An unfitted discretization of the Stokes problem robust to a pressure jump

Daniele Corti, Guillaume Delay, Miguel Fernández, Fabien Vergnet, Marina Vidrascu

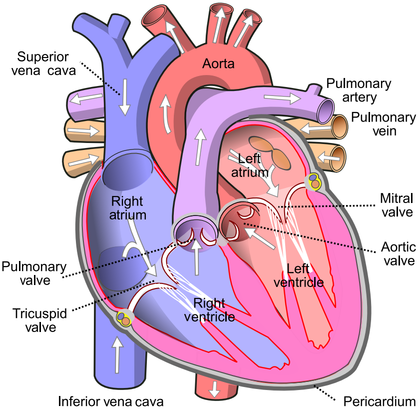
Laboratoire Jacques-Louis Lions, Sorbonne Université, Paris, France

Congrès d'Analyse NUMérique (CANUM) 2022
16^{th} June 2022

Outline

- A cardiology problem
- An unfitted numerical method
- Some numerical analysis elements
- Numerical simulations

A cardiology problem



Goal : run simulations involving left atrium, left ventricle, mitral valves

Technical difficulties

Several difficulties from Fluid-Structure Interaction (**FSI**) problems

- **two systems** : fluid and structure
- **contact** between several deformable solids
- **deformation of the fluid domain** over time
- high pressure jumps through the valves

Technical difficulties

Several difficulties from Fluid-Structure Interaction (**FSI**) problems

- **two systems** : fluid and structure $\rightarrow$ **splitting scheme** (separate resolution for fluid and solid)
- **contact** between several deformable solids
- **deformation of the fluid domain** over time
- high pressure jumps through the valves

Technical difficulties

Several difficulties from Fluid-Structure Interaction (**FSI**) problems

- **two systems** : fluid and structure $\rightarrow$ **splitting scheme** (separate resolution for fluid and solid)
- **contact** between several deformable solids $\rightarrow$ **contact algorithm** has to be considered
- **deformation of the fluid domain** over time
- high pressure jumps through the valves

Technical difficulties

two systems : fluid and structure $\rightarrow$ **splitting scheme** (separate resolution for fluid and solid), case of **thin-walled structures**

- [Kamensky, Hsu, Schillinger, Evans, Aggarwal, Bazilevs, Sacks, Hughes 15]
- [Boilevin-Kayl, Fernández, Gerbeau 19]
- [Boilevin-Kayl, Fernández, Gerbeau 19]
- [Fernández, Landajuela 20]
- [Annese, Fernández, Gastaldi 22]

Technical difficulties

contact between several deformable solids

- [Kamensky, Xu, Lee, Yan, Bazilevs, Hsu 19]
- [Mlika, Renard, Chouly 17]
- [Burman, Fernández, Frei 20]
- [Burman, Fernández, Frei, Gerosa 22]

Technical difficulties

Deformation of the fluid domain :

- standard method : **Arbitrary Lagrangian Eulerian** (ALE) [Hu, Patankar, Zhu 01]
  - enables to account for the domain deformation
  - but ... : we need to **fully remesh** when large deformations are applied $\rightarrow$ **not adapted when contact occurs**

Technical difficulties

Deformation of the fluid domain :

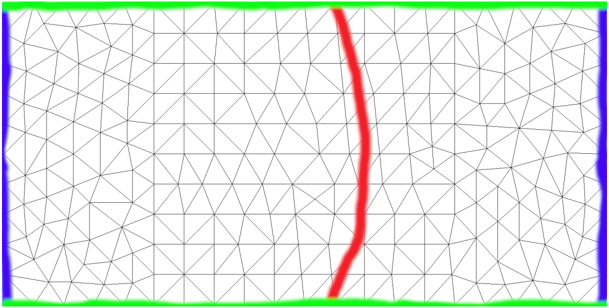
- standard method : **Arbitrary Lagrangian Eulerian** (ALE) [Hu, Patankar, Zhu 01]
  - enables to account for the domain deformation
  - but ... : we need to **fully remesh** when large deformations are applied $\rightarrow$ **not adapted when contact occurs**
- a more recent approach : **extended finite elements** (XFEM) [Groß, Reusken 07]
  - the mesh does not need to fit the boundary / interface
  - optimal convergence rates are established
  - but ... : needs to double the degrees of freedom for the cut cells $\rightarrow$ **matrix size depends on the position of the interface** + needs to handle small cuts

Technical difficulties

Deformation of the fluid domain :

- standard method : **Arbitrary Lagrangian Eulerian** (ALE) [Hu, Patankar, Zhu 01]
  - enables to account for the domain deformation
  - but ... : we need to **fully remesh** when large deformations are applied $\rightarrow$ **not adapted when contact occurs**
- a more recent approach : **extended finite elements** (XFEM) [Groß, Reusken 07]
  - the mesh does not need to fit the boundary / interface
  - optimal convergence rates are established
  - but ... : needs to double the degrees of freedom for the cut cells $\rightarrow$ **matrix size depends on the position of the interface** + needs to handle small cuts
- another try : **fictitious domain** :
  - similar to XFEM, but we do not double the degrees of freedom in the cut cells
  - gain : the matrix size is fixed along the whole simulation
  - main drawbacks : we do not have optimal convergence rates + the velocity is more sensitive to the pressure

The domain and its triangulation



- $\Omega = \Omega_1 \cup \Sigma \cup \Omega_2 \subset \mathbb{R}^d$ bounded polygonal, $d \in \{2, 3\}$
- Σ : immersed interface
- Γ_D : Dirichlet boundary (top, bottom)
- Γ_N : Neumann boundary (left, right)
- $\mathcal{T}_h$ a triangulation of Ω (not fitted to Σ)
- $\mathcal{S}_h$: a discretization of Σ

The Stokes problem

- We want to find $(\mathbf{u}, p)$ solution to

$$\begin{aligned} -\mathrm{div}\ \sigma(\mathbf{u}, p) &= \mathbf{f} \text{ in } \Omega_1 \cup \Omega_2 \\ \mathrm{div}\ \mathbf{u} &= 0 \text{ in } \Omega_1 \cup \Omega_2 \\ \mathbf{u} &= \mathbf{v}_s \text{ on } \Sigma \\ \mathbf{u} &= 0 \text{ on } \Gamma_D \\ \sigma(\mathbf{u}, p)\mathbf{n} &= \mathbf{g}_N \text{ on } \Gamma_N \end{aligned}$$

  with $\sigma(\mathbf{u}, p) := \nu \nabla u - pI$.

- We want **a good approximation of** $[\![\sigma(\mathbf{u}, p)\mathbf{n}]\!]$ through the interface.

We consider the following FE spaces ($\mathbb{P}^1$-$\mathbb{P}^1$-$\mathbb{P}^1$ FE method)

$$
\begin{aligned}
\mathbf{V}_h &:= \{\mathbf{v}_h \in H^1(\Omega;\mathbb{R}^d) \mid \mathbf{v}_h = 0 \text{ on } \Gamma_D \text{ and } \mathbf{v}_h|_T \in \mathbb{P}^1(T;\mathbb{R}^d) \quad \forall T \in \mathcal{T}_h\} \\
Q_h &:= \{q_h \in H^1(\Omega) \mid q_h|_T \in \mathbb{P}^1(T) \quad \forall T \in \mathcal{T}_h\} \\
\mathbf{\Lambda}_h &:= \{\boldsymbol{\mu}_h \in H^1(\Sigma;\mathbb{R}^d) \mid \boldsymbol{\mu}_h|_S \in \mathbb{P}^1(S;\mathbb{R}^d) \quad \forall S \in \mathcal{S}_h\}
\end{aligned}
$$

Find $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{\Lambda}_h$ s.t.

$$
\begin{aligned}
a_h(\mathbf{u}_h,\mathbf{v}_h) - b_h(\mathbf{v}_h,p_h) + c_h(\mathbf{v}_h,\boldsymbol{\lambda}_h) &= \ell_h(\mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h \\
b_h(\mathbf{u}_h,q_h) + s_h^{BP}(p_h,q_h) &= 0 && \forall q_h \in Q_h \\
-c_h(\mathbf{u}_h,\boldsymbol{\mu}_h) + s_h^{BH}(\boldsymbol{\lambda}_h,\boldsymbol{\mu}_h) &= -c_h(\mathbf{v}_s,\boldsymbol{\mu}_h) && \forall \boldsymbol{\mu}_h \in \mathbf{\Lambda}_h
\end{aligned}
$$

with

$$
\begin{aligned}
a_h(\mathbf{w}_h,\mathbf{v}_h) &:= \nu(\nabla\mathbf{w}_h,\nabla\mathbf{v}_h)_\Omega \\
b_h(\mathbf{w}_h,q_h) &:= (\mathrm{div}\ \mathbf{w}_h,q_h)_\Omega \\
c_h(\mathbf{w}_h,\boldsymbol{\mu}_h) &:= (\mathbf{w}_h,\boldsymbol{\mu}_h)_\Sigma \\
\ell_h(\mathbf{w}_h) &:= (\mathbf{f},\mathbf{v}_h)_\Omega + (\mathbf{g}_N,\mathbf{v}_h)_{\Gamma_N}
\end{aligned}
$$

Stabilization terms

We need the following stabilization terms:

- **Brezzi–Pitkäranta** stabilization for the pressure [Brezzi, Pitkäranta 84]

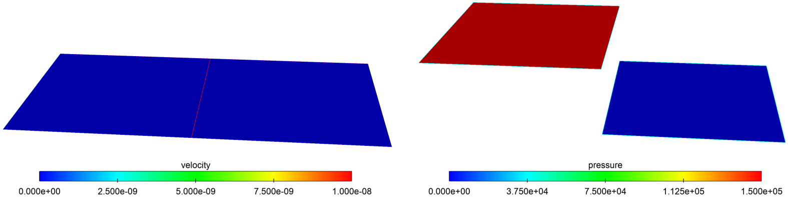
$$s_h^{BP}(p_h, q_h) := \frac{\gamma_p h^2}{\nu}(\nabla p_h, \nabla q_h)_\Omega$$

  with $\gamma_p = 0.1$ in the sequel
- **Barbosa–Hughes** stabilization for the multiplier [Barbosa, Hughes 91]

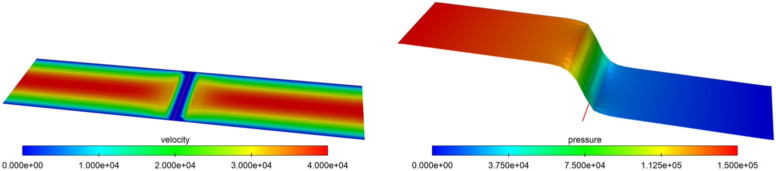
$$s_h^{BH}(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h) := \frac{\gamma_\lambda h}{\nu}(\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h)_\Sigma$$

  with $\gamma_\lambda = 0.01$ in the sequel
- **inf-sup condition** for the bilinear form

Test case:



Results:



Similar results for:

- P1–P0
- P2–P0
- Crouzeix–Raviart, ...

An interpretation of the problem

- We expect the approximation

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_\Omega + \|p-p_h\|_\Omega \leq C(\inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\nabla(\mathbf{u}-\mathbf{v}_h)\|_\Omega + \inf_{q_h\in Q_h} \|p-q_h\|_\Omega)$$
$$\leq C(|\mathbf{u}|_{H^{1+\gamma}(\Omega)} + |p|_{H^\gamma(\Omega)})h^\gamma$$

with $\gamma < \frac{1}{2}$ because of **jumps through the interface**

- We need to represent the pressure jump in the discrete space

Main idea

Enrich the pressure FE space with an Heavyside function

$$\chi_1(\mathbf{x}) := \begin{cases} 1, & \mathbf{x} \in \Omega_1 \\ 0, & \mathbf{x} \in \Omega_2 \end{cases}$$

$$\widetilde{Q}_h := Q_h \oplus \text{Span}(\chi_1)$$

The new formulation is : Find $(\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{V}_h \times \widetilde{Q}_h \times \boldsymbol{\Lambda}_h$ s.t.

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) + c_h(\mathbf{v}_h, \boldsymbol{\lambda}_h) &= \ell_h(\mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h \\ b_h(\mathbf{u}_h, q_h) + s_h^{BP}(p_h, q_h) + \widetilde{s}_h^{BH}((\boldsymbol{\lambda}_h, p_h), (0, q_h)) &= 0 && \forall q_h \in \widetilde{Q}_h \\ -c_h(\mathbf{u}_h, \boldsymbol{\mu}_h) + \widetilde{s}_h^{BH}((\boldsymbol{\lambda}_h, p_h), (\boldsymbol{\mu}_h, 0)) &= -c_h(\mathbf{v}_s, \boldsymbol{\mu}_h) && \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h \end{aligned}$$

where

$$\widetilde{s}_h^{BH}((\boldsymbol{\lambda}_h, p_h), (\boldsymbol{\mu}_h, q_h)) := \frac{\gamma_\lambda h}{\nu} (\boldsymbol{\lambda}_h - [\![p_h]\!] \mathbf{n}, \boldsymbol{\mu}_h - [\![q_h]\!] \mathbf{n})_\Sigma$$

Another formulation

This problem can be rewritten under the form : Find $(\mathbf{u}_h, \widetilde{p}_h, \widehat{p}_h, \boldsymbol{\lambda}_h) \in \mathbf{V}_h \times Q_h \times \mathbb{R} \times \boldsymbol{\Lambda}_h$ s.t.

$$\begin{aligned}
a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, \widetilde{p}_h) + c_h(\mathbf{v}_h, \boldsymbol{\lambda}_h) - d_h(\mathbf{v}_h, \widehat{p}_h) &= \ell_h(\mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h \\
b_h(\mathbf{u}_h, \widetilde{q}_h) + s_h^{BP}(\widetilde{p}_h, \widetilde{q}_h) &= 0 && \forall \widetilde{q}_h \in Q_h \\
-c_h(\mathbf{u}_h, \boldsymbol{\mu}_h) + \widetilde{s}_h^{BH}((\boldsymbol{\lambda}_h, \widehat{p}_h \chi_1), (\boldsymbol{\mu}_h, 0)) &= -c_h(\mathbf{v}_s, \boldsymbol{\mu}_h) && \forall \boldsymbol{\mu}_h \in \boldsymbol{\Lambda}_h \\
d_h(\mathbf{u}_h, \widehat{q}_h) + \widetilde{s}_h^{BH}((\boldsymbol{\lambda}_h, \widehat{p}_h \chi_1), (0, \widehat{q}_h \chi_1)) &= d_h(\mathbf{v}_s, \widehat{q}_h) && \forall \widehat{q}_h \in \mathbb{R}
\end{aligned}$$

where

$$d_h(\mathbf{u}_h, \widehat{q}_h) := \widehat{q}_h \int_{\partial\Omega_1} \mathbf{u}_h \cdot \mathbf{n}$$

This enrichment can be seen as globally **imposing mass conservation** in Ω_1

Similar idea in [Hisada , Washio 16] (in japanese)

Inf-sup condition

We define

$$\begin{aligned}&\mathcal{A}_h((\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h))\\ &:= a_h(\mathbf{w}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, r_h) + c_h(\mathbf{v}_h, \boldsymbol{\zeta}_h) + b_h(\mathbf{w}_h, q_h) - c_h(\mathbf{w}_h, \boldsymbol{\mu}_h)\\ &\qquad + s_h^{BP}(r_h, q_h) + \widetilde{s}_h^{BH}((\boldsymbol{\zeta}_h, r_h), (\boldsymbol{\mu}_h, q_h))\end{aligned}$$

The solution fulfills

$$\mathcal{A}_h((\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h)) = \ell_h(\mathbf{v}_h) - c_h(\mathbf{v}_s, \boldsymbol{\mu}_h)$$

$$|||\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h|||^2 := \|\nabla \mathbf{w}_h\|_\Omega^2 + \|r_h\|_\Omega^2 + h\|\boldsymbol{\zeta}_h\|_\Sigma^2$$

Inf-sup condition

There exists a constant $\beta > 0$ independent from h such that for all $(\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h) \in \mathbf{U}_h \times \widetilde{Q}_h \times \boldsymbol{\Lambda}_h$

$$\beta|||\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h||| \leq \sup_{(\mathbf{v}_h, q_h, \boldsymbol{\mu}_h) \in \mathbf{U}_h \times \widetilde{Q}_h \times \boldsymbol{\Lambda}_h \setminus \{(0,0,0)\}} \frac{\mathcal{A}_h((\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h))}{|||\mathbf{v}_h, q_h, \boldsymbol{\mu}_h|||}$$

proof : similar arguments as the ones in [Fournié, Lozinski 18]

Main steps of the proof (1/2)

- Denote

$$S := \sup_{(\mathbf{v}_h, q_h, \boldsymbol{\mu}_h) \in \mathbf{U}_h \times \widetilde{Q}_h \times \boldsymbol{\Lambda}_h \setminus \{(0,0,0)\}} \frac{\mathcal{A}_h((\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h))}{|\!|\!| \mathbf{v}_h, q_h, \boldsymbol{\mu}_h |\!|\!|}$$

- **Step 1 :** velocity and stabilization terms

$$\begin{aligned} &\nu \|\nabla \mathbf{w}_h\|_\Omega^2 + \frac{\gamma_p h^2}{\nu} \|\nabla r_h\|_\Omega^2 + \frac{\gamma_\lambda h}{\nu} \|\boldsymbol{\zeta}_h - [\![r_h]\!] \mathbf{n}\|_\Sigma^2 \\ &= a_h(\mathbf{w}_h, \mathbf{w}_h) + s_h^{BP}(r_h, r_h) + \widetilde{s}_h^{BH}((\boldsymbol{\zeta}_h, r_h), (\boldsymbol{\zeta}_h, r_h)) \\ &\qquad = \mathcal{A}_h((\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h), (\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h)) \leq S |\!|\!| \mathbf{w}_h, r_h, \boldsymbol{\zeta}_h |\!|\!| \end{aligned}$$

- **Step 2 :** pressure
  There exists $\mathbf{v}_p \in H_0^1(\Omega_1 \cup \Omega_2)$ such that and $\operatorname{div} \mathbf{v}_p = r_h - \overline{r_h}$ in Ω and $\|\mathbf{v}_p\|_{H^1(\Omega)} \leq C \|r_h - \overline{r_h}\|_\Omega$

$$\begin{aligned} \|r_h - \overline{r_h}\|_\Omega^2 &= (r_h - \overline{r_h}, \operatorname{div} \mathbf{v}_p)_\Omega \\ &= (r_h - \overline{r_h}, \operatorname{div} (\mathbf{v}_p - \mathbf{I}_h(\mathbf{v}_p)))_\Omega + (r_h - \overline{r_h}, \operatorname{div} \mathbf{I}_h(\mathbf{v}_p))_\Omega \end{aligned}$$

$$\begin{aligned} |(r_h - \overline{r_h}, \operatorname{div} (\mathbf{v}_p - \mathbf{I}_h(\mathbf{v}_p)))_\Omega| &= |(\nabla (r_h - \overline{r_h}), \mathbf{v}_p - \mathbf{I}_h(\mathbf{v}_p))_\Omega| \\ &\leq C h \|\nabla r_h\|_\Omega \|r_h - \overline{r_h}\|_\Omega \end{aligned}$$

Main steps of the proof (2/2)

- This gives

$$\|r_h - \overline{r_h}\|_\Omega^2 \leq CS|\!|\!|\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h|\!|\!|$$

- The mean pressure $\overline{r_h}$ can be estimated separately :

$$\|\overline{r_h}\|_\Omega^2 \leq CS|\!|\!|\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h|\!|\!|$$

  Combining both $\|r_h\|_\Omega^2 \leq CS|\!|\!|\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h|\!|\!|$

- **Step 3 :** Lagrange multiplier

$$\begin{aligned} h^{\frac{1}{2}}\|\boldsymbol{\zeta}_h\|_\Sigma &\leq h^{\frac{1}{2}}\|\boldsymbol{\zeta}_h - [\![r_h]\!]\mathbf{n}\|_\Sigma + h^{\frac{1}{2}}\|[\![r_h]\!]\mathbf{n}\|_\Sigma \\ &\leq h^{\frac{1}{2}}\|\boldsymbol{\zeta}_h - [\![r_h]\!]\mathbf{n}\|_\Sigma + C\|r_h\|_\Omega \\ &\leq CS|\!|\!|\mathbf{w}_h, r_h, \boldsymbol{\zeta}_h|\!|\!| \end{aligned}$$

- **Step 4 :** We conclude with Young's inequality

A priori bounds

Expected convergence rates

There exists $C > 0$ independent from h such that

$$|||u - u_h, p - p_h, \lambda - \lambda_h||| \leq Ch^{\gamma}(\|\mathbf{u}\|_{H^{1+\gamma}(\Omega)} + \|p - \widehat{J}_h(p)\|_{H^{\gamma}(\Omega)})$$

for every $\gamma < \frac{1}{2}$

proof: We define interpolation operators:

- velocity : $\mathbf{I}_h(\mathbf{u}) \in \mathbf{V}_h$
- pressure : $J_h(p) := \widetilde{J}_h(p) + \widehat{J}_h(p) \in Q_h \oplus \mathrm{Span}(\chi_1)$
- Lagrange multiplier : $\mathbf{L}_h(\boldsymbol{\lambda}) \in \boldsymbol{\Lambda}_h$

with $\widehat{J}_h(p) := \overline{[\![p]\!]}\chi_1 := ([\![p]\!], 1)_{\Sigma}\chi_1$ and $\widetilde{J}_h(p)$ a standard interpolation of $p - \widehat{J}_h(p)$

A priori bounds

- We compute

$$
\begin{aligned}
&\mathcal{A}_h((\mathbf{I}_h(\mathbf{u}) - \mathbf{u}_h, J_h(p) - p_h, \mathbf{L}_h(\boldsymbol{\lambda}) - \boldsymbol{\lambda}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h)) \\
&= \mathcal{A}_h((\mathbf{I}_h(\mathbf{u}), J_h(p), \mathbf{L}_h(\boldsymbol{\lambda})), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h)) - \ell_h(\mathbf{v}_h) + c_h(\mathbf{v}_s, \boldsymbol{\mu}_h) \\
&= \nu(\nabla(\mathbf{I}_h(\mathbf{u}) - \mathbf{u}), \nabla \mathbf{v}_h)_\Omega - (J_h(p) - p, \mathrm{div}\ \mathbf{v}_h)_\Omega \\
&+ (\mathrm{div}\ (\mathbf{I}_h(\mathbf{u}) - \mathbf{u}), q_h)_\Omega + (\mathbf{L}_h(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \mathbf{v}_h)_\Sigma - (\mathbf{I}_h(\mathbf{u}) - \mathbf{u}, \boldsymbol{\mu}_h)_\Sigma \\
&+ \frac{\gamma_p h^2}{\nu}(\nabla \widetilde{J}_h(p), \nabla \widetilde{q}_h)_\Omega + \frac{\gamma_\lambda h}{\nu}(\mathbf{L}_h(\boldsymbol{\lambda}) - \widehat{J}_h(p)\mathbf{n}, \boldsymbol{\mu}_h - \widehat{q}_h \mathbf{n})_\Sigma
\end{aligned}
$$

- We have

$$
(J_h(p) - p, \mathrm{div}\ \mathbf{v}_h)_\Omega = (\widetilde{J}_h(p) - (p - \widehat{J}_h(p)), \mathrm{div}\ \mathbf{v}_h)_\Omega
$$

$$
(\mathbf{L}_h(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \mathbf{v}_h)_\Sigma = (\mathbf{L}_h(\boldsymbol{\lambda} - \widehat{J}_h(p)\mathbf{n}) - (\boldsymbol{\lambda} - \widehat{J}_h(p)\mathbf{n}), \mathbf{v}_h)_\Sigma
$$

A priori bounds

- Then

$$
\begin{aligned}
&|\mathcal{A}_h((\mathbf{I}_h(\mathbf{u}) - \mathbf{u}_h, J_h(p) - p_h, \mathbf{L}_h(\boldsymbol{\lambda}) - \boldsymbol{\lambda}_h), (\mathbf{v}_h, q_h, \boldsymbol{\mu}_h))| \\
&\quad \leq Ch^{\gamma}(\|\mathbf{u}\|_{H^{1+\gamma}(\Omega)} + \|p - \widehat{J}_h(p)\|_{H^{\gamma}(\Omega)} + h^{\frac{1}{2}-\gamma}\|\boldsymbol{\lambda} - \widehat{J}_h(p)\mathbf{n}\|_{\Sigma}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \|\!|\mathbf{v}_h, q_h, \boldsymbol{\mu}_h|\!\|
\end{aligned}
$$

- With the inf-sup condition

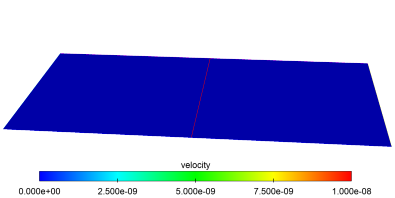
$$
\begin{aligned}
&\beta\|\!|\mathbf{I}_h(\mathbf{u}) - \mathbf{u}_h, J_h(p) - p_h, \mathbf{L}_h(\boldsymbol{\lambda}) - \boldsymbol{\lambda}_h|\!\| \\
&\quad \leq Ch^{\gamma}(\|\mathbf{u}\|_{H^{1+\gamma}(\Omega)} + \|p - \widehat{J}_h(p)\|_{H^{\gamma}(\Omega)} + h^{\frac{1}{2}-\gamma}\|\boldsymbol{\lambda} - \widehat{J}_h(p)\mathbf{n}\|_{\Sigma})
\end{aligned}
$$

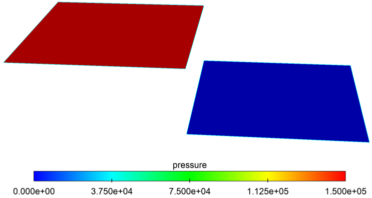
- We conclude with $\boldsymbol{\lambda} := [\![\sigma(\mathbf{u}, p)]\!]\mathbf{n} := [\![\nu\nabla\mathbf{u} - pI]\!]\mathbf{n}$ so

$$
\|\boldsymbol{\lambda} - \widehat{J}_h(p)\mathbf{n}\|_{\Sigma} \leq \nu\|[\![\nabla\mathbf{u}]\!]\|_{\Sigma} + \|[\![p - \widehat{J}_h(p)]\!]\|_{\Sigma}
$$

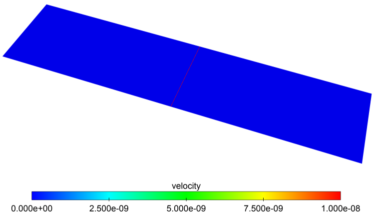
and approximation properties of the interpolations

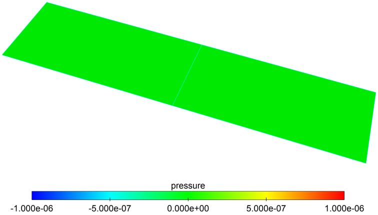
Test case :





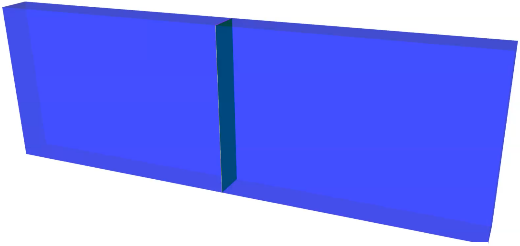
Results :

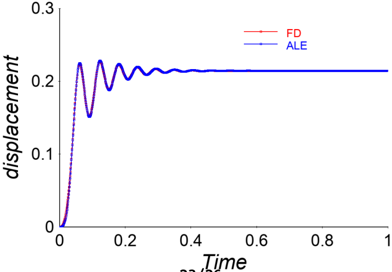




pressure jump : $[\![p_h]\!] = 1.5 \times 10^5$

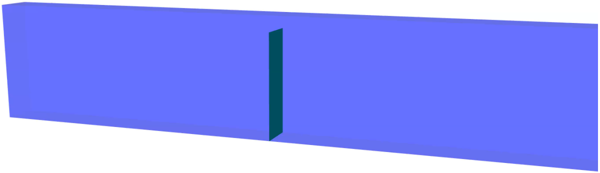
From now on : **Navier–Stokes** with symmetric stress tensor
$\sigma(\mathbf{u}, p) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) - pI$
Application to FSI : **closed valve** in a 3d setting





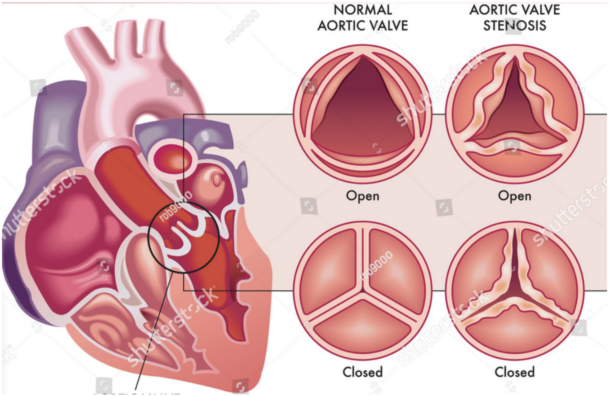
Numerical simulations (3/4)

Application to FSI : **open valve** in a 3d setting



Numerical simulations (4/4)

Application to FSI : simulation of the **aortic valve**



Wrap up

Achievements

- a scheme robust to high pressure jumps
- simulations involving Fluid-Structure Interaction
- results similar to ALE

Perspectives

- contact
- more realistic geometries
- mitral valves + atrium + ventricle

Thank you for your attention !