Control of the heat equation with shapes

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Control with constraints

Consider the following control problem: $\Omega \subset \mathbb{R}^N$, bounded, regular,

$$\begin{cases} y_t - \Delta y = u(t,x), \\ y = 0 \text{ on } \partial\Omega, \\ u(t,\cdot) \in \mathcal{U}_c, \quad \forall t \in [0,T]. \end{cases} \tag{1}$$

The control must fulfill a set of time-independent constraints given by $\mathcal{U}_c$. For instance, positivity, mass constraint...
Depending on $\mathcal{U}_c$:

- What controllability properties?
- What do the controls look like?

A particular constraint set: 1-shapes

$$\mathcal{U}^1_{\text{shape}} = \{\chi_\omega, \quad |\omega| \leq m_L\}, \quad m_L < |\Omega|.$$

Shape control

We work on "generalised" shapes: let $m_L < |\Omega|$,

$$\mathcal{U}_{\text{shape}} = \{M\chi_\omega, \quad M \geq 0, \quad |\omega| \leq m_L.\}$$

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Theorem (Pouchol, TrÃ©lat, Z. 2021)

Let $m_L < |\Omega|$, $T > 0$, $\varepsilon > 0$. Any ***nonnegative*** *$y_1 \in L^2(\Omega)$ is ε-**approximately** reachable from 0 in time T with controls u such that*

$$u(t,\cdot) \in \mathcal{U}_{shape} \quad f.a.e\ t \in (0,T).$$

Controllability results with two kinds of controls:

$$u(t,\cdot) = M(t)\chi_{\omega(t)},$$

$$u(t,\cdot) = M\chi_{\omega(t)}.$$

We can't expect better: comparison principle!

Summary

The unconstrained case

Well-known NSC: unique continuation.

$$\dot{y} = Ay + Bu, \quad L_T u := \int_0^T e^{(T-t)A} Bu(t)dt$$

Condition: injectivity i.e. $L_T^* p = 0 \implies p = 0, \quad \forall p.$

Dual of: $(\langle p, L_T u\rangle = 0, \forall u \in U) \implies p = 0$ i.e. $\mathrm{Im}(L_T)$ is dense.

Focus on *one* target y_f: $y_f \in \overline{\mathrm{Im}(L_T)}$.

$$\forall p, \, (\langle L_T u, p\rangle = 0 \; \forall u \in U) \implies \langle y_f, p\rangle = 0.$$

Geometrical interpretation: there exists no *(strict) separating hyperplane* between $\{y_f\}$ and $\mathrm{Im}(L_T)$.

Constrained case

Same system, but with the constraint

$$u(t) \in \mathcal{U}, \quad \forall t \in [0, T].$$

Using the *separating hyperplane* interpretation:

$$\forall p, \quad \text{for any } \alpha, \varepsilon \text{ s.t. } \langle p, L_T u \rangle \leq \alpha - \varepsilon, \quad \forall u \in \mathcal{U},$$

$$\langle p, y_f \rangle < \alpha + \varepsilon.$$

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$$\langle p, y_f\rangle < \alpha + \varepsilon.$$

Take $\alpha = \sup_{u\in\mathcal{U}} \langle p, L_T u\rangle + \varepsilon$:

$$\langle p, y_f\rangle < \sup_{u\in\mathcal{U}} \langle p, L_T u\rangle + 2\varepsilon$$

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Take $\alpha = \sup_{u \in \mathcal{U}} \langle p, L_T u \rangle + \varepsilon$:

$$\langle p, y_f \rangle \leq \sup_{u \in \mathcal{U}} \langle L_T^* p, u \rangle$$

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Take $\alpha = \sup_{u \in \mathcal{U}} \langle p, L_T u \rangle + \varepsilon$:

$$\langle p, y_f \rangle \leq \sigma_{\mathcal{U}}(L_T^* p)$$

Support function of $\mathcal{U}$.

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Optimal control approach

Incorporate constraints and obtain special properties by characterizing optimal controls for a certain cost.

See also:
Lions (1992), Kunisch Wang ('13), Berrahmoune ('14 '19), Ervedoza ('20), Biccari-Zuazua ('22)

The HUM method (J.-L. LIONS)

Control system

$$\dot{y} = Ay + Bu, \quad L_T u := \int_0^T e^{(T-t)A} Bu(t)dt$$

Find controls by solving

$$\inf_{u \text{ adm.}} \frac{1}{2} \int_0^T \|u(t)\|_U^2 dt.$$

Usually, HUM consists in minimizing the following functional (**dual problem**):

$$J_{\text{HUM}}(p_T) = \frac{1}{2} \int_0^T \|L_T^* p_T\|_U^2 dt + \langle y_0, e^{TA^*} p_T \rangle - \langle y_T, p_T \rangle + \varepsilon \|p_T\|$$

The optimal $p_T^\star \neq 0$ solves:

$$DJ_{\text{HUM}}(p_T^\star) = 0$$

The solution to the dual problem gives the control.

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$$\underbrace{y_T - \varepsilon \frac{p_T^\star}{\|p_T^\star\|}}_{\text{close enough}} = e^{TA} y_0 + L_T \left(\underbrace{L_T^* p_T^\star}_{\text{HUM control}} \right) dt$$

A variation on HUM

Now we try to solve

$$\inf_{u \text{ adm.}} \frac{1}{2} \sup_{[0,T]} \|u(t)\|_U^2.$$

The functional becomes:

$$J_{\mathrm{HUM}}(p_T) = \frac{1}{2}\left(\int_0^T \|L_T^* p_T\|_U dt\right)^2 + \langle y_0, e^{TA^*} p_T\rangle - \langle y_T, p_T\rangle + \varepsilon\|p_T\|$$

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Change the functional: change the control you get. **Geometrical interpretation?**

The convex constrained case

General **convex** functional:

$$J(p_T) = F^*(L_T^* p_T) + \langle y_0, e^{TA^*} p_T \rangle - \langle y_T, p_T \rangle + \varepsilon \|p_T\|.$$

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General **convex** functional:

$$J(p_T) = F^*(L_T^* p_T) + \sigma_{\overline{B}(y_f,\varepsilon)}(p_T).$$

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General **convex** functional:

$$J(p_T) = F^*(L_T^* p_T) + G^*(p_T).$$

In general, differential calculus not available!
But with convexity come other tools...
Subdifferential calculus:

$$DJ(p_T^\star) = 0 \text{ becomes } 0 \in \partial J(p_T^\star)$$

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The computation of the optimal control is not as direct as before...BUT we get the existence of a $u^\star$

$$u^\star \in \partial F^*(L_T^* p_T^\star), \quad L_T u^\star \in \partial G^*(-p_T^\star)$$

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IF we can say that

$$\partial(F^* \circ L_T^* + G^* \circ (-Id))(p_T^\star) = L_T \partial F^*(L_T^* p_T^\star) - \partial G^*(-p_T^\star)$$

Fenchel-Rockafellar duality

The real question: what does

$$J(p_T) = F^*(L_T^* p_T) + G^*(p_T)$$

have to do with the optimal control problem

$$\inf_{\substack{u \text{ admissible} \\ u(t,\cdot)\in\mathcal{U}}} C(u)?$$

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$$\inf_{u \in L^2 L^2} C(u) + \delta_{\mathcal{U}}(u) + \delta_{B_\varepsilon(y_f)}(L_T u)?$$

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$$\inf_{u \in L^2 L^2} F(u) + G(L_T u)?$$

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Conjugation:

$$f^*(p) = \sup_{u \in U} \langle p, u \rangle - f(u).$$

For convex functions:

$$f^{**} = f.$$

Fenchel-Rockafellar duality

Let E and $\tilde{E}$ be two Hilbert spaces, F and G convex proper functions on E and $\tilde{E}$ resp., and $L \in L(E, \tilde{E})$.

Primal optimization problem: (our optimal control problem)

$$\pi = \inf_{u \in E} F(u) + G(Lu).$$

Using convex conjugation, we can derive its associated **dual problem:**

$$d = \sup_{p \in \tilde{E}} \left(-F^*(L^*p) - G^*(-p)\right) = - \inf_{p \in \tilde{E}} \left(F^*(L^*p) + G^*(-p)\right).$$

Relationship between the values:

Theorem (Fenchel-Rockafellar)

Weak duality $\pi \geq d$ always holds. Moreover, if there exists $\bar{p} \in \tilde{E}$ such that F^* is continuous at $L^*\bar{p}$ and $G^*(-\bar{p}) < +\infty$, then strong duality holds ie

$$\pi = d \quad \text{and} \quad \pi \text{ is attained if finite.}$$

Optimality conditions

Relationship between the solutions.
Link between the primal and dual problems: the Lagrangian

$$\mathcal{L}(u,p) := \langle p, Lu \rangle + F(u) - G^*(p).$$

Proposition

If strong duality holds, the following are equivalent:

(i) $(u^\star, p^\star)$ *is a pair of solutions to the primal and dual problems.*

(ii) $(u^\star, -p^\star)$ *is a saddle point of* $\mathcal{L}$.

Saddle point:

$$\underbrace{u^\star}_{\text{Optimal control}} \in \operatorname{argmin}_{u \in E} \mathcal{L}(u, -p^\star)$$

$$-\underbrace{p^\star}_{\text{Dual minimizer}} \in \operatorname{argmax}_{p \in F} \mathcal{L}(u^\star, p),$$

To find shape controls, we need to find the right **dual** problem!

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Summary

Choosing the dual problem

Dual problem

$$-\inf_{p_T \in L^2} F^*(L_T^* p_T) + G^*(-p_T)$$

$$\begin{cases} p_t + \Delta p = 0, \\ p = 0 \quad \text{on } \partial\Omega, \\ p(T,\cdot) = p_T \in X := L^2, \end{cases} \qquad \text{control } u \in E := L^2(0,T;L^2).$$

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- Strong duality: F^* continuous in $L_T^* 0 = 0$, $G^*(0) < +\infty$.
- $F^*(L_T^* \cdot) + (\delta_{B_\varepsilon(y_f)})^*(-\cdot)$ has a minimum reached at $p_T^\star$.

$$(\delta_{B_\varepsilon(y_f)})^*(-p) = -\langle p_T, y_f\rangle + \varepsilon \|p_T\|_{L^2} < +\infty.$$

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- To find shapes: we know that there is at least one optimal control

$$u^\star \in \partial F^*(L_T^* p_T^\star).$$

The set of shape controls

$$\text{Shapes:} \quad \mathcal{U}^1_{\text{shape}} = \{\chi_\omega, \quad |\omega| \le m_L\}, \quad m_L < |\Omega|.$$

$$\text{Convex hull:} \quad \overline{\mathcal{U}_L} := \{u \in L^2,\ 0 \le u \le 1 \text{ and } \int_\Omega u \le m_L.\}$$

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Remember the support function:

$$v \in X, \quad \sigma_{\overline{\mathcal{U}_L}(v) :=} \sup_{u \in \overline{\mathcal{U}_L}} \langle u, v \rangle$$

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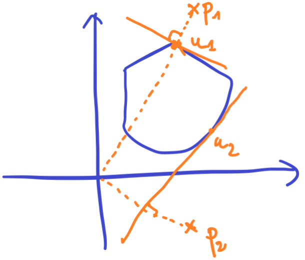


$\tilde{u} \in K$ maximizer

$$\langle \tilde{u}, v \rangle - \delta_K(\tilde{u}) = \sigma_K(v)$$

$$v \in \partial \delta_K(\tilde{u}), \quad \tilde{u} \in \partial \sigma_K(v)$$

Scalar products and bathtubs

Lemma (relaxed bathtub principle)

Let $v \in L^2$. Consider the maximisation problem

$$\sup_{u \in \overline{\mathcal{U}_L}} \langle u, v \rangle_{L^2}, \quad \overline{\mathcal{U}_L} := \left\{ u \in L^2(\Omega),\ 0 \leq u \leq 1 \text{ and } \int_\Omega u \leq m_L \right\}.$$

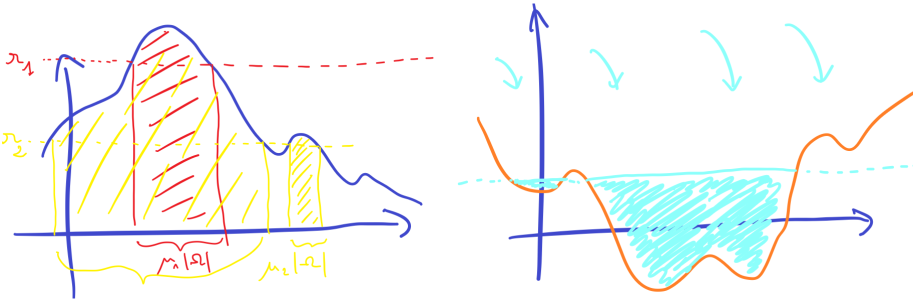
Let $r^\star = \max(0, \inf_{r \in \mathbb{R}} \{ |\{v > r\}| \leq m_L \})$. *The maximisers are given by*

$$u^\star := \chi_{\{v > r^\star\}} + c(x) \chi_{\{v = r^\star\}}, \tag{2}$$

where c is any measurable function such that $0 \leq c \leq 1$ *and*

$$\begin{cases} \int_{\{v = r^\star\}} c = m_L - |\{v > r^\star\}| & \text{if } r^\star > 0 \\ \int_{\{v = r^\star\}} c \leq m_L - |\{v > r^\star\}| & \text{if } r^\star = 0 \end{cases}$$

Bathtub principle



The maximisers of the bathtub principle can be characteristic functions:

$$\partial\sigma_{\mathcal{U}_L}(v) = \left\{ u^\star := \chi_{\{v>r^\star\}} + \underline{c(x)\chi_{\{v=r^\star\}}} \right\}.$$

Back to the dual problem

From the *static* bathtub principle to our *dynamic* optimisation problem: how do we choose $F^* : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$?

$$u^\star \in \partial F^*(L_T^* p_T^\star)$$

$$F^* = \qquad \sigma_{\mathcal{U}_L}$$

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Problem: linear growth of $\sigma_{\mathcal{U}_L}$ does not ensure that the dual problem has a minimum:

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Quadratic growth does! **(When** $y_f \geq 0$**)**

The optimal control problem

Simply compute the conjugate of the F^* we have chosen: we get cost **and** constraints!

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$$F(u) = \underbrace{\frac{1}{2}\sup_{t\in[0,T]} \max\left(\|u(t)\|_\infty, \frac{\|u(t)\|_1}{m_L}\right)^2}_{\text{Cost}} + \underbrace{\delta_{u\geq 0}}_{\text{Relaxed constraints}}$$

The dual problem corresponds to **an optimal control problem**, and both have solutions !

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Do we have shapes?

Getting shapes with the optimality conditions

The optimal control is characterized by the minimizer of the dual problem

$$u^\star(t) \in \partial F^*(L_T^* p_T^\star), \quad \forall t \in [0, T]$$

Technical computation of subdifferentials ($\int_0^T$ and square):

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$$F^*(p) = \frac{1}{2} \left(\int_0^T \sigma_{\mathcal{U}_L}(p(t)) dt \right)^2$$

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Non-empty, contain characteristic functions!

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Bathtub principle:

$$\partial\sigma_{\mathcal{U}_L}(p^\star(t)) = \left\{\chi_{\{p^\star(t)>r^\star(t)\}} + c(x)\chi_{\{p^\star(t)=r^\star(t)\}}\right\}.$$

Shapes for the heat equation

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Heat equation: analytic-hypoelliptic operator! Level sets have **zero measure.**
Finally,

$$u^\star(t) \in \mathcal{U}_{\text{shape}}, \quad f.a.e\ t \in [0,T].$$

Bonus

- General result using convex analysis and FR duality: Hilbert balls are strictly convex $\rightarrow$ **the optimal control is unique** in both cases.
- With another choice of F^*:

$$F^*(p_f) = \frac{1}{2}\int_0^T \sigma_{\mathcal{U}_L}(p(t))^2 dt$$

we get controls of the form

$$u^\star(t) \in \sigma_{\mathcal{U}_L}(p^\star(t))\partial\sigma_{\mathcal{U}_L}(p^\star(t)).$$

- For the heat equation: positive minimal control time appears if one restricts the shapes to a subdomain $\omega \subset \Omega$.

Bonus bonus

For the first cost:

$$C(u) = \frac{1}{2} \sup_{t\in[0,T]} \max\left(\|u(t)\|_\infty, \frac{\|u(t)\|_1}{m_L}\right)^2,$$

amplitude of the optimal control does not depend on time, but on the final time T: it solves the minimal norm problem

$$M^*(T) := \inf\{C(u), \quad u \in L^2L^2,\ u \geq 0,\ \|L_T u - y_f\|_2 \leq \varepsilon\},$$

which turns out to be equivalent to the time optimal control problem:

$$T^*(\lambda) = \inf\{T > 0,\ \exists u \in L^2L^2,\ u \geq 0,\ C(u) \leq \lambda,\ \|L_T u - y_f\|_2 \leq \varepsilon\},$$

for $\lambda > 0$.

$$M^*(T) \xrightarrow[T\to+\infty]{} \mu > 0, \quad M^*(T) \xrightarrow[T\to 0]{} +\infty$$

$$M^* \circ T^* = I_{(\mu,+\infty)}, \quad T^* \circ M^* = I_{(0,+\infty)}.$$

Conclusion and remarks

- Approximate controllability of the heat equation *from* 0 *to nonnegative states*, with shapes. In particular, this means any nonnegative state is approx reachable with nonnegative controls!
- General method: find the right **dual problem** to ensure some properties of the optimal controls we find.
- Amplitude of the shapes

$$M(T, m_L, y_T\varepsilon) = \int_0^T \int_{p^\star(t,\cdot) > r^\star(t)} p^\star(t,x)\,dx\,dt$$

  Focus on T: study of a minimal time problem.
- Adaptable to other equations, other constraints.

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