Sobolev Anisotropic inequalities with monomial weights

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Introduction

Aim: Establish anisotropic Sobolev inequalities in $\mathbb{R}^n$ with a monomial weight in the general setting of rearrangement invariant spaces.

Monomial weights

$$d\mu(x) := x^A dx = |x_1|^{A_1} \cdots |x_n|^{A_n} dx. \tag{1}$$

where $A = (A_1, A_2, \ldots, A_n)$ is a vector in $\mathbb{R}^n$ with $A_i \geq 0$ for $i = 1, \ldots, n$,

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- Bakry, Gentil and Ledoux: (Analysis and Geometry of Markov Diffusion Operators, Grundlehren der Mathematischen Wissenschaften, vol. 348. Springer, Berlin (2013))

For all smooth, compactly supported function $f \in \mathbb{R}^{N-1} \times \mathbb{R}^{+}$.

$$\left[\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}_+} |u(x)|^{\frac{2(N+a)}{N+a-2}} x_N^a dx\right]^{\frac{N+a-2}{2(N+a)}} \leq S(N,a)\left[\int_{\mathbb{R}^{N-1}}\int_{\mathbb{R}_+} |\nabla u(x)|^2 x_N^a dx\right]^{\frac{1}{2}}.$$

$a \geq 0$, $N + a > 2$

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- X. Cabre and X. Ros Oton, Regularity of stable solutions up to dimension 7 in domains of double revolution. Commun. Partial Differ. Equ. 38 (2013), 135–154.

Motivated by an open question raised by Haim Brezis, Cabre and Ros–Oton studied the problem of the regularity of stable solutions to reaction-diffusion problems of double revolution and derived

Proposition 1.7. *Let $a > -1$ and $b > -1$ be real numbers, being positive at least one of them, and let*

$$D = 2 + a + b.$$

Let u be a nonnegative Lipschitz function with compact support in $\mathbb{R}^2$ such that

$$u_s \leq 0 \quad \text{and} \quad u_t \leq 0 \quad \text{in} \quad (\mathbb{R}_+)^2,$$

with strict inequality when $u > 0$. Then, for each $1 \leq q < D$ there exist a constant C, depending only on a, b, and q, such that

$$\left(\int_{(\mathbb{R}_+)^2} s^a t^b |u|^{q^*} \, dsdt \right)^{1/q^*} \leq C \left(\int_{(\mathbb{R}_+)^2} s^a t^b |\nabla u|^q dsdt \right)^{1/q}, \tag{1.10}$$

where $q^* = \frac{Dq}{D-q}$.

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- X. Cabre and X. Ros-Oton, *Sobolev and isoperimetric inequalities with monomial weights*. J. Differential Equations, **255** (2013), 4312–4336.

(1.4) $$\mathbb{R}^n_* = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_i > 0 \text{ whenever } A_i > 0\}$$

and

$$B^*_r = B_r(0) \cap \mathbb{R}^n_*.$$

For each $1 \le p < \infty$, let $W^{1,p}_0(\mathbb{R}^n, x^A dx)$ be the closure of the space of $C^1_c(\mathbb{R}^n)$ under the norm $\left(\int_{\mathbb{R}^n} x^A(|u|^p + |\nabla u|^p)dx\right)^{1/p}$.

Theorem 1.3. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and $1 \le p < D$ be a real number. Then,*

(a) *There exists a constant C_p such that for all $u \in C^1_c(\mathbb{R}^n)$,*

(1.5) $$\left(\int_{\mathbb{R}^n_*} x^A |u|^{p_*} dx\right)^{\frac{1}{p_*}} \le C_p \left(\int_{\mathbb{R}^n_*} x^A |\nabla u|^p dx\right)^{\frac{1}{p}},$$

where $p_* = \frac{pD}{D-p}$ and x^A is given by (1.1).

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Trudinger–Moser inequalities with monomial weights

Theorem 1.7. *Let A be a nonnegative vector in $\mathbb{R}^n$, $D = A_1 + \cdots + A_n + n$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then, for each $u \in C_c^1(\Omega)$,*

$$\int_\Omega \exp\left\{\left(\frac{c_1|u|}{\|\nabla u\|_{L^D(\Omega, x^A dx)}}\right)^{\frac{D}{D-1}}\right\} x^A dx \le C_2 m(\Omega),$$

where $m(\Omega) = \int_\Omega x^A dx$, and c_1 and C_2 are constants depending only on D.

- Nguyem, V.H.; Sharp weighted Sobolev and Gagliardo-Nirenberg inequalities on half-spaces and consequences, Proc. Lond. Math. Soc. 111, 127–148 (2015)
- Gurka, P. Hauer, D: More insights into the Trudinger–Moser inequality with monomial weight. Calc. Var. Partial Differential Equations 60 (2021)

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Introduction

Hardy Sobolev type inequalities with monomial weight

$$\left(\int_{\mathbb{R}^N} \left|x^B u(x)\right|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} \left|x^A \nabla u(x)\right|^{p} \mathrm{d}x\right)^{\frac{1}{p}}$$

for all $u \in C_c^\infty(\mathbb{R}^N)$.

- Castro, H: Hardy-Sobolev type inequalities with monomial weights. Ann. Mat. Pura Appl. 196, 579–598 (2017).

Introduction

The main purpose of this paper is to obtain some anisotropic Sobolev inequalities on $\mathbb{R}^n$ with monomial weight x^A in the general setting of rearrangement invariant spaces (*e.g.* L^p, Lorentz, Orlicz, Lorentz-Zygmund, etc...).

$$d\mu(x) := x^A dx = |x_1|^{A_1} \cdots |x_n|^{A_n} dx. \tag{2}$$

We observe that when $p > 1$ and $A_i < p - 1$ for all $i = 1, \cdots, n$ x^A belongs to the Muckenhoupt class A_p, but, in general the monomial weight does not satisfy the Muckenhoupt condition.

Basics

Distribution function of f

$$\mu_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}| \ \ (t \in \mathbb{R}).$$

The **decreasing rearrangement** f_μ^* of f:

$$f_\mu^*(s) = \inf\{t \geq 0 : \mu_f(t) \leq s\}.$$

$$f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds.$$

Oscillation of f

$$O_\mu(f,t) := f_\mu^{**}(t) - f_\mu^*(t)$$

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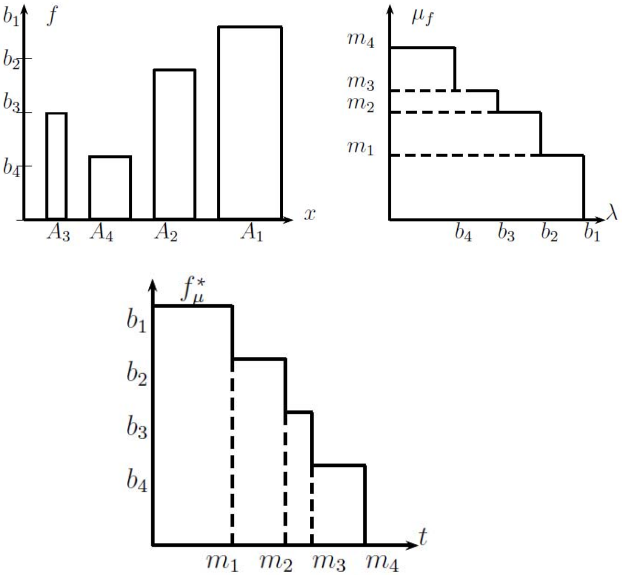

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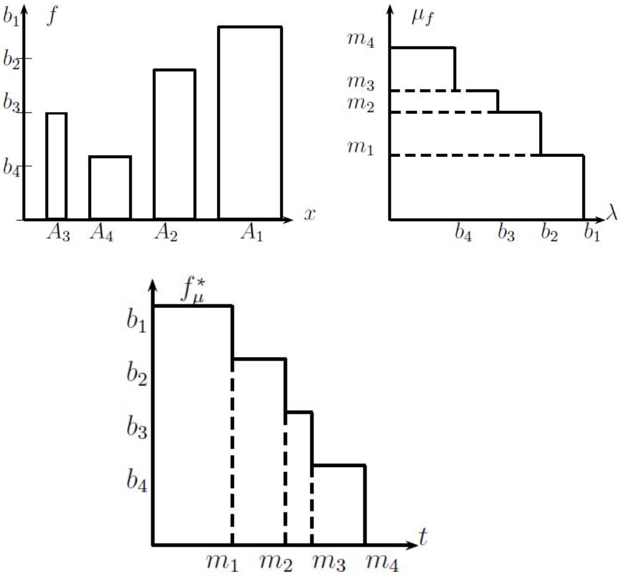

r.i. spaces

We say that a Banach function space $X = X(\mathbb{R}^n)$ on $(\mathbb{R}^n, \mu)$ is **rearrangement-invariant (r.i.) space**, if $g \in X$ implies that $f \in X$ for all $\mu-$measurable functions f such that $f^*_\mu = g^*_\mu$, and $\|f\|_X = \|g\|_X$.
Basic property:

- $$\int_{\mathbb{R}^n} |u(x)w(x)|\, d\mu \leq \int_0^\infty u^*_\mu(t) w^*_\mu(t)\, dt.$$
- $$\int_0^r f^*_\mu(s) ds \leq \int_0^r g^*_\mu(s) ds \quad \forall r > 0 \Rightarrow \|f\|_X \leq \|g\|_X$$

A r.i. space $X(\mathbb{R}^n)$ can be represented by a r.i. space on $(0, +\infty)$, with Lebesgue measure, $\bar{X} = \bar{X}(0, \infty)$, such that

$$\|f\|_X = \|f^*_\mu\|_{\bar{X}},$$

r.i. spaces

Lorentz spaces: $(0 < p < \infty,\ 0 < q \leq \infty)$

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty \left(t^{1/p} f_\mu^*(t) \right)^q \frac{dt}{t} \right)^{1/q}.$$

$$\|f\|_{L^{p,p}} = \left(\int_0^\infty \left(t^{1/p} f_\mu^*(t) \right)^p \frac{dt}{t} \right)^{1/p} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, d\mu \right)^{1/p}.$$

Lorentz-Zygmung spaces: $(0 < q \leq \infty)$

$$\|f\|_{L^{\infty,q}(logL)^{-1}} = \left(\int_0^\infty \left(\frac{f_\mu^{**}(t)}{1 + log^+\left(\frac{1}{t}\right)} \right)^q \frac{dt}{t} \right)^{1/q}.$$

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The generalized Lorentz spaces $\Lambda^{p,q}(w)$

$$\|f\|_{\Lambda^{p,q}(w)} := \left(\int_0^\infty \left(t^{1/p} f_\mu^*(t) \right)^q w(t) \frac{dt}{t} \right)^{1/q} < \infty. \qquad (3)$$

The Gamma space $\Gamma^p(w)$

$$\|f\|_{\Gamma^p(w)} := \left(\int_0^\infty f_\mu^{**}(s)^p w(s) ds \right)^{1/p} < \infty. \qquad (4)$$

Orlicz spaces:

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left(\frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

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If X is a r.i. space, the $p-$**convexification** $X^{(p)}$ of X, is the r.i. space defined $X^{(p)} = \{f : |f|^p \in X\}$ endowed with the following norm

$$\|f\|_{X^{(p)}} = \||f|^p\|_X^{1/p}, \ 1 \leq p < \infty. \tag{5}$$

Main Result

Let A be a non negative vector in $\mathbb{R}^n$, $D = A_1 + A_2, \cdots + A_n + n$.

$$d\mu(x) := x^A dx = |x_1|^{A_1} \cdots |x_n|^{A_n} dx.$$

There are equivalent

- (Poincaré inequality)

$$\|f\|_{L_\mu^{\frac{D}{D-1}}} \preceq \sum_{i=1}^{n} \|f_{x_i}\|_{L_\mu^1}. \tag{6}$$

- $$\int_0^t \left(O_\mu(f,\cdot)(\cdot)^{-\frac{1}{D}}\right)^*(s)ds \preceq \int_0^t \prod_{i=1}^{n} \left[\left(\frac{d}{ds}\int_{\{|f|>f_\mu^*(s)\}} \left|f_{x_i}\right| d\mu\right)^*(\tau)\right]^{\frac{A_i+1}{D}} d\tau$$

Main Result

X r.i. space, then

$$\left\| O_\mu(f,t)t^{-\frac{1}{D}} \right\|_{\bar{X}} \preceq \left\| \prod_{i=1}^{n} \left[\left(\frac{d}{ds} \int_{\{|f|>f_\mu^*(s)\}} |f_{x_i}|\, d\mu \right)^* (\tau) \right]^{\frac{A_i+1}{D}} (t) \right\|_{\bar{X}} . \tag{7}$$

Given $p_1, \cdots, p_n \geq 1$, let $\overline{p}$ be the weighted harmonic mean between $p_1, \cdots, p_n$, *i.e.*

$$\frac{1}{\overline{p}} = \frac{1}{D} \sum_{i=1}^{n} \frac{A_i + 1}{p_i}, \tag{8}$$

then

$$\| O_\mu(f,t)t^{-\frac{1}{D}} \|_{\bar{X}^{(\overline{p})}} \preceq \prod_{i=1}^{n} \| f_{x_i} \|_{X^{(p_i)}}^{\frac{A_i+1}{D}} , \tag{9}$$

$$(\tilde{f}_{x_i})^*_\mu(t) := \left(\frac{d}{ds}\int_{\{|f|>f^*_\mu(s)\}} |f_{x_i}|\, d\mu\right)^*(t).$$

Lemma

Let $f \in W^{1,1}_0(\mathbb{R}^n, \mu)$, then

$$\int_0^t (\tilde{f}_{x_i})^*_\mu(\tau)d\tau \leq \int_0^t |f_{x_i}|^*_\mu(\tau)d\tau, \quad (t \geq 0)$$

therefore for any r.i space X on $(\mathbb{R}^n, \mu)$ we have that

$$\left\|(\tilde{f}_{x_i})^*_\mu\right\|_{\bar{X}} \leq \left\||f_{x_i}|^*_\mu\right\|_{\bar{X}} = \|f_{x_i}\|_X .$$

Lemma

Let X be a r.i. space and let $0 \leq \theta_i \leq 1$ such that $\sum_{i=1}^n \theta_i = 1$, then

$$\|\prod_{i=1}^n |f_i|^{\theta_i}\|_X \leq \prod_{i=1}^n \|f_i\|_X^{\theta_i}.$$

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From oscillation to embeddings

Main difficulty: the oscillation depends neither on the growth of f^* nor on f^{**}.

There is a weight w (which depends on X and D)

$$\|f^{**}(t)w(t)\|_{\bar{X}^{(\overline{p})}} \preceq \|O_\mu(f,t)t^{-\frac{1}{D}}\|_{\bar{X}^{(\overline{p})}},$$

Examples: L^p

Let $p_1, \cdots, p_n \geq 1$ and $f \in \; C_c^1(\mathbb{R}^n)$.
i) If $\;\overline{p} < D$, then

$$\|f\|_{L_\mu^{\overline{p}^*,\overline{p}}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}}, \tag{10}$$

where $\overline{p}$ is defined as in (8) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\overline{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(t)}{1+\ln\frac{1}{t}}\right)^D \frac{dt}{t}\right)^{1/D} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}. \tag{11}$$

iii) If $\overline{p} > D$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}. \tag{12}$$

Examples: $L^{p,q}$

Let $f \in C_c^1(\mathbb{R}^n)$, $p_1, \cdots, p_n \geq 1$ and $q_1, \cdots, q_n \geq 1$.
i) If $\bar{p} < D$, then

$$\|f\|_{L_\mu^{\overline{p}^*,\bar{q}}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}},$$

where $\bar{p}$ and $\bar{q}$ are defined as in (8) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\bar{p} > D$, then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

iii) If $\bar{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(s)}{1+\ln\frac{1}{s}}\right)^{\bar{q}} \frac{ds}{s}\right)^{1/\bar{q}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{L_\mu^{p_i,q_i}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1+L^\infty}.$$

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Examples: Lorentz-Zygmund spaces

Let $f \in C_c^1(\mathbb{R}^n)$, $p_1, \cdots, p_n \geq 1$, $q_1, \cdots, q_n \geq 1$ and $\alpha \in \mathbb{R}$.
i) If $\overline{p} < D$, then

$$\|f\|_{L_\mu^{\overline{p}^*, \overline{q}}(\log L)^\alpha} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i, q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}}.$$

where $\bar{p}$ and $\bar{q}$ are defined as in (8) and $\overline{p}^* = \frac{\overline{p}D}{D-\overline{p}}$.
ii) If $\overline{p} > D$, then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i, q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1 + L^\infty}.$$

iii) If $\overline{p} = D$, then

$$\left(\int_0^1 \left(\frac{f_\mu^{**}(s)}{1+\ln\frac{1}{s}}\right)^{\bar{q}} \left(1+\ln\frac{1}{s}\right)^\alpha \frac{ds}{s}\right)^{1/\bar{q}} \preceq \prod_{i=1}^n \|f_{x_i}\|_{L_\mu^{p_i, q_i}(\log L)^\alpha}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1 + L^\infty}$$

General r.i. spaces

Boyd indices:

$$\bar{\alpha}_X = \inf_{s>1} \frac{\ln h_X(s)}{\ln s} \quad \text{and} \quad \underline{\alpha}_X = \sup_{s<1} \frac{\ln h_X(s)}{\ln s},$$

where $h_X(s)$ denotes the norm of the compression/dilation operator E_s on $\bar{X}$, defined for $s > 0$, by $E_s f(t) = f^*(\frac{t}{s})$.

Examples

If $X = L^p_\mu$, $X = L^{p,q}_\mu$ or $X = L^{p,q}_\mu(logL)^\alpha$, then $\bar{\alpha}_X = \underline{\alpha}_X = \frac{1}{p}$.

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Examples

If $X = L^p_\mu$, $X = L^{p,q}_\mu$ or $X = L^{p,q}_\mu(logL)^\alpha$, then $\bar{\alpha}_X = \underline{\alpha}_X = \frac{1}{p}$.

General r.i. spaces

Let X be a r.i. space on $(\mathbb{R}^n, \mu)$ and $f \in C_c^1(\mathbb{R}^n)$. If $p_1, \cdots, p_n \geq 1$, then

$$\|t^{-1/D}[f_\mu^{**}(t) - f_\mu^*(t)]\|_{\bar{X}^{(\bar{p})}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{D}\sum_{i=1}^{n} \frac{A_i+1}{p_i},$$

i) if $\underline{\alpha}_X > \frac{\overline{p}}{D}$, then

$$\|t^{-1/D} f_\mu^{**}(t)\|_{\bar{X}^{(\bar{p})}} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}};$$

ii) if $\bar{\alpha}_X < \frac{\overline{p}}{D}$, then

$$\|f\|_{L^\infty} \preceq \prod_{i=1}^{n} \|f_{x_i}\|_{X^{(p_i)}}^{\frac{A_i+1}{D}} + \|f\|_{L_\mu^1 + L^\infty}.$$

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Optimality

Given $p \geq 1$,

$$A(p) = \left\{ \hat{q} = (q_1, \cdots, q_n) \in \mathbb{R}^n : q_i \geq 1 \text{ for } i = 1, \cdots, n \text{ and } \frac{1}{D} \sum_{i=1}^{n} \frac{A_i + 1}{q_i} = \frac{1}{p} \right\}.$$

Let X, Y be two r.i. spaces on $(\mathbb{R}^n, \mu)$ and $p \geq 1$. The next statements are equivalent:

i) The Hardy type operator $\bar{Q}_D f(t) := \int_t^\infty s^{1/D} |f(s)| \frac{ds}{s}$ is bounded from $\bar{X}^{(p)}$ to $\bar{Y}$.

ii) There is a positive constant c, such that for all $\hat{q} \in A(p)$ and $u \in C_c^1\left(\overline{\mathbb{R}_*^n}\right)$ we get

$$\|u\|_Y \leq c \prod_{i=1}^{n} \|u_{x_i}\|_{X^{(q_i)}}^{\frac{A_i+1}{D}}. \qquad (13)$$

iii) For all $u \in C_c^1\left(\overline{\mathbb{R}_*^n}\right)$ we get

$$\|u\|_Y \preceq \|t^{-1/D}[u_\mu^{**}(t) - u_\mu^*(t)]\|_{\bar{X}^{(p)}}.$$

- Feo, F.; Martín, J.; Posteraro, M. R. Sobolev anisotropic inequalities with monomial weights. J. Math. Anal. Appl. 505 (2022), no. 1, Paper No. 125557, 30 pp.

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Thank you very much for your attention.

Questions?