A new semi-algebraic two-grid method for Oseen problems

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Overview

Oseen problems

The Oseen problem appears in

- viscous incompressible flows with convection
- Picard linearisation of Navier-Stokes equations

and can be written as:

$$\begin{aligned} \nu\Delta\vec{u} + \vec{v}\cdot\nabla\vec{u} + \nabla p &= \vec{f}\,, \\ \nabla\cdot\vec{u} &= 0, \end{aligned}$$

in Ω with the boundary conditions on $\partial\Omega \;=\; \partial\Omega_D \cup \partial\Omega_N$:

$$\begin{aligned} \vec{u} \;&=\; \vec{w} \text{ on } \partial\Omega_D\,, \\ \frac{\partial\vec{u}}{\partial n} - \vec{n}p \;&=\; \vec{s} \text{ on } \partial\Omega_N. \end{aligned}$$

State-of-the-art

Discretizations lead to the **saddle-point** system:

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \begin{pmatrix} \mathrm{f} \\ \mathrm{g} \end{pmatrix}$$

- F: convection-diffusion block (non symmetric)
- B and B^T: discrete divergence and gradient
- C: stabilization block (0 for naturally stable methods)
- Variable convective flow: no linear-time solver robust wrt both the mesh size and the viscosity ν (*i.e.* Reynolds number)
- AMG methods: struggle with the C pressure block: 0 or small elements.

AMG methods

Multigrid methods are efficient solvers for elliptic PDEs.
Two main components: the **smoother** and the **coarse-grid correction**

- **smoother**: simple iterative scheme (Jacobi, GS) corrects high frequency part of the error
  $\Rightarrow$ the error well approximated on a coarser grid
- **coarse-grid construction**: prolongation operator P ($\mathrm{v} = P\mathrm{v}_c$)
  $\Rightarrow$ coarse matrix $A_c = P^T AP$
- **coarse-grid correction**: amounts to solve system with A_c as system matrix.

Multigrid (MG): recursive application.
Algebraic multigrid (AMG): uses only the system matrix A
$\Rightarrow$ **black-box** approach.

Aggregation

- Classic AMG: P interpolation operator $\Rightarrow$ often leads to denser A_c
- Aggregation: keep P as simple as possible and then A_c as sparse as possible
- In practice: definition of small subsets of the unknowns (= aggregates)
- Coarse unknowns are identified as aggregates
- Prolongation defined as constant by aggregate $\Rightarrow$ P logical matrix and P_{ij} indicates whether the i^{th} (fine) unknown belongs to the j^{th} aggregate.
- A_c keeps the same sparsity pattern as A.

Transform-then-solve approach

Algebraic transformation of the system to give it a more "suitable" structure:

- Sign of the pressure equation is changed
- Change of variables:

$$\begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \begin{pmatrix} I & -D_{\mathrm{u}}^{-1} B^T \\ & I \end{pmatrix} \begin{pmatrix} \hat{\mathrm{u}} \\ \hat{\mathrm{p}} \end{pmatrix} ,$$

where D_{u} is typically the diagonal of the F block. The matrix system becomes:

$$A = \begin{pmatrix} F & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} I & -D_{\mathrm{u}}^{-1} B^T \\ & I \end{pmatrix} = \begin{pmatrix} F & (I - F D_{\mathrm{u}}^{-1}) B^T \\ -B & C + B D_{\mathrm{u}}^{-1} B^T \end{pmatrix}$$

Transform-then-solve approach

$$A = \begin{pmatrix} F & (I - FD_{\mathrm{u}}^{-1})B^T \\ -B & C + BD_{\mathrm{u}}^{-1}B^T \end{pmatrix}$$

Diagonal blocks: structure compatible with AMG methods:

- F: convection-diffusion matrix
- $\widehat{C} = C + BD_{\mathrm{u}}^{-1}B^T$: similar to discrete Laplacian.

Stokes problems: approach robust and efficient when combined with aggregation-based algebraic multigrid.
Aggregation: unknown-based coarsening:

- Velocity: based on F
- Pressure: based on $\widehat{C}$.

From unknown-based to point-based coarsening

What about Navier-Stokes problems?

- Theory (to follow): NS problems require point-based coarsening instead of unknown-based coarsening
- Point-based coarsening: coarsening of the grid points and not of the unknowns
- The coarsening of each type of unknown is deduced from the one of the grid points

$\Rightarrow$ same coarsening for each type of unknown.

Convergence theory: context

Theoretical framework:

- Two-grid method
- Jacobi-type smoother $\omega^{-1}D \;=\; \omega^{-1}\begin{pmatrix} D_{\mathrm{u}} & \\ & D_p \end{pmatrix}$
- Prolongation of the form $P \;=\; \begin{pmatrix} P_{\mathrm{u}} & \\ & P_p \end{pmatrix}$
- Galerkin coarse grid correction $A_c \;=\; P^T A P$
- One (pre)-smoothing step only.

Theorem 1

- For SPD G, $c_{\mathrm{ap}}(G, P, D) = \sup_{\mathrm{v}\neq 0} \frac{\mathrm{v}^H D\left(I - P\left(P^T D\, P\right)^{-1} P^T D\right) \mathrm{v}}{\mathrm{v}^H G\, \mathrm{v}}$ (characterize the convergence of the two-grid method for G)
- $A_S = \frac{1}{2}\left(A + A^T\right)$ and $(A^{-1})_S = \frac{1}{2}(A^{-1} + A^{-T})$
- K: right-preconditioned matrix (with two-grid preconditioner)

For any α, β such that

$$c_{\mathrm{ap}}\left(A_S\, ,\, P\, ,\, D\right) \leq \alpha^{-1} \quad \text{and} \quad c_{\mathrm{ap}}\left(D(A^{-1})_S D\, ,\, P\, ,\, D\right) \leq \beta^{-1}\, ,$$

there holds

$$\min_{\tau} \|I - \tau\, K\|_{P^\perp} \leq \|I - \tfrac{\beta}{\omega}\, K\|_{P^\perp} \leq \sqrt{1 - \alpha\beta}$$

where $\|M\|_{\mathcal{V}} = \max_{\mathrm{z}\in\mathcal{V}\setminus\{0\}} \frac{\|M\,\mathrm{z}\|}{\|\mathrm{z}\|}$.

Theorem 2

Let the above assumptions hold. There holds,

$$c_{\mathrm{ap}}\left(A_S\,,\ P\,,\ D\right)\ \leq\ \frac{\max\left(c_{\mathrm{ap}}\left(F_S\,,\ P_{\mathrm{u}}\,,\ D_{\mathrm{u}}\right)\,,\ c_{\mathrm{ap}}\left(\widehat{C}\,,\ P_p\,,\ D_p\right)\right)}{1-\frac{\sqrt{\gamma_{\mathrm{u}}}}{2}}\,,$$

$$c_{\mathrm{ap}}\left(D(A^{-1})_S D\,,\ P\,,\ D\right)\ \leq\ \zeta\ =\ \frac{\gamma_p+1+\sqrt{(\gamma_p-1)^2+\gamma_{\mathrm{u}}\,\gamma_p}}{2\left(1-\frac{\gamma_{\mathrm{u}}}{4}\right)}\,,$$

where γ_{u} and γ_p are such that

$$\lambda_{\max}\left(D_{\mathrm{u}}^{-1/2}\ \left(\tfrac{1}{2}(F^{-1}+F^{-T})\right)^{-1} D_{\mathrm{u}}^{-1/2}\right)\ \leq\ \gamma_{\mathrm{u}}$$

and

$$\lambda_{\max}\left(D_p^{-1/2} C\, D_p^{-1/2} + \Pi\, D_p^{-1/2} B\, F_S^{-1} B^T D_p^{-1/2}\Pi\right)\ \leq\ \gamma_p\ ,$$

with $\Pi\ =\ \left(I - D_p^{1/2} P_p \left(P_p^T D_p P_p\right)^{-1} P_p^T D_p^{1/2}\right)$.

Discussion of Theorem 2

Bounds depend on 4 constants:

- $c_{\mathrm{ap}}\big(F_S\,,\ P_{\mathrm{u}}\,,\ D_{\mathrm{u}}\big)$
- $c_{\mathrm{ap}}\big(\widehat{C}\,,\ P_p\,,\ D_p\big)$
- $\gamma_{\mathrm{u}} = \lambda_{\max}\left(D_{\mathrm{u}}^{-1/2}\ \left(\frac{1}{2}(F^{-1}+F^{-T})\right)^{-1} D_{\mathrm{u}}^{-1/2}\right)$
- $\gamma_p \ = \ \lambda_{\max}\left(D_p^{-1/2} C\, D_p^{-1/2} + \Pi\, D_p^{-1/2} B\, F_S^{-1} B^T D_p^{-1/2} \Pi\right).$

Keep constants bounded:

- γ_{u}: nicely bounded if $D_{\mathrm{u}} \ = \ \mathrm{diag}(F)$
- c_{ap} constants: nicely bounded if prolongation such that the two-grid method (with D) is good for F and $\widehat{C}$
- γ_p: depends on the coarsening of the pressure unknowns via Π.

Evaluation of the constants

- Numerical evaluation for model problem: staggered grid, constant convection flow, finite differences,
- Results for aggregation type prolongation:

		Linewise $c_{\mathrm{ap}}(\widehat{C})$	Boxwise $c_{\mathrm{ap}}(\widehat{C})$	Linewise γ_p	Boxwise γ_p
$\theta = 0$	$\nu = 10^{-2}$	9.1	5.7	3.5	17.0
	$\nu = 10^{-4}$	9.1	5.7	6.0	192.8
	$\nu = 10^{-6}$	9.1	5.7	6.1	219.6
$\theta = \pi/4$	$\nu = 10^{-2}$	9.1	5.7	1.3	1.3
	$\nu = 10^{-4}$	9.1	5.7	1.3	1.3
	$\nu = 10^{-6}$	9.1	5.7	1.3	1.3

Table: Evaluation of the constants related to the convergence analysis (mesh size $h = 1/16$).

Local Fourier Analysis (LFA)

- Cheap tool to compute exact quantities related to two-grid methods (asymptotic convergence factor)
- Several (constraining) hypotheses (constant coefficient, uniform (Cartesian) grid, periodic boundary conditions)
- Heuristic in nature but recognized as reliable
- Each operator can be associated to its eigenvalue (= symbol)

LFA for constant convection problem

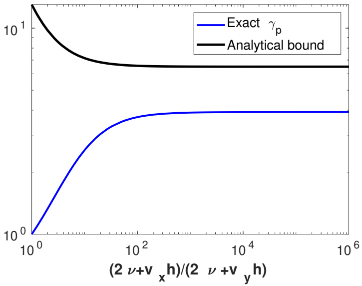
Constant convection problem: LFA to confirm numerical results.

- Parameters of the problem:
  - Convection wind $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$,
  - Viscosity ν,
  - Mesh size h.
- Analytical bound ($v_x > v_y$):
  - $\gamma_p \leq \frac{13}{2}\left(1 + \frac{2\nu + v_y h}{2\nu + v_x h}\right) < 13$ if linewise coarsening along main component of $\vec{v}$,
  - $\gamma_p \propto \left(1 + \frac{2\nu + v_x h}{2\nu + v_y h}\right)$ if boxwise coarsening.

LFA computation

Proof of uniform convergence for constant convection flow if coarsening in the direction of the flow.

Linewise coarsening



Boxwise coarsening

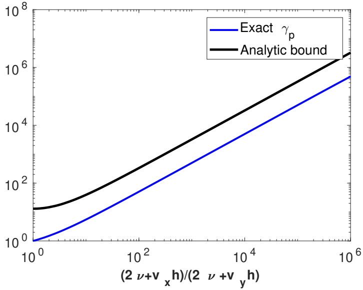


Figure: γ_p and its analytical bound for linewise and boxwise coarsening.

Proposed two-grid method: aggregation

Coarsening of velocity: driven by convection (bound c_{ap})
$\Rightarrow$ algebraic aggregation based on F
Coarsening of pressure: driven by convection (bound γ_p)
$\Rightarrow$ algebraic aggregation **NOT** based on $\widehat{C} \;=\; C + BD_{\mathrm{u}}^{-1}B^T$

> We need coarsening of pressure unknowns driven by convection $\Rightarrow$
> Point-based coarsening

Use of a geometric artifice:

- Build auxilary convection-diffusion matrix X on pressure points
- Algebraically aggregate the points based on auxilary matrix X.

$\Rightarrow$ Similar aggregation pattern to velocity

Numerical experiments: settings

- Convection-diffusion problems on staggered grid (FD): constant and recirculating flows
- Smoother: 1 SOR iteration for pre- and post-smoothing with $\omega = 2/3$
- Point-based aggregation defined by applying AGMG software to auxilary matrix X: quality-aware aggregation:
  $\Rightarrow$ aligns aggregates along the main component of the flow
- Galerkin coarse grid matrix
- Two-grid method used as a preconditioner for GCR (mathematically equivalent to GMRES).

Numerical experiments: constant convection flow

	$h = 1/128$				$h = 1/256$			
$\theta \backslash \nu$	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$\theta = 0$	14	21	22	22	15	19	22	22
$\theta = \pi/16$	14	22	23	23	11	20	22	22
$\theta = \pi/4$	10	11	12	12	9	11	11	11

Table: Constant convection flow: number of iterations until $\|r_{rel}\| \leq 10^{-6}$.

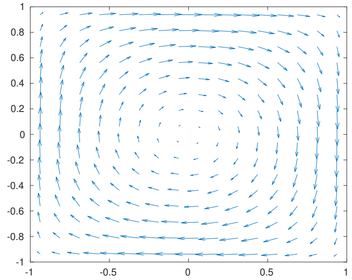
Numerical experiments: recirculating flows

$$\text{2D1: } \vec{v} = \begin{pmatrix} x(1-x)(2y-1) \\ -(2x-1)y(1-y) \end{pmatrix}$$

$$\text{2D2: } \vec{v} = \begin{pmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{pmatrix}$$

$$\text{ESW}^1\text{: } \vec{v} = \begin{pmatrix} 2y(1-x^2) \\ -2x(1-y^2) \end{pmatrix}$$

Recirculating Flow ESW



[1] ESW: Elman, Silvester, Wathen, 2005.

Numerical experiments: recirculating flow (cont'd)

Flow \ ν	$h = 1/128$ 10^{-2}	10^{-4}	10^{-6}	10^{-8}	$h = 1/256$ 10^{-2}	10^{-4}	10^{-6}	10^{-8}
2D1	15	36	48	49	14	30	49	51
2D2	14	39	47	49	15	37	46	53
ESW	14	40	42	42	13	38	44	44

Table: Recirculating flows: number of iterations until $\|r_{rel}\| \leq 10^{-6}$.

Conclusions and perspectives

Conclusions:

- Convergence theory in norm for Oseen problems
  - Proof of uniform bound on the convergence for constant convection flow if point-based coarsening driven by the convection flow
- Setting-up the two-grid method
- Numerical results suggest uniform bound on the convergence also for recirculating flows.

Perspectives:

- Transition from two- to multigrid
- Full algebraization of the method
- Application to Navier-Stokes equations (Picard linearization).

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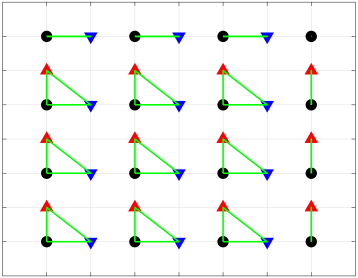
Questions?

Technical difficulty: staggered grid

- Staggered grid: often used for Oseen problems because $C = 0$ but point-based coarsening uneasy.
- Solution: gathering of neighboring unknowns into points as on the figure.
- Points located on the pressure unknowns.



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