.

iii) $A(n, k)$, $n \geq 2$, $1 \leq k \leq n$ is the total number of alternating permutations $(a_1, \ldots, a_n)$ with $k = a_1 < a_2$. For convenience $A(1, 1)$ is defined to be 1 and $A(n, k)$ to be 0 for $n \leq 0$, $k \leq 0$ or $k > n$.

Lemma. $A(n+1, k) = \sum_{i=1}^{n+1-k} A(n, i)$ *for* $n \geq 1, k \geq 1$.

Proof. For $k > n + 1$ the truth of the assertion follows immediately from the definition and the summation convention. For fixed k, $1 \leq k \leq n + 1$ the correspondence

$$(k, a_2, \ldots, a_{n+1}) \leftrightarrow (a'_2, \ldots, a'_{n+1})$$

where $a'_i = \begin{cases} n+1-a_i \text{ if } a_i < k \\ n+2-a_i \text{ if } a_i > k \end{cases}$, $i = 2, \ldots, n+1$ is one to one between the particular permutations $(k, a_2, \ldots, a_{n+1})$ of $(1, \ldots, n+1)$ and all the permutations of $(1, \ldots, n)$. Since $(k, a_2, \ldots, a_{n+1})$ is alternating with $k < a_2$ if and only if $(a'_2, \ldots, a'_{n+1})$ is alternating with $a'_2 < n + 2 - k$ and $a'_2 < a'_3$ we obtain the desired result.

The above recurrence relation is that of KEMPNER referred to earlier.

Lemma.

$$A(n, k+1) = A(n, k) - A(n-1, n-k) \quad \textit{for} \quad n \geq 2,\ k \geq 1.$$

Proof. From the preceding lemma we have

$$A(n, k+1) - A(n, k) =$$

$$= \sum_{i=1}^{n-k-1} A(n-1, i) - \sum_{i=1}^{n-k} A(n-1, i) = -A(n-1, n-k).$$

Entringer's idea is to consider

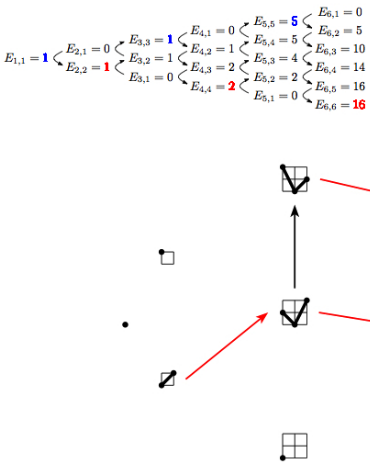
$$\mathcal{DU}_{n,k} := \{\sigma \in \mathcal{DU}_n \mid \sigma_1 = k\}, \quad \mathcal{UD}_{n,k} := \{\sigma \in \mathcal{UD}_n \mid \sigma_1 = k\}$$

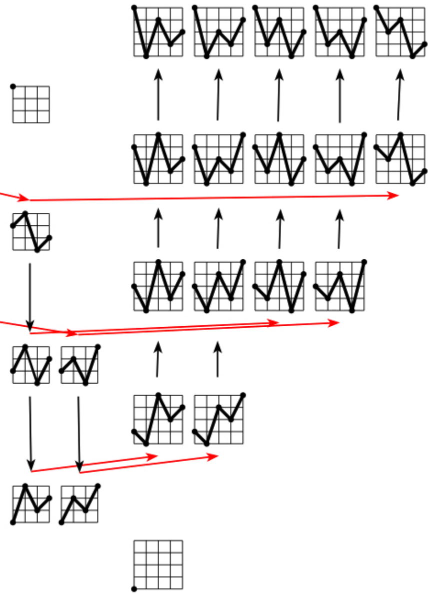
and $E_{n,k} = \#\mathcal{DU}_{n,k} = \#\mathcal{UD}_{n,n+1-k}$.

Lemma.

$$E_{n,k} = E_{n,k-1} + E_{n-1,n+1-k}$$

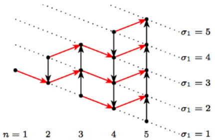
Seidel-Entringer-Arnol'd Triangle (H. & Wanner, 2019)

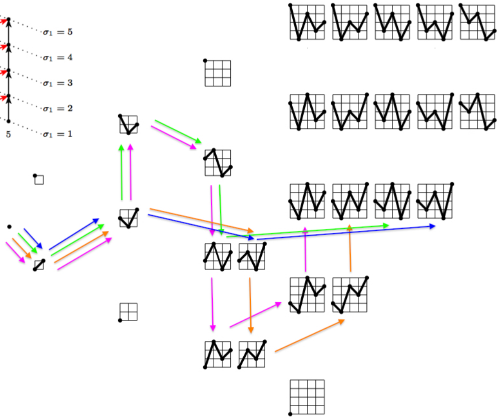


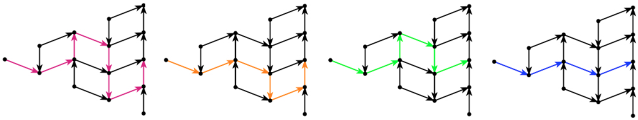


$\rightsquigarrow$ combinatorial proof of $L_i = E_i \frac{a^i}{i!}$.

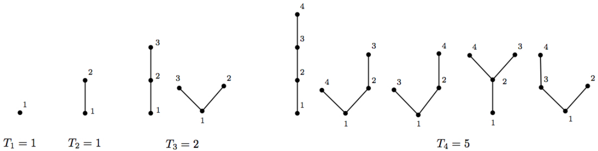
Paths in a Directed Graph (Arnol'd, 1991)







Proposition. Bijection between alt. perm. in a node and paths to this node.



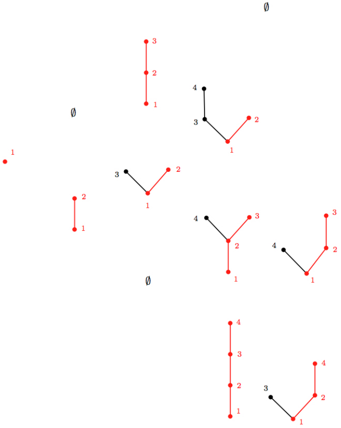
Proposition 1. (Foata & Schützenberger, 1971) $T_n = E_n$.

$T_{n,k} = \#\{T \in \mathcal{T}_n \mid \mathrm{emc}(T) = k\}$
where emc is the end of the minimal chain.

Proposition 2. (Poupard, 1982) $T_{n,k} = E_{n,k}$.

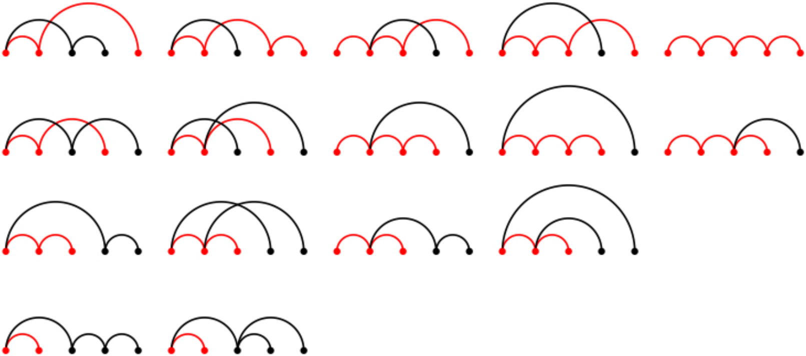
Proposition 3. (Gelineau & al., 2010)
There exists a bijection $\Psi : \mathcal{DU}_{n,k} \to \mathcal{T}_{n,k}$ such that

$$\mathrm{first}(\sigma) = \mathrm{emc}(\Psi(\sigma)).$$

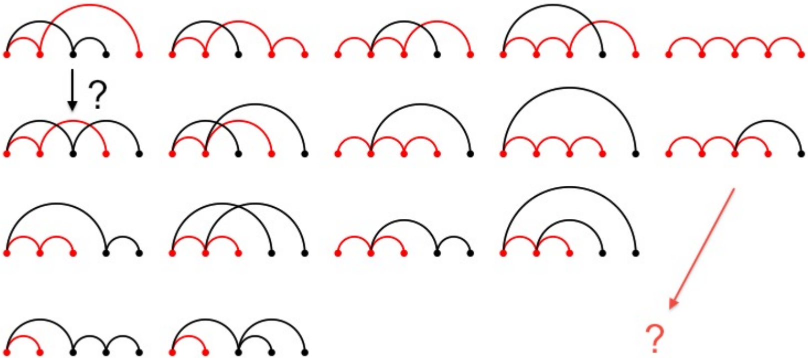


(His proof doesn't use paths in the graph !)

The Set $\mathcal{T}_5$ organised

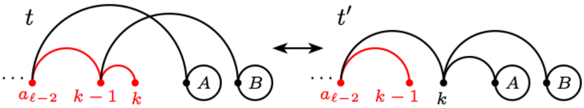


The Set $\mathcal{T}_5$ organised

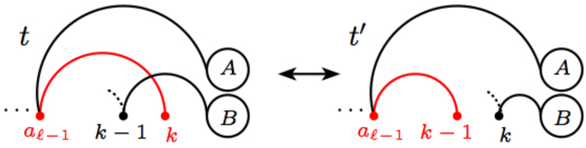


Reduction(-Amplification): $t \in \mathcal{T}_{n,k}$ with minimal chain $(1, 2, \ldots, a_\ell = k)$.

- R0) $k = 2$ no modifications !
- R1) $k > 2$, $a_\ell = k$, $a_{\ell-1} = k - 1$ and $\min(A) \leq \min(B)$ $(\min(\emptyset) = \infty)$:

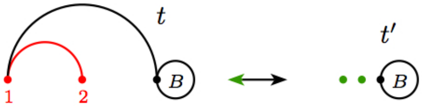


- R2) $k > 2$, $a_\ell = k$, $a_{\ell-1} < k - 1$:

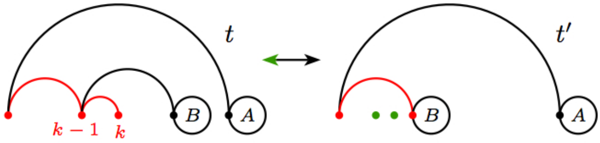


Deletion(-Insertion):

- $k = 2$:



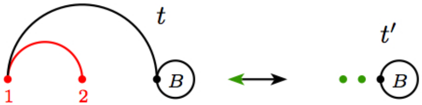
- $k \geq 3$, $a_\ell = k$, $a_{\ell-1} = k-1$ (otherwise apply R2) and $\min(A) > \min(B)$ (otherwise apply R1):



Always possible to add the points $\{1, 2\}$, $\{2, 3\}$,. . .,$\{k-1, k\}$ with $k-1 = \mathrm{emc}(t')$.

Deletion(-Insertion):

- $k = 2$:



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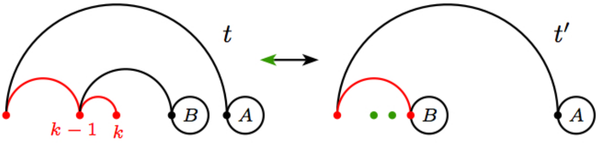
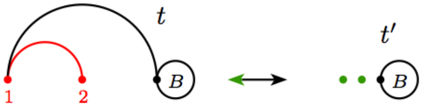


Always possible to add the points $\{1, 2\}$, $\{2, 3\}$,. . .,$\{k - 1, k\}$ with $k-1 = \text{emc}(t')$.

Example. $\sigma = 748591623 \in \mathcal{DU}_{9,7}$. How to construct $\Psi(\sigma)$?

Deletion(-Insertion):

- $k = 2$:



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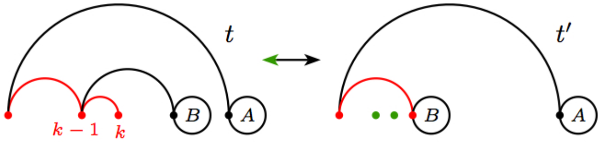

Always possible to add the points $\{1, 2\}$, $\{2, 3\}$,. . .,$\{k - 1, k\}$ with $k-1 = \mathrm{emc}(t')$.

Example. $\sigma = 748591623 \in \mathcal{DU}_{9,7}$. How to construct $\Psi(\sigma)$?

Watch the video animation of my friend Martin Anderegg !

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Joh. Bernoulli
L. Euler
J. L. Lagrange
J. Fourier
D. André
R. C. Entringer
V. Arnol'd
Y. Gelineau

THANKS!