Numerical solving of an optimal control problem in large time in presence of a turnpike: a missile guidance problem

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CANUM 2022, Evian Les Bains June 2022

June 15, 2022

Missile guidance problem

Cruise missile trajectory optimization in the vicinity of a target.

Cruise missile: a self propelled endo-atmospheric aerial vehicle maneuvering using its aerodynamic surfaces.

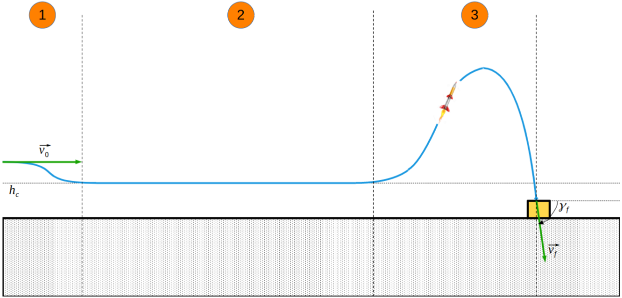
To avoid exposure to ennemy radars, it flies most of the time at very low altitude, except in the very vicinity of the target.

It aims to hit a ground target under specified flight conditions (speed, flight path angle).

Study limited to the vertical plane (most critical).

Typical trajectory shape

1. initial transitory phase: the vehicle joins the cruise altitude
2. "cruise phase": the missile remains close to h_c
3. "bunt phase": climb and dive onto the target



Dynamical system modelling

$$\dot{x} = v.\cos\gamma \tag{1a}$$

$$\dot{h} = v.\sin\gamma \tag{1b}$$

$$\dot{v} = \frac{T_{\max}.(1 + C_s.v).u_1 - D(h,v)}{m} - g.\sin\gamma \tag{1c}$$

$$\dot{\gamma} = \frac{L(h,v)}{m.v} - \frac{g.\cos\gamma}{v} \tag{1d}$$

$$\dot{m} = -C_s.T_{\max}.u_1 \tag{1e}$$

The aerodynamic forces (drag & lift) are given by:

$$L(h,v) = \bar{q}(h,v).S.C_l \tag{2a}$$

$$D(h,v) = \bar{q}(h,v).S.(C_d + k_{cz}.C_l^2) \tag{2b}$$

Dynamical system modelling

$\bar{q} := \frac{1}{2}\rho(h).v^2$ is the dynamic pressure, $T_{\max}$ is the maximum thrust, C_l, C_d and k_{cz} are aerodynamic coefficients, S is the cross section of the missile and $\rho(.)$ is the air density.

The control is in dimension 2: the thrust throttle $u_1(.)$ and the lift coefficient $C_l := u_2(.)$.

The controls are bounded due to the vehicle physical limitations: $u_1(t) \in [\eta, 1]$, with $\eta \in]0, 1[$ and $|u_2(t)| \leqslant u_2^m$.

Notations: $\xi := (x \;\; h \;\; v \;\; \gamma \;\; m)^T$, the dynamics f is given by (1) $u := (u_1, u_2)$ and $U := [\eta, 1] \times [-u_2^m, u_2^m]$

Optimal control problem

$$
\textbf{(OCP)}\begin{cases}
\min\limits_{(t_f, u\in U)} \displaystyle\int_0^{t_f} \left(k_0 + k_1.\frac{(h(s)-h_c)^2}{h_c^2} + k_2.u_2(s)^2\right) ds \\
\dot{\xi}(s) = f(\xi(s), u(s)) \qquad \forall s \in [0, t_f] \\
\xi(0) = \xi_0, \quad \xi(t_f) = \xi_f \\
u(s) \in U \qquad \forall s \in [0, t_f]
\end{cases} \tag{3}
$$

where:

- $\xi_0 = (x_0 \;\; h_0 \;\; v_0 \;\; \gamma_0 \;\; m_0)^T$ and $\xi_f = (x_f \;\; h_f \;\; v_f \;\; \gamma_f \;\; *)^T$ are the prescribed initial and final states.
- h_c is the cruise altitude
- $(k_0, k_1, k_2) \in \mathbb{R}^3$ is the weight triple in the performance index.

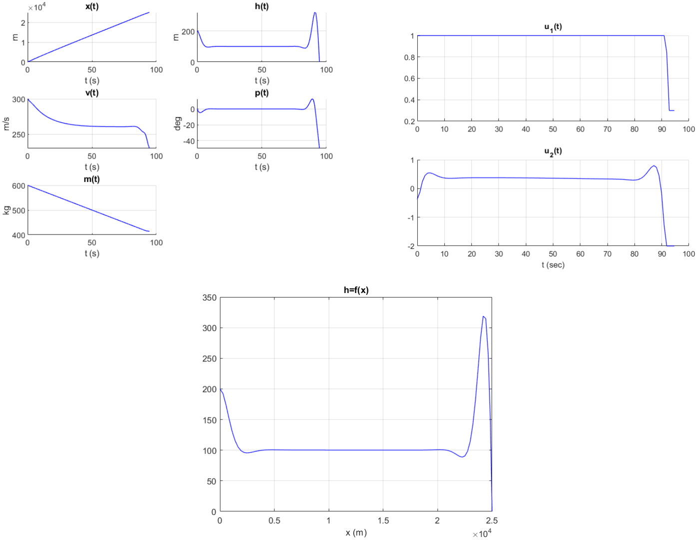
Direct method

▷ Get the global structure of the optimal control by solving the problem using a direct method (*Ipopt* solver under the automatic differentiation code *AMPL*)

▷ Numerical values of the boundary conditions:

$$(x_0, h_0, v_0, \gamma_0, m_0) = (0\text{m}, 200\text{m}, 300\text{m/s}, 0^\circ, 600\text{kg})$$
$$(x_f, h_f, v_f, \gamma_f, m_f) = (25000\text{m}, 0\text{m}, 230\text{m/s}, -50^\circ, *)$$
$$k_0 = k_1 = k_2 = 0.1$$

Numerical results using Ipopt under AMPL



Comments

▷ Crank Nicholson numerical scheme. Very reasonable computation time (5sec) on the NEOS server with a "naive" initialisation.

▷ Main output: $t_f \simeq 94.6s$, $t_s \simeq 92s$, Cost $\simeq 14.3$

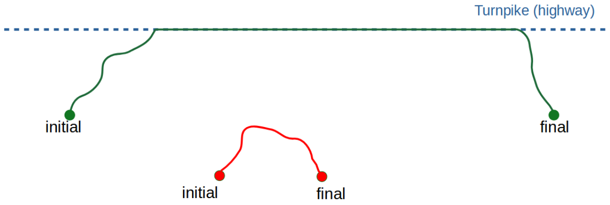
▷ The optimal structure of the $u_1(.)$ is of bang-bang nature $u_1 = 1$ then $u_1 = \eta$. In what follows, we exploit this without proving it.

▷ *Partial turnpike phenomenon* on h et γ state variables due to the cruise altitude constraint in the cost.

Turnpike property

Turnpike property

- The solution of an optimal control problem in large time should spend most of its time near a steady-state.
- In infinite horizon the solution should converge to that steady-state.



Recent literature

Exponential turnpike → *Trelat, Zuazua, JDE 2015* (finite dim.); *Trélat, Zhang, Zuazua, SICON 2018* (infinite dim.)

Measure turnpike results for dissipative systems → *Bonvin Faulwasser 2016; Grune 2018; Trélat Zhang MCSS 2018*

Partial turnpike → *Trélat 2021; Aftalion Trélat 2021*

Pontryagin Maximum Principle

The Hamiltonian: $H(\xi, p, p^0, u) := \langle p, f(\xi, u)\rangle + p^0.f^0(\xi, u)$

The optimal triple $(\xi^\star(.), p^\star(.), u^\star(.))$ of (OCP) checks:

$$\dot{\xi}^\star(t) = \frac{\partial H}{\partial p}(\xi^\star(t), p^\star(t), p^0, u^\star(t)), \tag{4a}$$

$$\dot{p}^\star(t) = -\frac{\partial H}{\partial \xi}(\xi^\star(t), p^\star(t), p^0, u^\star(t)) \tag{4b}$$

$$u^\star(t) = \arg\max_{v \in U} H(\xi^\star(t), p^\star(t), p^0, v) \tag{4c}$$

Extremal state constraints: $\xi^\star(0) = \xi_0$ and $\xi^\star(t_f^\star) = \xi_f$

Transversality condition: $p_m^\star(t_f^\star) = 0$

Terminal condition: $H(\xi^\star(t_f^\star), p^\star(t_f^\star), p^0, u^\star(t_f^\star)) = 0$

Assumption: $p^0 = -1 \rightarrow$ no abnormal case.

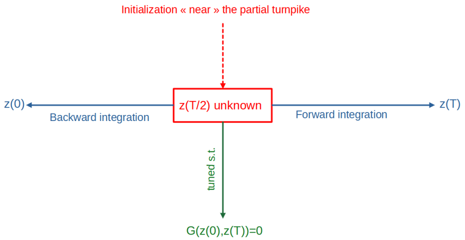
Shooting function

From the maximisation condition (4c): $u = \Pi(\xi, p)$, the numerical solving of (4) is equivalent to find $z = (\xi, p)$ s.t:

$$\dot{z}(t) = F(z(t)) \text{ and } G(z(0), z(t_f)) = 0 \tag{5}$$

Usual implementation: $z(0)$ unknown tuned such that (5) is true.

- ▷ When turnpike, approach losing its efficiency when T large.
- ▷ New variant (see *Trelat, Zuazua, JDE 2015*):



Model reduction

Exploiting the partial turnpike-like behaviour:

$\forall t \in [\tau, t_f - \tau], \;\; h(t) \simeq h_c$:

$$(1\text{b}) \Rightarrow \dot{h}(t) \simeq 0 \Rightarrow \gamma(t) \simeq 0 \tag{6a}$$

$$(1\text{d}) \Rightarrow \dot{\gamma}(t) \simeq 0 \Rightarrow \bar{u}_2 \simeq \frac{m.g}{\bar{q}(h_c, v).S} \tag{6b}$$

The system behavior can be approximated in dimension 3 by setting $h(t) = h_c$ and $\gamma(t) = 0$.

To numerically solve (OCP), we perform a continuation on the final state over three intermediate control problems of increasing dimension and complexity.

$(OCP)_3$

$$
(\mathbf{OCP}_3) \begin{cases} \min\limits_{u \in [\eta, 1]} t_f \\ \dot{\phi}(s) = f_3(\phi(s), u(s)) \qquad \forall s \in [0, t_f] \\ \phi(0) = \phi_0, \ \ \phi(t_f) = \phi_f \end{cases}
$$

- $\phi := (x \ \ v \ \ m)^T$, $u := u_1$
- $f_3 := \begin{pmatrix} v \\ \dfrac{T_{\max}.(1 + C_s v)}{m} u - \dfrac{D_0(h_c, v)}{m} \\ -C_s.T_{\max}.u \end{pmatrix}$
- $D_0(h, v) := \bar{q}(h_c, v).S.C_d$ is the first order drag force.
- $\phi_0 = (x_0 \ \ v_0 \ \ m_0)^T$ and $\phi_f = (x_f \ \ v_f \ \ *)^T$.

$(OCP)_4^\theta$

Add γ variable.

$$(\mathbf{OCP}_4^\theta)\begin{cases}\displaystyle\min_{(u_1,u_2)\in U}\int_0^{t_f}\left(k_0+k_2u_2^2(t)\right)dt\\ \dot\psi(s)=f_4^\theta(\psi(s),u_1(s),u_2(s)) \qquad \forall s\in[0,t_f]\\ \psi(0)=\psi_0,\;\;\psi(t_f)=\psi_f\end{cases}$$

- $\phi := (x\;\; v\;\; \gamma\;\; m)^T,\; u := (u_1,u_2)$, $\theta\in[0,1]$
- $$f_4^\theta := \begin{pmatrix} v\cos(\theta.\gamma)\\ \dfrac{T_{\max}.(1+C_s v).u_1 - D(h,v,u_2)}{m} - \theta.g.\sin(\gamma)\\ \dfrac{L(h,v,u_2)}{m.v} - \dfrac{g.\cos(\gamma)}{v}\\ -C_s.T_{\max}.u\end{pmatrix}$$
- $\psi_0 = (x_0\;\; v_0\;\; \gamma_0\;\; m_0)^T$ and $\psi_f = (x_f\;\; v_f\;\; \gamma_f\;\; *)^T$.

$(OCP)_5^\delta$

Add h variable.

$$
(\mathbf{OCP}_5^\delta)\begin{cases}\min\limits_{(u_1,u_2)\in U}\displaystyle\int_0^{t_f}\left(k_0+k_1.\frac{(h(t)-h_c)^2}{h_c^2}+k_2u_2^2(t)\right)dt\\ \dot{\xi}(s)=f_5^\delta(\xi(s),u_1(s),u_2(s)) \qquad \forall s\in[0,t_f]\\ \xi(0)=\xi_0,\quad \xi(t_f)=\xi_f\end{cases}
$$

- $\xi := (x \;\; h \;\; v \;\; \gamma \;\; m)^T$, $\delta\in[0,1]$
- $f_5^\delta := \begin{pmatrix} v\cos(\gamma)\\ v.\sin(\gamma)\\ \dfrac{T_{\max}.(1+C_s v).u_1-D(\delta.h,v,u_2)}{m}-\theta.g.\sin(\gamma)\\ \dfrac{L(\delta.h,v,u_2)}{m.v}-\dfrac{g.\cos(\gamma)}{v}\\ -C_s.T_{\max}.u\end{pmatrix}$

Continuation strategy

1. **(OCP$_3$)**: Standard shooting on the final state from $(x_3^i, v_3^i) = (1000\text{m}, 285\text{m/s})$ to $(x_3^f, v_3^f) = (3000\text{m}, 270\text{m/s})$:

$$x_3^h(\lambda) = x_3^i + \lambda.(x_3^f - x_3^i)$$
$$v_3^h(\lambda) = v_3^i + \lambda.(v_3^f - v_3^i)$$

   for $\lambda \in [0, 1]$. Trivial initial guess $(p(0), t_s, t_f)$.

2. **(OCP$_4^\theta$)**: Set $(x_4^i, v_4^i) = (x_3^f, v_3^f)$, $\gamma_4^i = 0^\circ$ and $\theta = 0$. Standard shooting on the final state and θ from $(x_4^i, v_4^i, \gamma_4^i, 0)$ up to $(x_4^f, v_4^f, \gamma_4^f, 1)$:

$$x_4^h(\lambda) = x_4^i + \lambda.(x_4^f - x_4^i)$$
$$v_4^h(\lambda) = v_4^i + \lambda.(v_4^f - v_4^i)$$
$$\gamma_4^h(\lambda) = \gamma_4^i + \lambda.(\gamma_4^f - \gamma_4^i)$$
$$\theta^h(\lambda) = \lambda.$$

   with $\lambda \in [0, 1]$, $(x_4^f, v_4^f, \gamma_4^f) = (10000\text{m}, 260\text{m/s}, -20^\circ)$.

Continuation strategy

3. $(\mathbf{OCP}_5^{\delta})$: Set $(x_5^i, h_5^i, v_5^i, \gamma_5^i) = (x_4^f, h_f, v_4^f, \gamma_4^f)$. Shooting from the middle on the final state and δ from $(x_5^i, h_5^i, v_5^i, \gamma_5^i, 0)$ up to $(x_5^f, h_5^f, v_5^f, \gamma_5^f, 1)$:

$$
\begin{aligned}
x_5^h(\lambda) &= x_5^i + \lambda.(x_5^f - x_5^i)\\
h_5^h(\lambda) &= h_5^i + \lambda.(h_5^f - h_5^i)\\
v_5^h(\lambda) &= v_5^i + \lambda.(v_5^f - v_5^i)\\
\gamma_5^h(\lambda) &= \gamma_5^i + \lambda.(\gamma_5^f - \gamma_5^i)\\
\delta^h(\lambda) &= \lambda.
\end{aligned}
$$

for $\lambda \in [0,1]$, $(x_5^f, h_5^f, v_5^f, \gamma_5^f) = (15000\text{m}, h_f, 240\text{m/s}, -35^\circ)$. The initialization is done using the estimated adjoint state and times from $(\mathbf{OCP}_4^1)$, setting $p_h\left(\frac{t_f}{2}\right) = 0.5$.

Continuation strategy

4. $(\mathbf{OCP}_5^1)$: Triple shooting method from the middle: two additional time knots t_{1b}, t_{1f} are added to the initial one $\frac{t_f}{2}$ such that $t_{1b} = \frac{t_f}{4}$ and $t_{1f} = \frac{3.t_f}{4}$.

$$\begin{aligned}
x^h_{5m}(\lambda) &= x^i_{5m} + \lambda.(x_f - x^i_{5m}) \\
h^h_{5m}(\lambda) &= h^i_{5m} + \lambda.(h_f - h^i_{5m}) \\
v^h_{5m}(\lambda) &= v^i_{5m} + \lambda.(v_f - v^i_{5m}) \\
\gamma^h_{5m}(\lambda) &= \gamma^i_{5m} + \lambda.(\gamma_f - \gamma^i_{5m})
\end{aligned}$$

with $\lambda \in [0,1]$, $(x^i_{5m}, h^i_{5m}, h^i_{5m}, \gamma^i_{5m}) = (x^f_5, h^f_5, v^f_5, \gamma^f_5)$.

Homotopy algorithm

Initialization: $\lambda = 0$, $z = z_0$, $s = 0.1$
while $s \geqslant s_{\min}$ and $\lambda \leqslant 1$ **do**
  $\lambda = \lambda + s$
  $\phi = \phi_i + \lambda.(\phi_f - \phi_i)$
  Look for $\tilde{z}$ zero of the shooting function
  if success **then**
    $z \leftarrow \tilde{z}$
    $s \leftarrow \min(2.s, s_{\max})$
  else
    $\lambda \leftarrow \lambda - s$
    $s \leftarrow \frac{s}{2}$
  end if
end while

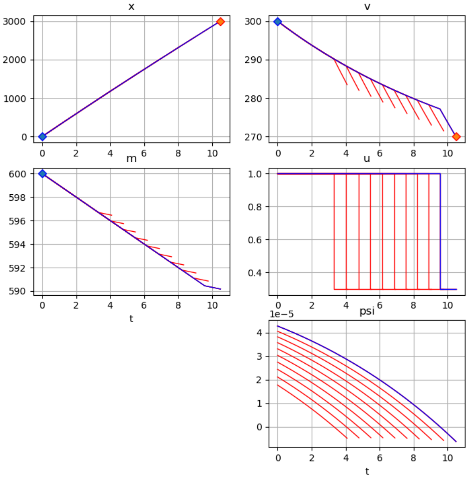
Numerical implementation

$\triangleright$ Implementation in Python language on a "standard +" desktop computer.

$\triangleright$ The *scipy.optimize* package associated to the *hybrd* method $\rightarrow$ zeros of the shooting function.

$\triangleright$ The *scipy.integrate.odeint* package $\rightarrow$ integration of the ODE.

3d model


x
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596
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592
590
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1.0
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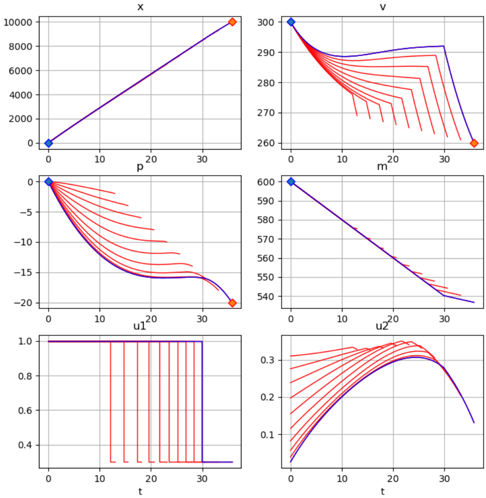
4d model


x
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t
u2
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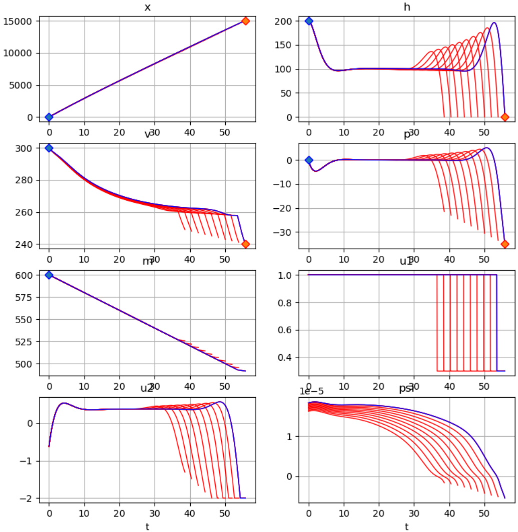
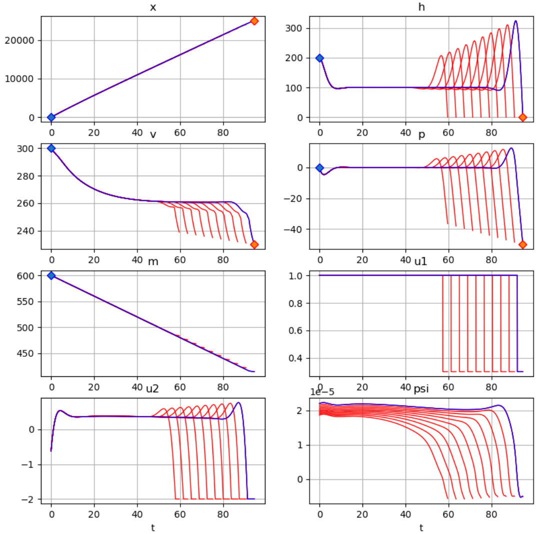
t

5d model (simple shooting)



5d model (triple shooting)



Comments

Dim	Unknowns	Proc. time	$\|\|F(z)\|\|$	Shooting
3	5	0.15 s	10^{-29}	standard & simple
4	6	0.5 s	10^{-22}	standard & simple
5	12	11 s	10^{-16}	middle & simple
5	32	32 s	10^{-10}	middle & triple

▷ The continuation with the "classical" shooting method fails for $x_f \geqslant 8000m$.

▷ For $x_f \geqslant 15000m$, it is required to implement a multiple shooting method in order to increase the stability.

▷ The "junctions" between the models remain to be automatized and optimized...

Thank you for your attention!

Technological parameters of the missile

Variable	Value	Unit
d	0.65	m
C_d	0.4	na
$T_{\max}$	5000	N
C_s	4.10^{-4}	$kg.s^1.N^{-1}$
g	9.81	$m.s^{-2}$
ρ_0	1.225	kg/m^3
h_r	7314	m
η	0.3	n.a
k_{cz}	0.05	n.a
u_2^m	2	n.a
h_c	100	m