Existence of entropy solutions in BV^s for a non-conservative and non-strictly hyperbolic diagonal system modelling 2 dislocations in cristallography

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Analyse de dislocations, chair: Ahmad El Hajj

The model

$$\partial_t u_i + \left(\sum_{j=1}^{d} A_{ij} u_j\right) \partial_x u_i = 0 + \varepsilon \partial_x^2 u_i, \qquad i = 1, \ldots, d, \quad A \geq 0$$

Existence for "$\varepsilon = 0$" by vanishing viscosity
TVD estimates: $TV\, u_i^\varepsilon \leq TV\, u_i(0,.)$
Maryam Al Zohbi, Ahmad El Hajj, Mustapha Jazar, Nonlinearity 2021

- d Burgers type equtions, shock wave so $u_i \partial_x u_i =$ heaviside $\times$ Dirac ???
- Meaning of the limit equation ? Halmiton-Jacobi viscosity solution

2 dislocations

$$\partial_t u + (u - v)\partial_x u = 0, \tag{1}$$
$$\partial_t v + (v - u)\partial_x v = 0. \tag{2}$$

- eigenvalues $\mathcal{D} = u - v$, $-\mathcal{D}$ pb $u = v$
- Riemann invariants, u and v
- 2×2 hyperbolic systems have many entropies (Lax)
- Looking for TVD entropy solutions $U = (u, v)$
  $\forall\ \nabla^2_U \eta \geq 0$, q the associated entropy flux,
  $\partial_t \eta(U) + \partial_x q(U) \leq 0$ in the sense of distribution.
- All vanishing viscosity solutions are entropy solutions (Lax)

The key conservative equation

linear entropy $\mathcal{S} = u + v$

$$\partial_t \mathcal{S} + \partial_x \left(\frac{\mathcal{D}^2}{2} \right) = 0 \tag{3}$$

1. Define the speed of shock-wave: Rankine-Hugoniot
2. $v = constant$ (3) yields (1): $u_t + (u - v)u_x = 0$
3. $u = constant$ (3) yields (2): $v_t + (v - u)v_x = 0$

$\rightarrow$ Riemann problem $U(0, x) = U_\pm, \ \pm x > 0$,
selfsimilar solution $U = U(x/t)$, $(U_l = U_-, \ U_r = U_+)$
"almost like" a Temple system:
$\rightarrow$ elementary solutions on horizontal or vertical lines, plane (u, v)

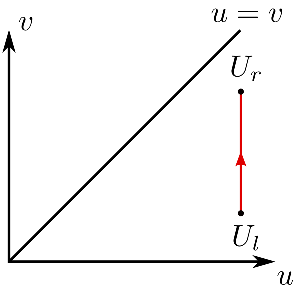
Elementary waves: 3 families

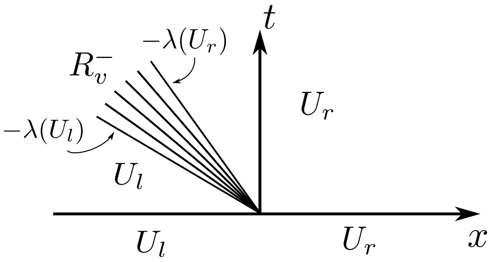
- $v = constant$,
  solve a Burgers equation $u_t + (u - constant)^2_x/2 = 0$
  unique entropy solution so the vanishing viscosity solution
  Rarefaction (R) or Shock-wave (S) $u_- < u_+$ or $u_- > u_+$
- $u = constant$ idem
- On the diagonal $\Delta = \{u = v\}$
  a double contact discontinuity (DCD)
  (not a “Temple” solution)

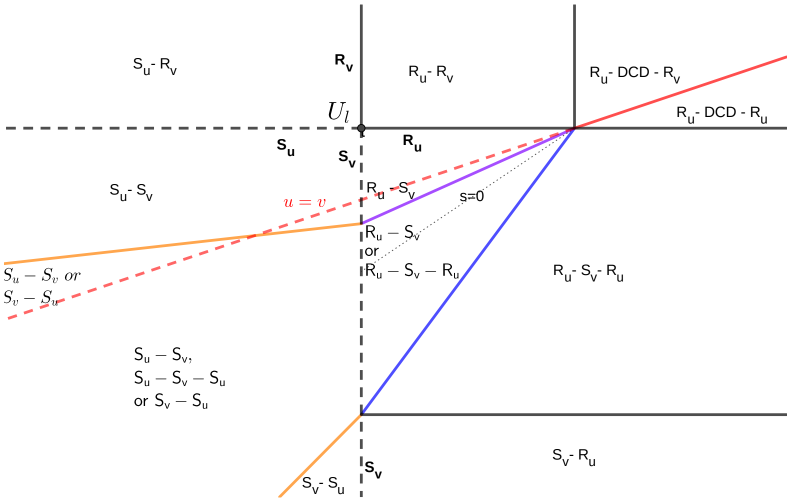
Proposition (M. Al Zohbi, S. J. 2022)

Elementary waves are the unique vanishing viscosity solutions

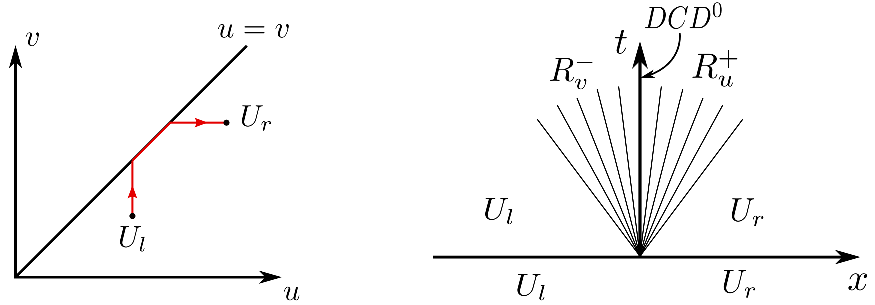
a Rarefaction with $u = constant$, unique



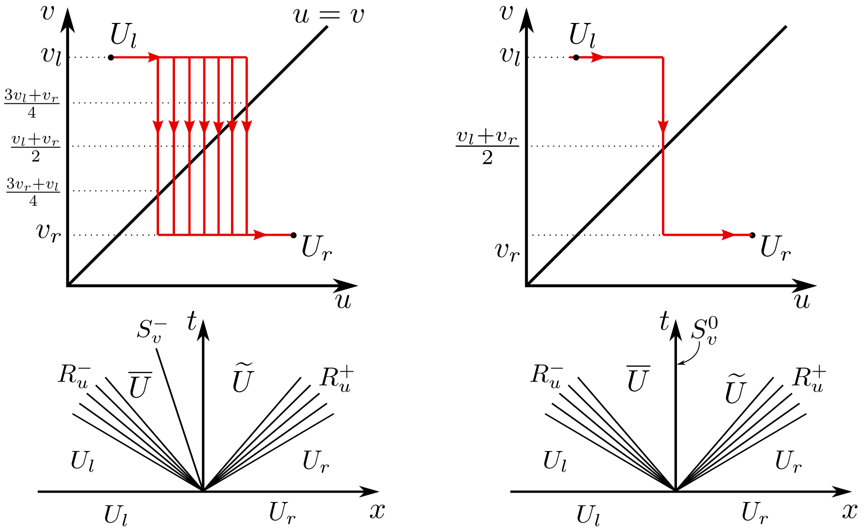



S_u- R_v
R_v
R_u- R_v
R_u- DCD - R_v
U_l
R_u- DCD - R_u
S_u
S_v
R_u
S_u- S_v
u = v
R_u - S_v
s=0
R_u − S_v
or
R_u − S_v − R_u
R_u- S_v- R_u
S_u − S_v or
S_v − S_u
S_u − S_v,
S_u − S_v − S_u
or S_v − S_u
S_v
S_v- S_u
S_v- R_u

3 waves: $R_v - DCD - R_v$, unique



3 waves: $R_u - S_v - R_u$, no uniqueness



Godunov Scheme and BV^s, $0 < s \leq 1$

$BV^s \simeq W^{s,1/s}_{loc}$ + "BV"-traces

$$TV^s u = \sup_{\cdots < x_i < x_{i+1} < \cdots} \sum_i |u(x_{i+1}) - u(x_i)|^{1/s}$$

Theorem (Existence in BV^s M. Al Zohbi, S. J. 2022)

If the initial data $u_0, v_0 \in BV^s(\mathbb{R})$, $0 < s \leq 1$, then there exists a TVD entropy solution of (1)-(2) such that:

$$u, v \in Lip^s([0,+\infty), L^1_{loc}(\mathbb{R}) \cap L^\infty([0,+\infty), BV^s(\mathbb{R})$$

- Initial data Approximated by piecewise constant data, mesh Δx
- CFL: $\sup |\mathcal{D}| \, \Delta t \leq \Delta x$
- BV and BV^s estimates A. P. Choudury, S. J. 2022

$\Rightarrow$ Continuity in time in L^1_{loc} in space and compactness

Meaning of equations

1. Only one equation: $\mathcal{S}_t + (\mathcal{D}^2)_x/2 = 0$ + entropy inequalities
2. 2×2 system for BV solutions,

$$\partial_t \mathcal{S} + \overline{\mathcal{D}} \partial_x \mathcal{D} = 0, \qquad \overline{\mathcal{D}} = \frac{D^- + D^+}{2} \tag{4}$$

$$\partial_t \mathcal{D} + \widetilde{\mathcal{D}} \partial_x \mathcal{S} = 0, \qquad [S]^2 \widetilde{\mathcal{D}} = [D]^2 \overline{D} \tag{5}$$

- BV Volpert calculus 1967, $[\mathcal{D}] = \mathcal{D}^+ - \mathcal{D}^-$, $[\mathcal{S}] = \mathcal{S}^+ - \mathcal{S}^-$
  Le Floch CPDE 1988, Dal Maso, Murat, Le Floch 1995
- Where D is continuous, $\overline{\mathcal{D}} = \mathcal{D} = \widetilde{\mathcal{D}}$
- System satisfied by
  - smooth solutions
  - solutions of Riemann problems

Prospects

1. Hamilton-Jacobi viscosity solutions?
2. Continuous solutions for non-decreasing initial data?
   El Hajj, Monneau JHDE 2010
3. Recovering unicity when the initial data does not cross the diagonal (rectangle)?
4. Vanishing viscosity solutions when $[U_-, U_+]$ crosses the diagonal $u = v$? uniqueness? numerics?
5. *d* dislocations: $\mathcal{S}_t + \left(\sum_i A_{ii} u_i^2/2 + \sum \sum_{i>j} A_{ij} u_i u_j \right)_x = 0$

   How many entropies?
   Generalization in φ-BV, if $\forall \lambda > 0,\ \varphi$- var$U_0(\lambda x) < +\infty$