Transport optimal dynamique, surfaces minimales et variables de Clebsch

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How to measure a distance between two images ?

Applications:

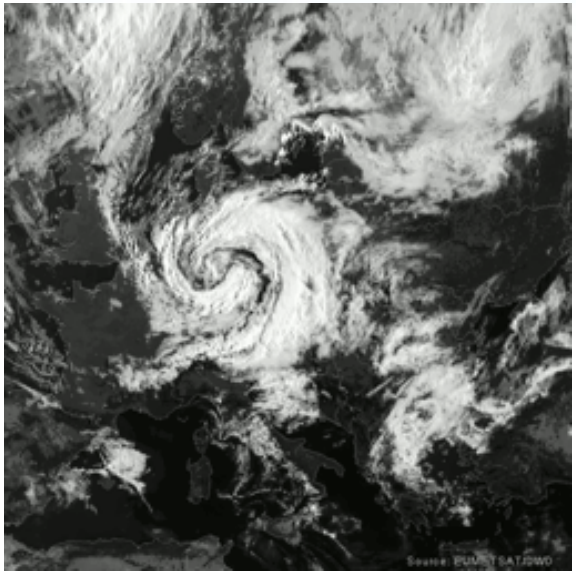
- Compare two images

  Applications to: image assimilation in weather prediction or other sciences

- Find interpolation between images, or extrapolating

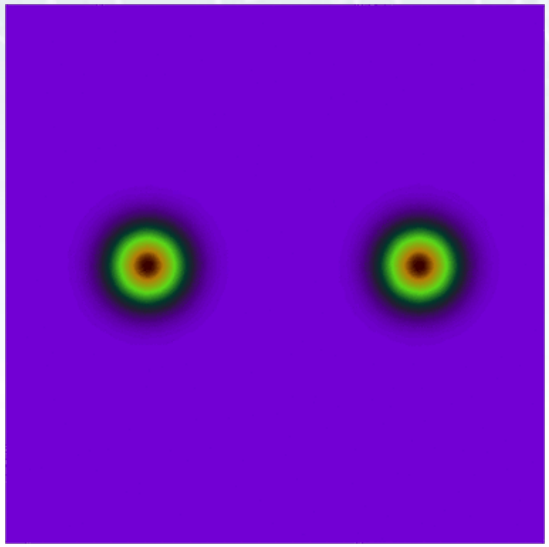
  Application to: movie creation from image sequences, extrapolation of tumor growth images ...

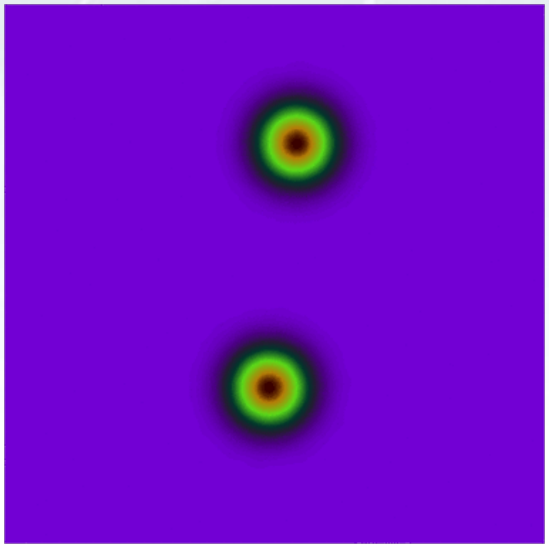
- Not so simple than in our Euclidean space ...



Images do not fit well in Euclidean space

Consider those two images (imagine biological cells for instance):

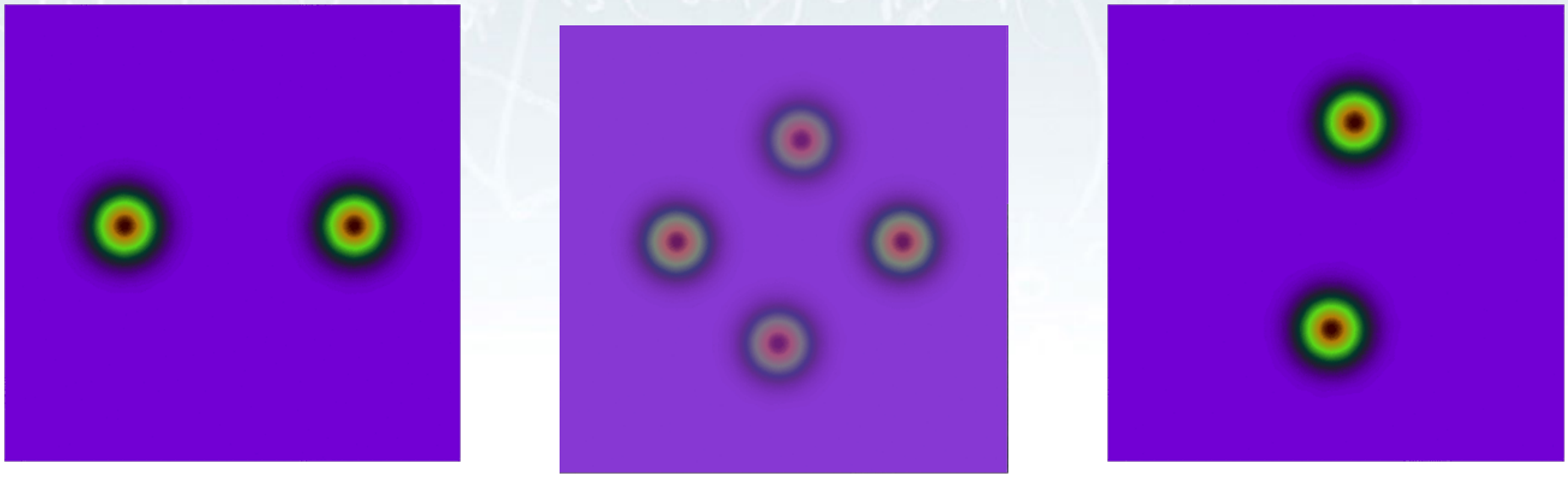




Proceeding as in Euclidean space to find a middle between two elements gives:

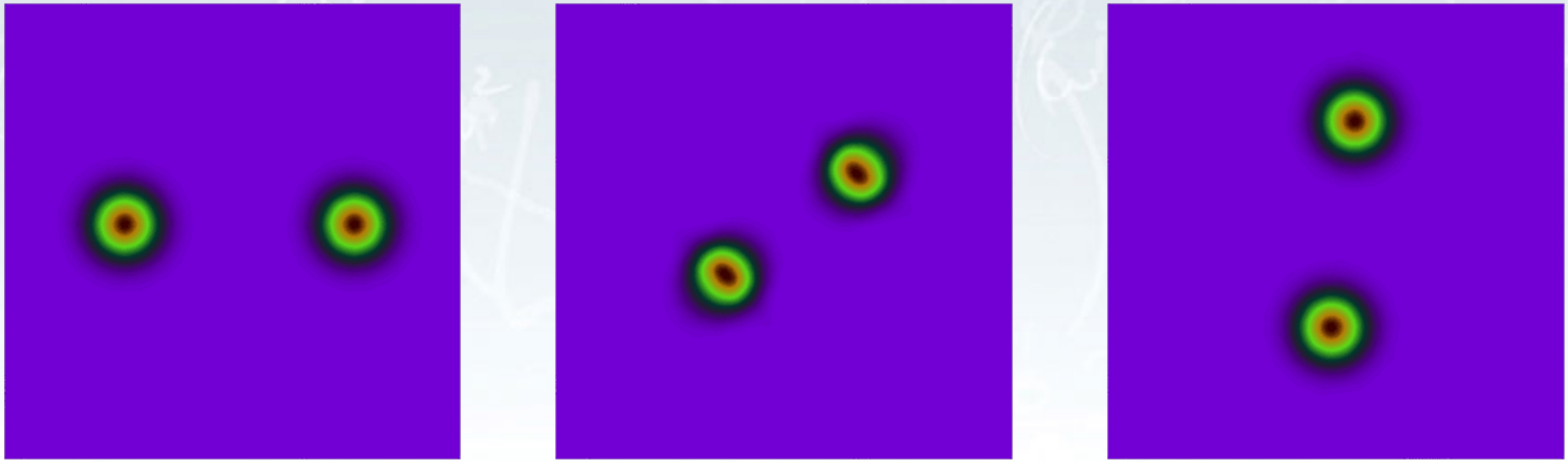
Spaces of images are not Euclidean

If it were, the middle image I would be obtained by averaging pixelwise the two images:



Is it a good notion of middle image ?
As well, L^p distance between two non-overlapping characteristic functions does not depend on the distance of their support.

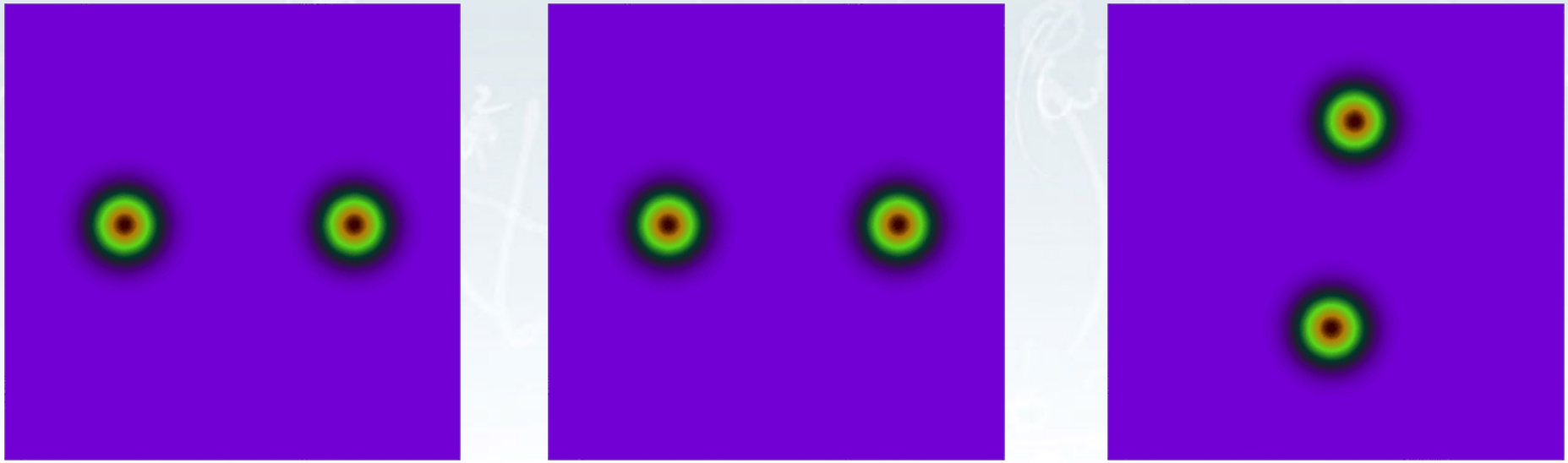
Where Physics comes into play ...



How to compute that image ?

Let's think about a sand pile ...

Optimal transport



Introduced by **Gaspard Monge** (1781) ...
... studied by many authors **Kantorovich, ... , Brenier, McCann, Villani, ...**

How to formalize this idea as an optimisation problem?

Quick intro to (static) L^2-optimal transport

Ω open bounded domain. μ, ν probability measures, absolutely continuous with respect to the Lebesgue measure, of densities ρ_0 and ρ_1, nonnegative on Ω.

- Monge problem: pushing μ to ν, through a transportation map which minimizes some cost. In the L^2 case,

$$\underset{T \in \Gamma(\rho_0, \rho_1)}{\operatorname{argmin}} \int \frac{1}{2} |T(x) - x|^2 \rho_0(x) dx$$

  where $\Gamma(\rho_0, \rho_1) = \{T : X \to Y,\ T\#\rho_0 dx = \rho_1 dx\}$. This infimum value defines $d_2(\rho_0, \rho_1)$, the L^2-KW distance between ρ_0 and ρ_1.

- Recent methods to solve efficiently this problem rely on the alternative Kantorovich form: Mérigot et al and Léger et al.

- The problem admits an unique solution $T(x)$ [Brenier, McCann], which is the gradient of a convex fonctional $\Psi : \Omega \to \mathbb{R}$, solution of the Monge-Ampère equation:

$$\det(D^2\Psi)\rho_1(\nabla\Psi(x)) = \rho_0(x).$$

  Numerical methods to solve this Monge-Ampère equation have been investigated in [Loeper-Rapetti '05, Dean-Glowinski '06, Benamou-Froese-Oberman '14]. This approach as well as other (Angenent-Haker-Tannenbaum '04, Iollo-Lombardi '11) strongly relies on the particular energy involved.

Dynamics formulation of optimal transport

- Let ρ_0, ρ_1 be two (positive) densities of unit mass on a domain $\Omega \subset \mathbb{R}^n$.
- Benamou-Brenier: Consider all functions $\rho(t,x) \geq 0$ and vector fields $v(t,x) \in \mathbb{R}^n$ solution of the continuity conditions with prescribed initial and final densities:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \qquad \rho(0,x) = \rho_0(x), \quad \rho(1,x) = \rho_1(x), \quad v \cdot n = 0 \text{ on } \partial\Omega \quad (1)$$

Theorem (Benamou-Brenier)

The following (non-convex) optimisation problem on ρ, v

$$d_2(\rho_0, \rho_1)^2 = \inf \int_0^1 \int_\Omega \frac{1}{2} \rho(t,x) |v(t,x)|^2 dx dt$$

where the inf *is taken on (ρ, v) verifying the (non-convex) constraint (1), defines a distance on the space of densities of unit mass coinciding with KW distance.*

- From now on, we are looking for this continuous sequence of densities $t \to \rho(t, \cdot)$ which links optimally ρ_0 to ρ_1 thus our problem is a space-time problem.

Convex formulation

Following Benamou & Brenier, set $m = \rho v$. The problem becomes

$$\inf_{(\rho,m)\in C_{\rho_0,\rho_1}} \int_0^1 \int_\Omega J(\rho(t,x), m(t,x))dxdt$$

where J is the (non strictly) convex proper l.s.c. function defined by

$$\forall(\rho, m) \in \mathbb{R} \times \mathbb{R}^n,\ J(\rho, m) = \begin{cases} \frac{|m|^2}{2\rho}, & \text{if } \rho > 0, \\ 0, & \text{if } (\rho, m) = (0, 0), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2}$$

and the affine set of constraints is

$$C_{\rho_0,\rho_1} \ = \ \{(\rho, m),\ \partial_t\rho + \operatorname{div} m = 0, \quad \rho(0,\cdot) = \rho_0,\ \rho(1,\cdot) = \rho_1,\ m \cdot n = 0 \text{ on } \partial\Omega\}$$

NB: 0 is an eigenvalue of the Hessian of J at $(\rho > 0, m)$, with (ρ, m) as eigenvector.

Numerical approximation

- This formulation is numerically solved by Benamou-Brenier (Numerische Math. '01) using an augmented Lagrangian method [Fortin-Glowinski] on an unstaggered discretization. We proved rigorously convergence of this algorithm under weaker assumptions (Hug-M.-Papadakis, JMAA '19) and extended it to more general energies (Hug-M.-Papadakis, 15').
  Requires one Poisson equation to solve in $\mathbb{R}^{n+1}$ per iteration.
- Oudet, Papadakis & Peyré adapted the primal dual algorithm of Chambolle-Pock '10 (CP) on a staggered discretization (SIAM Imaging '14).
  Faster (less iterations), but still requires one Poisson equation to solve in $\mathbb{R}^{n+1}$ per iteration.
- A fast algorithm for entropy regularization of the energy was introduced by Cuturi in 2013.
- In Henry, M. & Perrier we enhanced this algorithm using the CP algorithm on an Helmholtz decomposition of (ρ, m) in $\mathbb{R}^{n+1}$ (ICIP '15).
  Faster, a single Poisson equation to solve in $\mathbb{R}^{n+1}$ at initialization.
- Presently, this algorithm is further considered in a new setting which links it with minimal surfaces equation.
  Even faster, due to the simpler structure of the convex functional.

Roadmap

- Quick intro to Chambolle-Pock primal-dual algorithm
- Application to dynamic optimal transportation
- Helmholtz version of primal-dual for DOT
- Some numerical results and comparisons
- Link with minimal surfaces

Chambolle-Pock (CP) algorithm

- It applies to problems which could be written as

$$\min_{x \in X} F(Kx) + G(x)$$

  where X, Y are Hilbert spaces, $F : Y \to \mathbb{R}$ and $G : X \to \mathbb{R}$ are convex proper l.s.c. functionals, and K a linear continuous operator from X to Y.

- The min-max form is

$$\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle - F^*(y) + G(x)$$

  where $F^*(y) = \sup_{x \in X} \langle x, y \rangle - F(x)$ is the convex conjugate of F, which verifies $F^{**} = F$.

- Roughly speaking, the CP algorithm alternates gradient ascents of the concave functional $y \to \langle Kx, y \rangle - F^*(y)$ and gradient descents of the convex functional $x \to \langle Kx, y \rangle + G(x) = \langle x, K^* y \rangle + G(x)$.

- Its power lies in the ability to perform seemingless implicit iterations whenever F^* and G are easily proximable.

Chambolle-Pock (CP) algorithm

$$\min_{x\in X}\max_{y\in Y}\langle Kx,y\rangle - F^*(y) + G(x) = \min_{x\in X}\max_{y\in Y}\langle x,K^*y\rangle + G(x) - F^*(y)$$

In its simplest form (without θ parameter)

- Implicit ascent: $y^{k+1} = y^k + \sigma(Kx^k - \partial F^*(y^{k+1}))$ which corresponds to

$$y^{k+1} = (id + \sigma\partial F^*)^{-1}(y^k + \sigma Kx^k)$$

- Implicit descent: $x^{k+1} = x^k - \tau(K^*y^{k+1} + \partial G(x^{k+1})$ which corresponds to

$$x^{k+1} = (id + \tau\partial G)^{-1}(x^k - \tau K^*y^{k+1})$$

- Chambolle-Pock'10 (and numbers of following results by Peyré, Condat, ...) proved under assumption on $\sigma,\tau > 0$ such that $\sigma\tau\|K\|^2 < 1$ that this algorithm converges linearly to a solution of the min-max problem.
- Works well if the inverse maps can be easily computed. This is what we referred as easily proximable, since for a convex l.s.c. functional $H : Z \to \mathbb{R}$:

$$(id + \sigma\partial H)^{-1}(z) = \arg\min_{u\in Z}\frac{1}{2}\|u-z\|^2 + \sigma H(u) =: \mathrm{prox}_{\sigma H}(z)$$

Chambolle-Pock (CP) algorithm

Examples of easily proximable functionals:

- Indicator of a closed convex set C ($\chi_C = 0$ inside C, $+\infty$ outside):

$$\mathrm{prox}_{\chi_C}(z) = \arg\min_{u \in Z} \frac{1}{2}\|u - z\|^2 + \chi_C(u) = \arg\min_{u \in C} \frac{1}{2}\|u - z\|^2 = \mathrm{proj}_C(z)$$

- ℓ_1-norm in $\mathbb{R}^N$,

$$\mathrm{prox}_{\lambda\|\cdot\|_1}(z) = \arg\min_{u \in \mathbb{R}^N} \frac{1}{2}\|u - z\|^2 + \lambda\|u\|_1 = ([|z_i| - \lambda]_+ \,\mathrm{sgn}(z_i))_{1 \le i \le N}$$

It turns out that:

- J^* is the indicator function of a paraboloid set:

$$K = \left\{(a, b) \in \mathbb{R}^{n+1}, a + \frac{|b|^2}{2} \le 0\right\}$$

  and projection on a paraboloid set reduces to find roots of a polynomial of degree 3.

- Projection on the affine set of divergence-free vector fields C_{ρ_0,ρ_1} with given normal boundary values, amounts to solve a Poisson equation with corresponding Neumann boundary condition.

Papadakis / Peyré / Oudet algorithm

Alternates projection on the paraboloid and projection on divergence-free velocity fields. Here K, in the continuous setting, is the identity[1] $\mathbb{I}$.

Algorithm (Available on authors's GitHub)

– *Initialization :* $\tau > 0, \sigma > 0, \theta \in [0,1]$, $\mu^0 = (\rho^0, m^0), z^0 = \mathbb{I}\mu^0$ *given*
– *Iterations :*

$$z^{k+1} = \mathrm{proj}_K(z^k + \sigma \mathbb{I}\tilde{\mu}^k) \quad \textit{Paraboloid projection}$$
$$\mu^{k+1} = \mathrm{proj}_{C(\rho_0,\rho_1)}(\mu^k - \tau \mathbb{I}^* z^{k+1}) \quad \textit{Poisson equation}$$
$$\tilde{\mu}^{k+1} = \mu^{k+1} + \theta(\mu^{k+1} - \mu^k) \quad \textit{Acceleration}$$

Cost per iteration for an image size $d \times d$ pixels:
d^3 projections on K + one Poisson equation on a grid d^3.

[1] In the discrete implementation it is the interpolation operator between unstaggered and staggered grids.

Helmholtz decomposition

Idea: to avoid this Poisson equation, perform the minimization in the constraint set (which forbids also to go in the flat direction).
To work in C_{ρ_0,ρ_1}, we use the orthogonal decomposition of $L^2(Q)^{1+n}$, with $Q = (0,1) \times \Omega$,

$$(\rho, m) = \nabla \times \phi + \nabla h,$$

where unless otherwise stated we denote $\nabla = \nabla_{t,x}$. We have $\phi \in (H_0^1(Q))^3$, and $h \in H^1(Q)/\mathbb{R}$. Because (ρ, m) is divergence-free we obtain

$$\begin{cases} \Delta_{t,x} h = 0 & \text{in } Q, \\ \frac{\partial h}{\partial \nu_Q} = (\rho, m) \cdot \nu_Q & \text{on } \partial Q. \end{cases} \tag{3}$$

Then, knowing h, we can find the minimum of the energy expressed in the new unknown:

$$\mathcal{J}_h(\nabla \times \phi) = \int_0^1 \int_\Omega J_h(\nabla \times \phi(t,x)) dx dt, \tag{4}$$

where $J_h = J(\cdot + \nabla h)$ pointwise.

Our primal-Dual algorithm

The primal-dual algorithm described by Chambolle and Pock applies on (4):

$$\min_{\phi} \max_{z} \langle K\phi, z\rangle + \chi_0(\phi) - \mathcal{J}_h^*(z).$$

- $K = \nabla\times$ is the curl operator.
- $F^*(z) := J_h^*(z) = J^*(z) - \langle z, \nabla h\rangle$ thus $\mathrm{prox}_{\sigma J_h^*}(x) = \mathrm{prox}_{\sigma J^*}(x + \sigma\nabla h)$.
- $G := \chi_0 : X \to [0, +\infty)$ is the indicator function of $C_0 := \{\phi,\ \phi = 0 \text{ on } \partial Q\}$.

Algorithm (GitHub MATLAB codes + Python notebook)

– *Initialization :* $\tau, \sigma, \theta \in [0, 1]$, $(\phi^0, z^0 = K\phi^0, \tilde{\phi}^0 = \phi^0)$.
– *Iterations :*

$$z^{k+1} = \mathrm{prox}_{\sigma\mathcal{J}_h^*}(z^k + \sigma K\tilde{\phi}^k) = \mathrm{proj}_K(z^k + \sigma(\nabla\times\tilde{\phi}^k + \nabla h))$$
$$\phi^{k+1} = \mathrm{prox}_{\chi_0}(\phi^k - \tau K^* z^{k+1}) = \mathrm{proj}_{C_0}(\phi^k - \tau\nabla^*\times z^{k+1})$$
$$\tilde{\phi}^{k+1} = \phi^{k+1} + \theta(\phi^{k+1} - \phi^k)$$

Cost per iteration for an image size $d \times d$ pixels: d^3 projections on K.

A word on discretization: 1D+t case

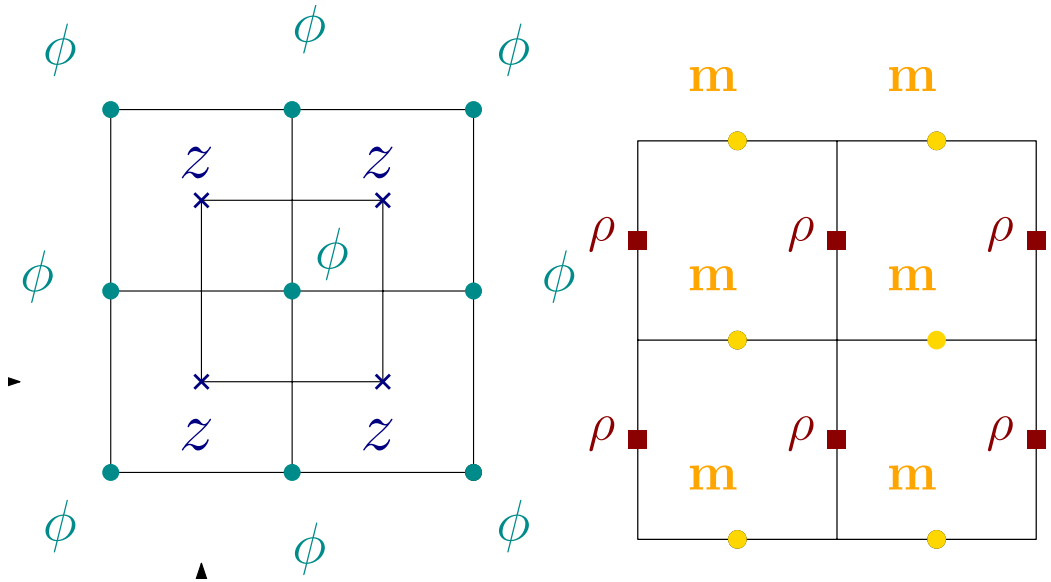
- **Centered grid.** The evaluation of the dual variable $z(t,x)$ is done on a regular grid G^{c1}, whereas the one of the primal variable $\phi(t,x)$ is done on a regular grid G^{c2}, defined by

$$G^{c1} = \{t_i, x_j\}_{1\leq i\leq M,\ 1\leq j\leq N}, \quad G^{c2} = \{t_{i-1/2}, x_{j-1/2}\}_{1\leq i\leq M+1,\ 1\leq j\leq N+1},$$

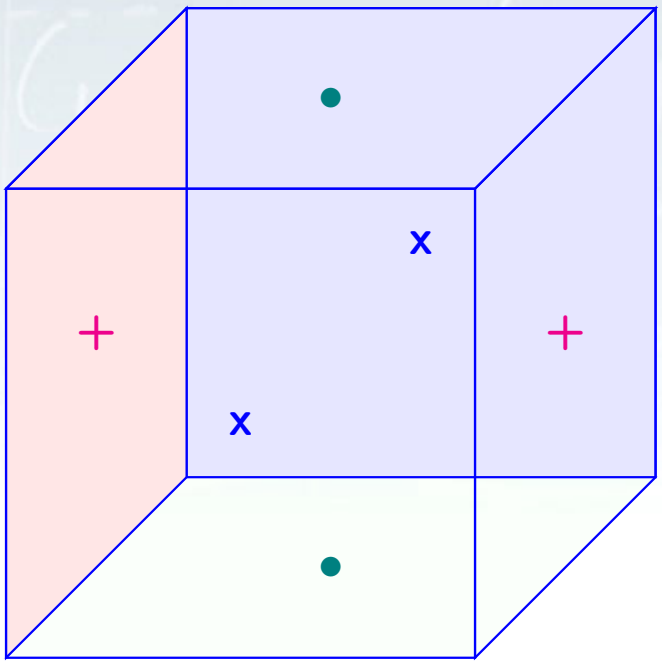
with $t_i = \frac{i}{M+1}$, $x_j = \frac{j}{N+1}$ the discrete locations in the domain $Q = (0,1)^2$.

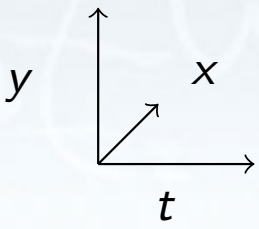
- **Staggered grid.** We now introduce the grid G^{s1}, which provides a discretization coherent with the divergence of (ρ, m) and which is defined by:

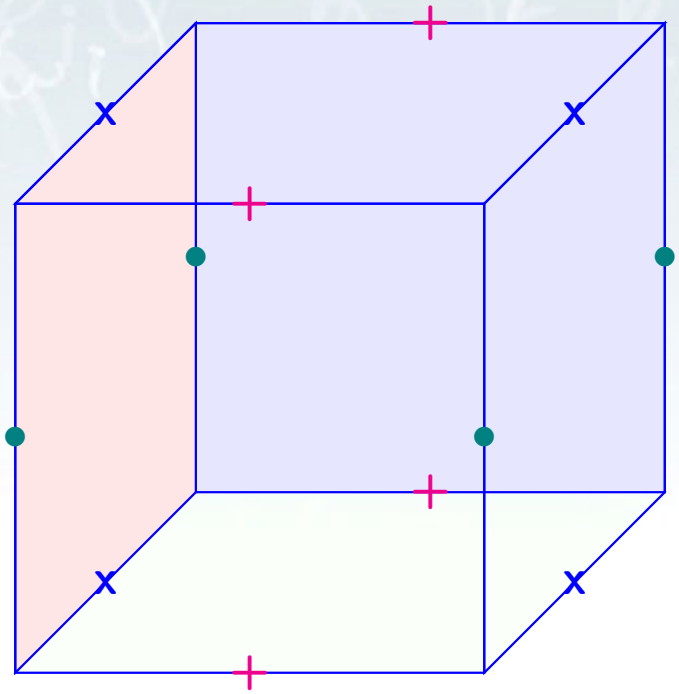
$$G_t^{s1} = \{t_{i-1/2},\ x_j\}_{1\leq i\leq M+1,\ 1\leq j\leq N}, \qquad G_x^{s1} = \{t_i,\ x_{j-1/2}\}_{1\leq i\leq M,\ 1\leq j\leq N+1}.$$



A word on discretization: 2D+t case





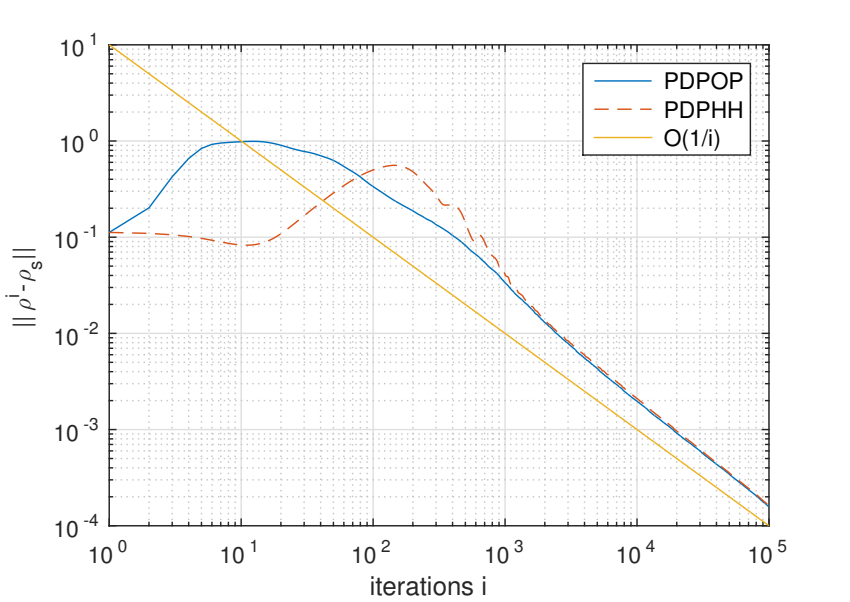


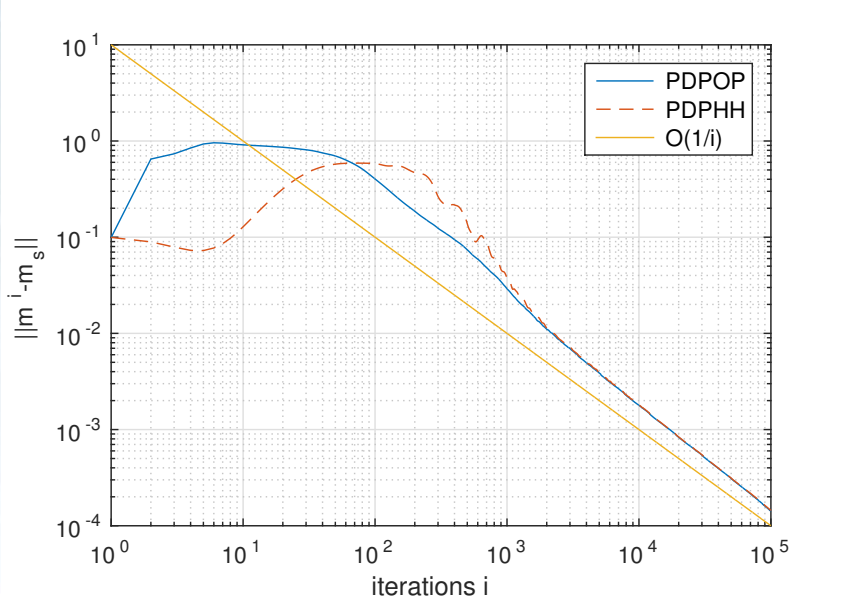
A grid G^{s_1} to evaluate $(\rho, m_1, m_2) = \nabla \times \phi + \nabla h$.

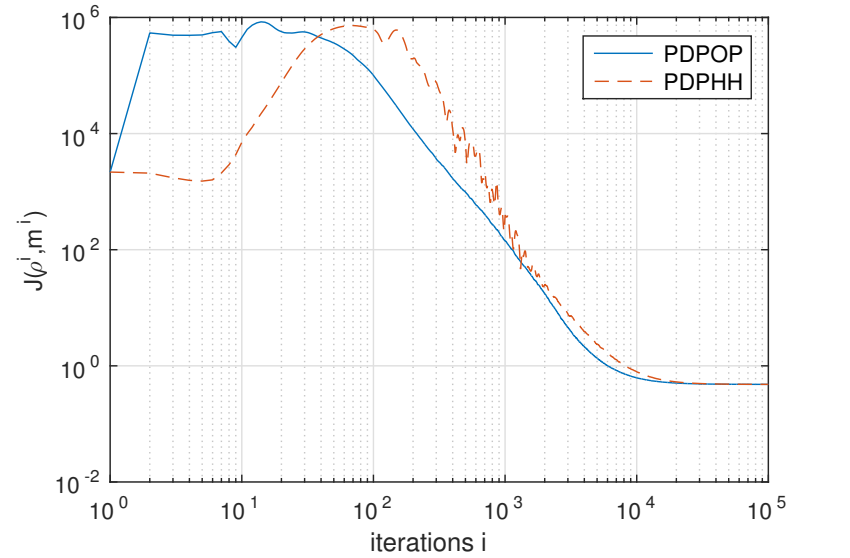
A grid G^{s_2} to define (ϕ_1, ϕ_2, ϕ_3) whose curl lives on the staggered grid G^{s_1}.

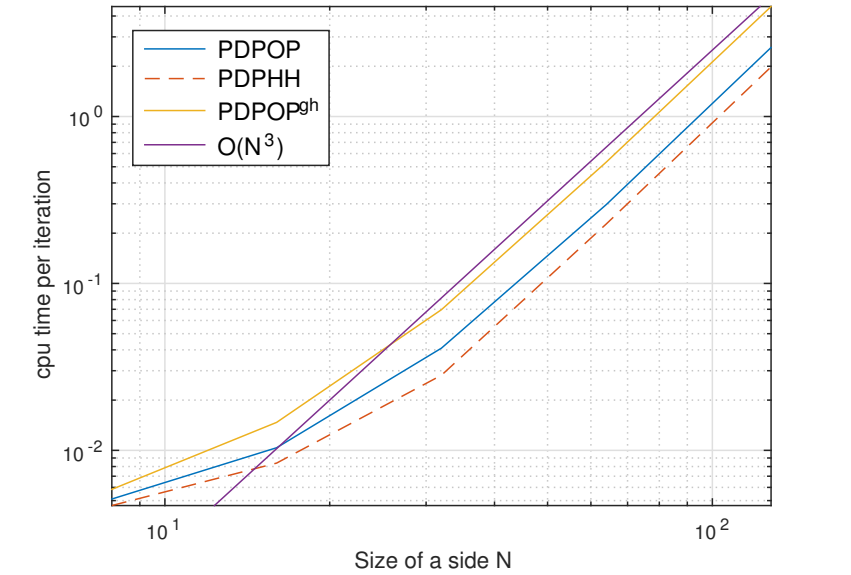
Figure: Staggered grids for 2D+t images

Numerical tests

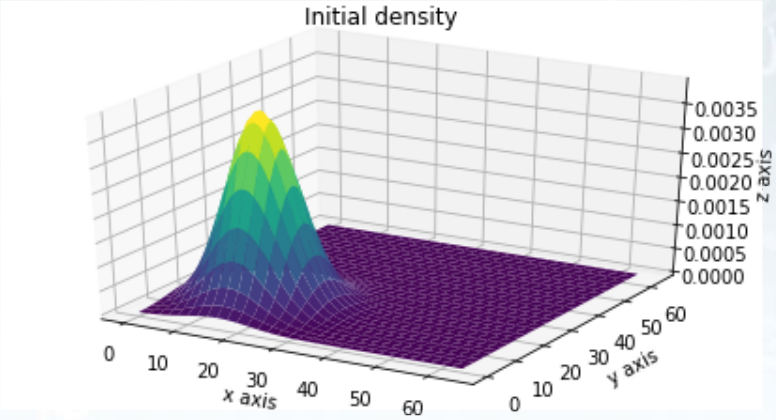


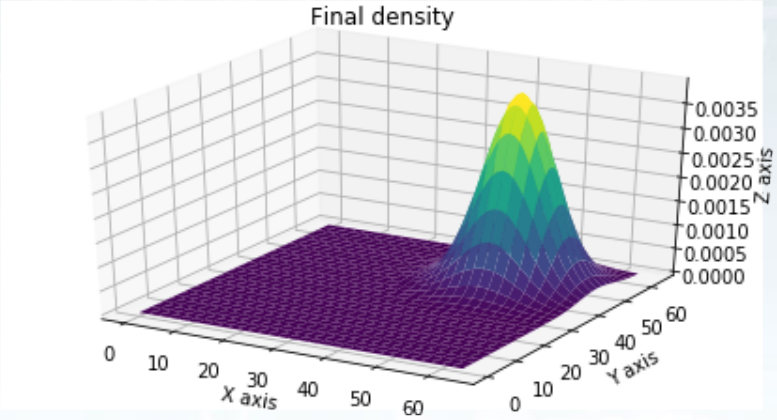


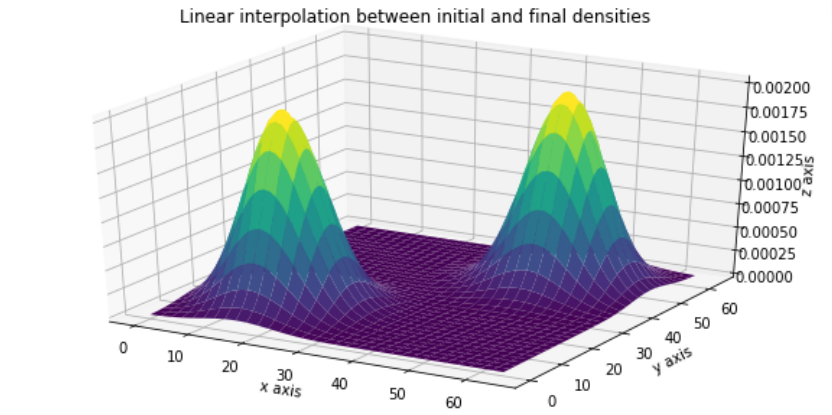


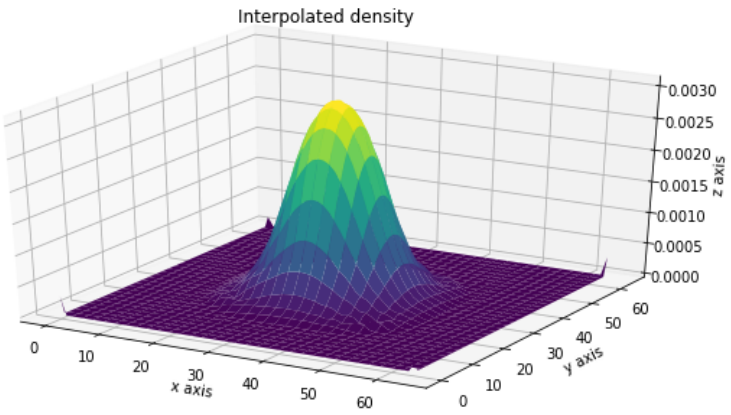


Output of (full centered) python code: time slices views









Output of (full centered) python code: spatio-temporal views

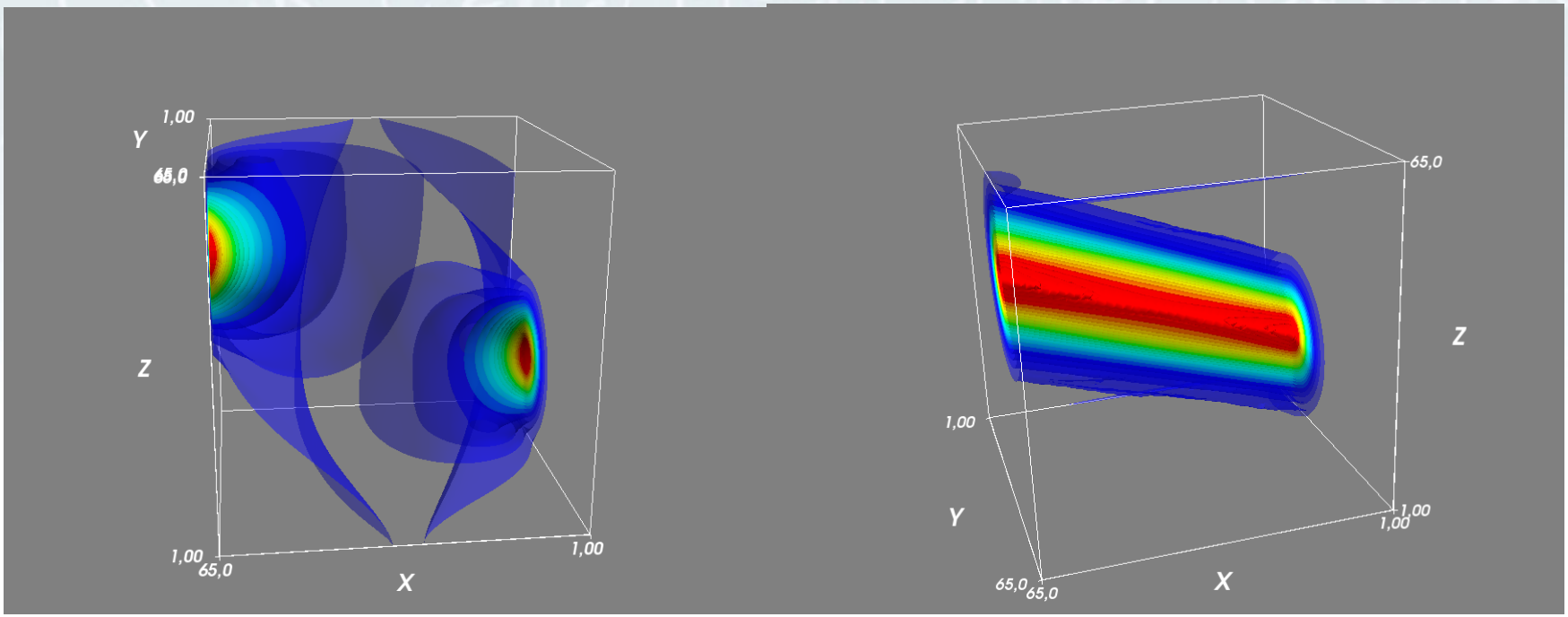


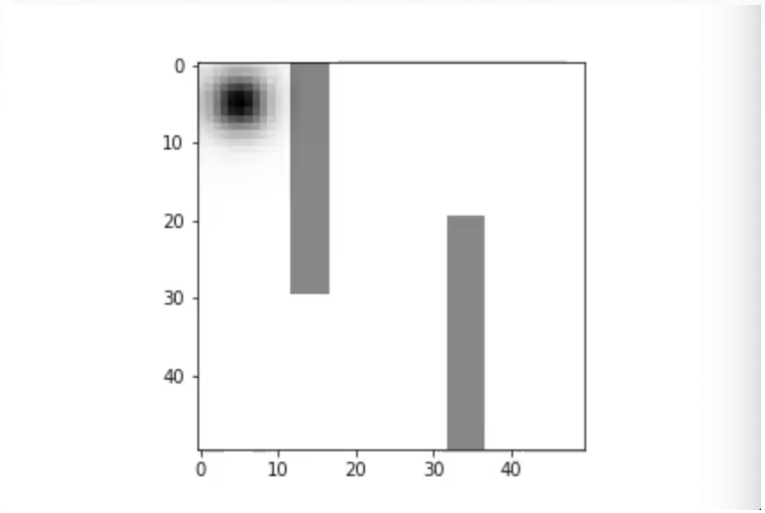
Figure: Left: initial state. Right: final state

Case with obstacles[2] (python notebook available)

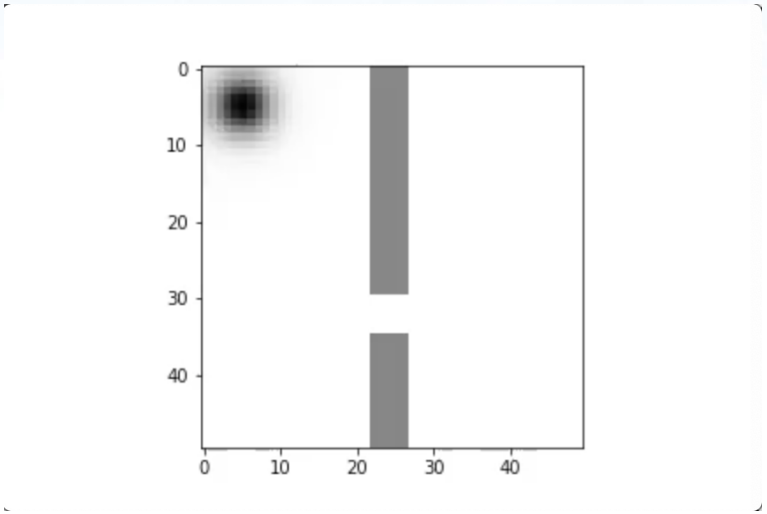
- Solve the minimization problem written in the space of constraint using Helmholtz decomposition, for a weighted functional with some weight A larger on the obstacles than outside. This amounts to look for a minimum of

$$\min_{(\rho,m)\in C_{\rho_0,\rho_1}} \int_0^1 \int_\Omega J_\alpha(\rho,m)dxdt, \qquad J_\alpha(\rho,m) = \frac{m^T A(t,x)m}{2\rho}$$

- Typical examples are isotropic stationnary or unstationary:



$A(t,x) = \alpha(x)\mathbb{I}_2$



$A(t,x) = \alpha(t,x)\mathbb{I}_2$

[2] For theory and more general constraints see (Hug, Papadakis, Maitre, M2AN '15)

Link with minimal surfaces equation: dimension 1 case

- Let us consider the case $n = 1$. Then the Helmholtz decomposition in $L^2(Q)^2$ reads:

$$(\rho, m) = \nabla \times \phi + \nabla h$$

with $\phi \in H_0^1(Q)$ and $h \in H^1(Q)/\mathbb{R}$. Here $\nabla \times \phi = (\partial_x \phi, -\partial_t \phi)$.

- Since (ρ, m) is divergence-free, h is harmonic, thus $\nabla h = \nabla \times \psi$ and $(\rho, m) = \nabla \times \Phi$.
- The Hessian matrices of $J(X_1, X_2) = \frac{X_2^2}{2X_1}$ and of $\bar{J}(X_1, X_2) = \sqrt{X_1^2 + X_2^2}$ are proportional:

$$\nabla^2 J = \frac{1}{X_1^3} \begin{pmatrix} X_2^2 & -X_1 X_2 \\ -X_1 X_2 & X_1^2 \end{pmatrix} \qquad \nabla^2 \bar{J} = \frac{1}{(X_1^2 + X_2^2)^{\frac{3}{2}}} \begin{pmatrix} X_2^2 & -X_1 X_2 \\ -X_1 X_2 & X_1^2 \end{pmatrix}$$

- Thus the OT problem is formally equivalent to

$$\min_{\Phi} \int_0^1 \int_\Omega \|\nabla \times \Phi\| dxdt = \min_{\Phi} \int_0^1 \int_\Omega \|\nabla \Phi\| dxdt = \min_{(\rho,m) \in C_{\rho_0, \rho_1}} \int_0^1 \int_\Omega \sqrt{\rho^2 + m^2} dxdt$$

whose Euler-Lagrange equation is

$$\mathrm{div}_{t,x} \frac{\nabla \Phi}{\|\nabla \Phi\|} = 0$$

Link with minimal surfaces equation: dimension 1 case

- Application with Primal-dual method:

Algorithm

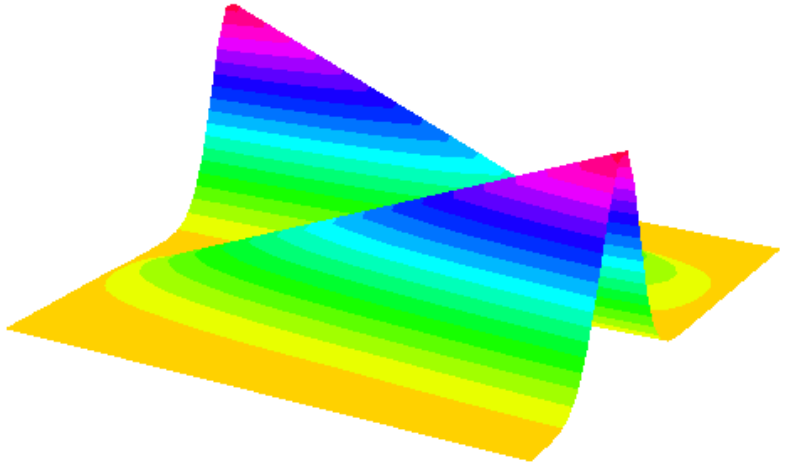
– *Initialization :* $\tau, \sigma, \theta \in [0,1]$, $(\Phi^0, z^0 = \nabla\Phi^0, \tilde{\Phi}^0 = \Phi^0)$.
– *Iterations :*

$$z^{k+1} = \mathrm{prox}_{\sigma\|\cdot\|_1^*}(z^k + \sigma K\tilde{\Phi}^k) = \mathrm{proj}_B(z^k + \sigma\nabla\tilde{\Phi}^k), \quad \mathrm{proj}_B(z) = \frac{z}{\max(\|z\|, 1)}$$

$$\Phi^{k+1} = \mathrm{prox}_{\chi_0}(\Phi^k - \tau\,\mathrm{div}\, z^{k+1}) = \mathrm{proj}_{C_0}(\Phi^k - \tau\,\mathrm{div}\, z^{k+1})$$

$$\tilde{\Phi}^{k+1} = \Phi^{k+1} + \theta(\Phi^{k+1} - \Phi^k)$$

- Cost per iteration for an "image" size d pixels: d^2 projections on the unit Ball.
- Solving the Euler-Lagrange equation by a fixed point iteration in FreeFEM++:



Link with minimal surfaces equation: dimension 2 case

- Not so simple: the hessian property does not hold anymore. Actually in $1D$ any convex cost gives the same transport, but this is not the case in higher dimensions.
- Let us consider the following energy:

$$\mathcal{E}(\rho, m) = \int_0^1 \int_\Omega \sqrt{\rho^2 + m^2} dxdt. \tag{5}$$

still minimized on the constraint set

$$C_{\rho_0,\rho_1} = \{(\rho, m),\ \partial_t \rho + \operatorname{div} m = 0, \quad \rho(0,\cdot) = \rho_0,\ \rho(1,\cdot) = \rho_1,\ m \cdot n = 0 \text{ on } \partial\Omega\}.$$

- Since $\sqrt{\rho^2 + m^2} = \rho\sqrt{1 + v^2}$, the associated cost is (verifies Ma-Trudinger-Wang property):

$$d_m(x, y) = \sqrt{1 + |x - y|^2} - 1$$

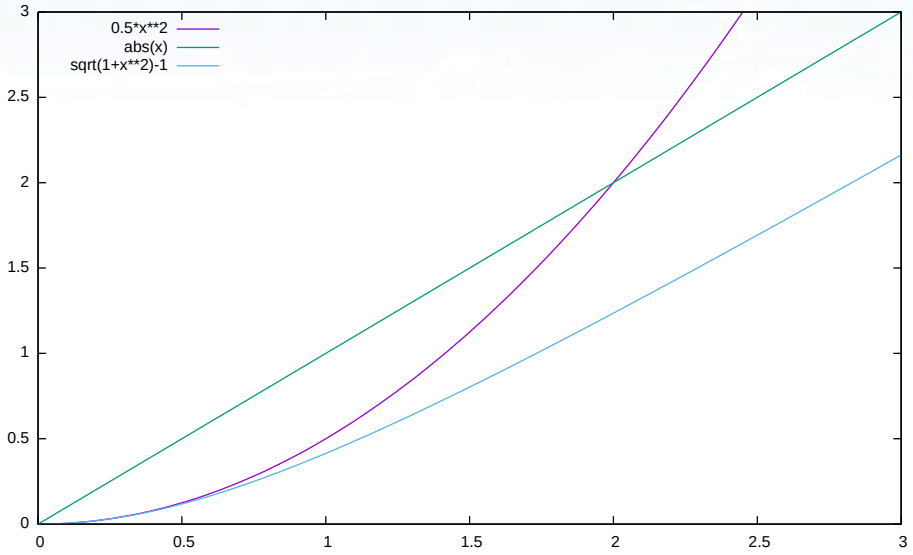
- For this cost, the Helmholtz decomposition leads to a vectorial minimal surface problem with no Poisson equation and a projection on a Euclidean ball:

$$\inf_{\phi \in (H_0^1(Q))^3} \int_0^1 \int_\Omega |\nabla \times \phi + \nabla h|_2 dxdt, \tag{6}$$

where the infimum is taken on $\phi \in (H_0^1(Q))^3$.

Link with minimal surfaces equation: dimension 2 case

- $d_m(x,y) = \sqrt{1+|x-y|^2} - 1$ is intermediate between L^1 and L^2.
- Indeed, $d_m \approx |x-y|$ for $|x-y| \gg 1$ while $d_m(x,y) \approx \frac{1}{2}|x-y|^2$ when $x \approx y$.
- Brenier[3] pointed out the the relativistic cost $d_r(x,y) = c^2\left(1-\sqrt{1-\frac{|x-y|^2}{c^2}}\right)$ ($+\infty$ if $|x-y|>c$), whose dual is $d_m(x,y) = c^2\left(\sqrt{1+\frac{|x-y|^2}{c^2}}-1\right)$ which interpolates L^1 and L^2 distances while c goes from 0 to $+\infty$.



[3] Brenier, Y. (2003). Extended Monge-Kantorovich theory. In Optimal transportation and applications (pp. 91-121). Springer, Berlin, Heidelberg.

Link with minimal surfaces equation: dimension 2 case

- The associated Lagrangian for the minimization of $\mathcal{E}$ on C_{ρ_0,ρ_1} is:

$$\mathcal{L}_{ms}(\rho, m, \psi) = \int_0^1 \int_\Omega \sqrt{\rho^2 + m^2} - \rho\partial_t\psi - m\cdot\nabla_x\psi dxdt - \int_\Omega \rho_0\psi(0,x) - \rho_1\psi(1,x)dx$$

  A saddle point (ρ, m, ψ) of $\mathcal{L}_{ms}$ is characterized by

$$\frac{\rho}{\sqrt{\rho^2+m^2}} = \partial_t\psi, \quad \frac{m}{\sqrt{\rho^2+m^2}} = \nabla_x\psi, \quad \partial_t\rho + \operatorname{div} m = 0,$$

  which leads to the Hamilton-Jacobi equation

$$|\nabla_{t,x}\psi| = 1.$$

- The counterpart of the Euler equation of L^2 OT is the stationary Euler equation in the (t, x) space,

$$(\nabla_{t,x}\psi \cdot \nabla_{t,x})\,\nabla_{t,x}\psi = 0. \tag{7}$$

  In terms of (ρ, m) our saddle point verifies:

$$\begin{cases} \partial_t\rho + \operatorname{div}_x m = 0 \\ \begin{pmatrix} \rho \\ m \end{pmatrix} \cdot \nabla_{t,x} \begin{pmatrix} \frac{\rho}{\sqrt{\rho^2+m^2}} \\ \frac{m}{\sqrt{\rho^2+m^2}} \end{pmatrix} = 0_{\mathbb{R}^3} \end{cases} \tag{8}$$

Euler and Clebsch variables in hydrodynamics

- Euler showed that any continuously differentiable vector field u of free divergence may be represented **locally** as

$$u = \nabla\phi_1 \times \nabla\phi_2 \qquad \text{(Euler form)}$$

  where ϕ_1 and ϕ_2 are scalar functions are called the Clebsch potentials (or Euler, Darboux and Pfaff[4]).
- Remark 1: this is a valuable although nonlinear and only local, generalization of the $1D + t$ situation, since $\nabla \times \phi \approx \nabla\phi \times (0,0,1)^T = \nabla\phi \times \nabla x_2$.
- Remark 2: Since for any vector field v, $\nabla \times v$ is divergence free, it can be written as

$$\nabla \times v = \nabla\phi_1 \times \nabla\phi_2 \qquad \text{(Euler form)}$$

  then $\nabla \times (v - \phi_1\nabla\phi_2) = 0$ so that any smooth vector field v can be written as

$$v = \phi_1\nabla\phi_2 + \nabla h.$$

  In that context, ϕ_1, ϕ_2, h are called Monge (!) potentials.
- Remark 3: used by e.g. Tom Hou to study singularity in Euler equations and Flavien Léger to link regularize optimal transport with Schrödinger Bridge problems.

[4] C. Truesdell, The Kinematics of Vorticity, K. Ohkitani, Nonlinearity 2018 and ref. therein

Link with minimal surfaces equation: dimension 2 case

- Our OT problem therefore boils down to minimize on ϕ_1, ϕ_2 the energy

$$\int_0^1 \int_\Omega \|\nabla_{t,x}\phi_1 \times \nabla_{t,x}\phi_2\| dxdt.$$

whose Euler-Lagrange equations are

$$\mathrm{div}_{t,x}\left(\frac{\nabla_{t,x}\phi_1 \times \nabla_{t,x}\phi_2}{\|\nabla_{t,x}\phi_1 \times \nabla_{t,x}\phi_2\|} \times \nabla\phi_i\right) = 0, \quad i = 1,2 \tag{9}$$

these equations are equivalent to a kind of minimal codim 2 surface equation:

$$\mathrm{div}_{t,x}\left(\|\nabla_{t,x}\phi_i\| \frac{P_{\nabla_{t,x}\phi_i^\perp}(\nabla_{t,x}\phi_j)}{\|P_{\nabla_{t,x}\phi_i^\perp}(\nabla_{t,x}\phi_j)\|}\right) = 0, \quad (i,j) \in \{(1,2),(2,1)\}. \tag{10}$$

- This formulation appears in the Level Set method to model a co-dimension 2 interface[5] as intersection of two hypersurfaces described by level functions ϕ_1 and ϕ_2.

5

Summary

- We proposed a method to solve the dynamic OT problem which relies on a parametrization of the constraint space, saving computational costs.
- This approach can cope with a large class of energies which could account for obstacles or physical priors in images.
- A parametrization in terms of Clebsch variables gives some insight on the geometry of the minimizers.