Boundary condition on porous boundary

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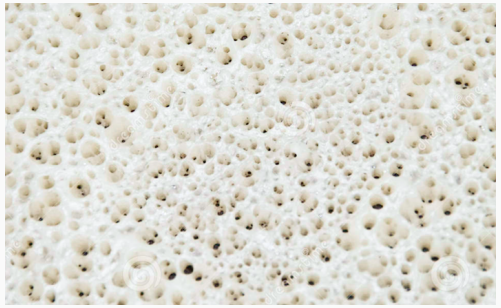
In fluid mechanics, different physical situations demand different boundary conditions:

- Impermeable wall $\Rightarrow$ **no-slip** $\boxed{\mathbf{u} = 0}$
- **Inlet/outlet** or moving boundary $\Rightarrow$ $\boxed{\mathbf{u} = \mathbf{g}}$
- **Pressure** boundary condition $\boxed{p = q \ \ \& \ \ \mathbf{u} \times \mathbf{n} = 0}$
- **Stress** boundary condition $\boxed{\mathbf{T}\mathbf{n} = \mathbf{q}}$
- **Flux** boundary condition $\boxed{\int_\Gamma \mathbf{u} \cdot \mathbf{n} = F}$
- **Slip** (Navier) boundary condition $\boxed{\dfrac{\partial u_\tau}{\partial \mathbf{n}} + \alpha\, u_\tau = 0}$
- **Porous boundary ?**

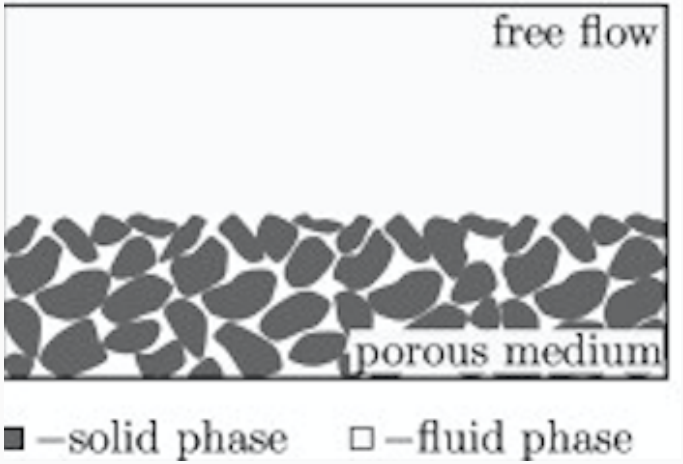
Effective condition on a porous boundary

Situation: Viscous flow in domain with porous boundary



What boundary condition corresponds to porous boundary?

Interface condition between a free flow and a porous medium

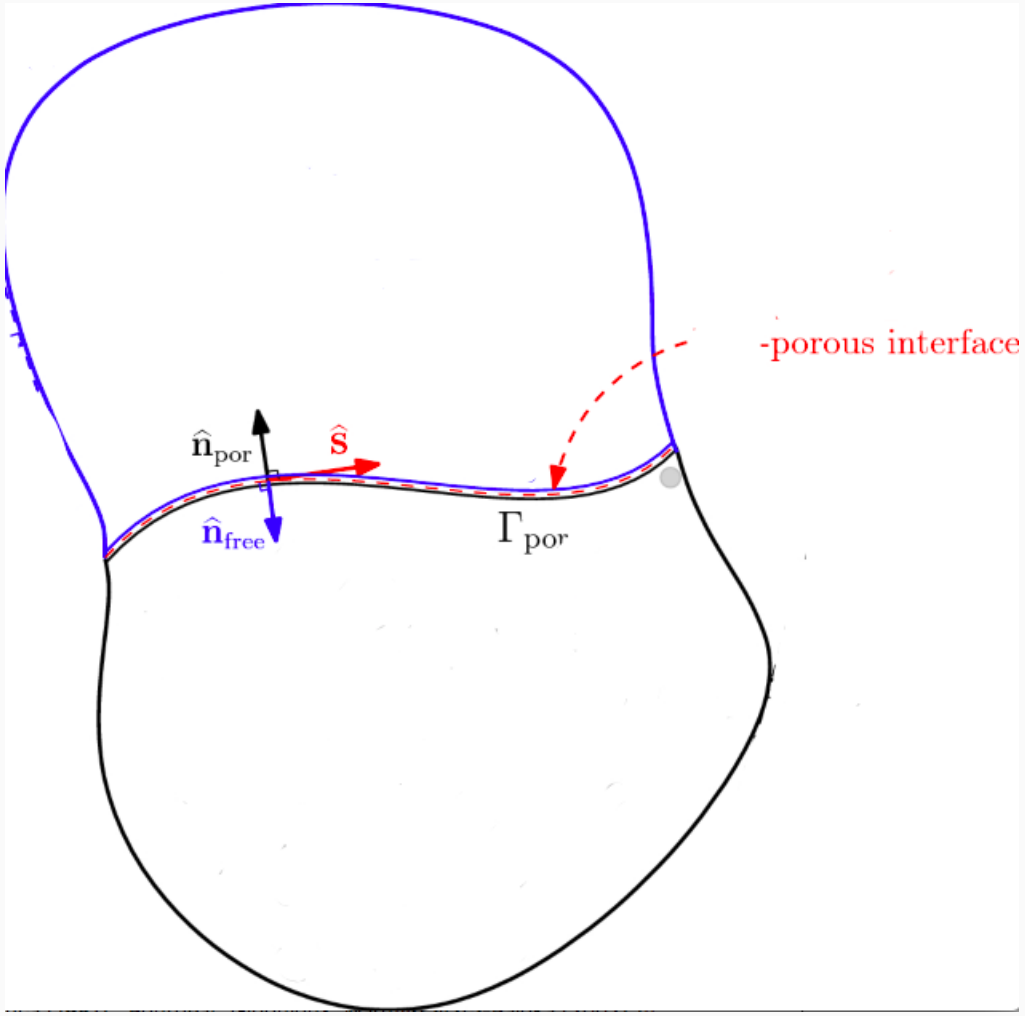


Effective condition is the Beavers-Joseph condition:

$$\boxed{\frac{\partial u_\tau}{\partial \mathbf{n}} = \frac{\alpha}{\sqrt{K}}(u_\tau - U) \quad , \quad u_n = 0} \tag{1}$$

(Beavers & Joseph (1967), Jäger & Mikelić (1996))

Porous interface inside a fluid domain



Porous interface inside a fluid domain

Effective condition:

$$\boxed{\mathbf{u} = \mathbf{c} = const. \ , \ \ [p] = p^{+} - p^{-} = \lambda\, \mathbf{c}} \tag{2}$$

(Sanchez-Palencia (1985), Conca (1987) , Bourgeat, Gipouloux, Marušić-Paloka (2002))

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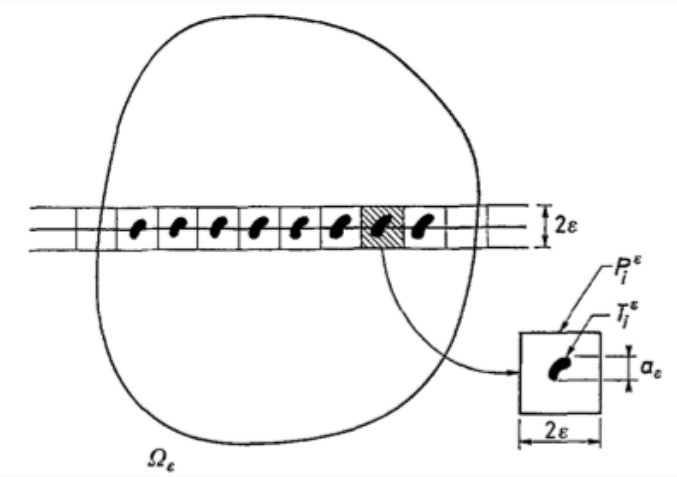
Remark

The above law can be seen as the Darcy law on the porous wall

$$\mathbf{u} = k\,(p^+ - p^-) \ .$$

Porous interface inside a fluid domain

The case of high porosity (array of small obstacles distributed on a surface) was studied by G.Allaire in 1991



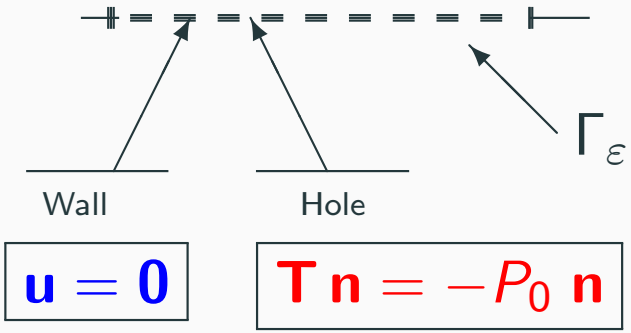
Brinkman-type of the effective model is obtained

$$-\mu\Delta\mathbf{u} + \nabla p + \mathbf{M}\,\mathbf{u}\,\delta_\Sigma = 0$$

where δ_Σ is a measure concentrated on the surface Σ.

Description of the problem

The idea is to start with the Stokes system in a domain with boundary that has periodically distributed holes. On the solid part of the boundary we impose the standard no-slip condition, while on each hole we impose an appropriate dynamic condition: the value of the stress. The goal is to obtain the effective model by studying the convergence of the homogenization procedure, as the period of the porous boundary tends to zero.



The domain's geometry

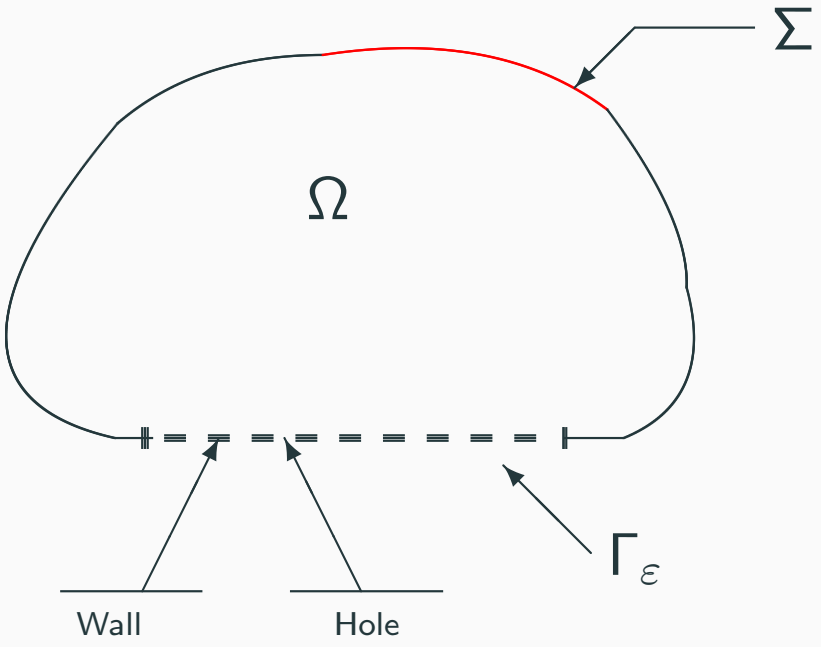


Figure: Domain with porous boundary.

$$\Gamma_\varepsilon = \Gamma_\varepsilon^D \cup \Gamma_\varepsilon^N$$

$$\Gamma_\varepsilon^D \ - \ \text{wall} \ , \ \Gamma_\varepsilon^N \ - \ \text{holes} \ .$$

The governing system

As indicated in the Introduction, we assume that the fluid flow is governed by the Stokes system. We add the corresponding boundary conditions to obtain:

$$-\mu\,\Delta\mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{0}\ ,\ \ \operatorname{div}\mathbf{u}^\varepsilon = 0\ \text{ in }\ \Omega\,, \tag{3}$$

$$\text{the dynamic b.c.}\ \ :\{\ \mathbf{T}(\mathbf{u}^\varepsilon, p^\varepsilon)\,\mathbf{n} = -P_0\,\mathbf{n}\ \text{ on }\ \Gamma_\varepsilon^N \tag{4}$$

$$\text{the kinematic b.c.}:\begin{cases} \mathbf{u}^\varepsilon = 0\ \text{ on } \Gamma_\varepsilon^D \cup S\,, \\ \mathbf{u}^\varepsilon = \mathbf{g}\ \text{ on }\ \Sigma\ -\ \text{the inflow} \end{cases} \tag{5}$$

$$\mathbf{T}(\mathbf{u}^\varepsilon, p^\varepsilon) = -2\mu\mathbf{e}(\mathbf{u}^\varepsilon) + p^\varepsilon\ \mathbf{I}\quad -\ \text{the stress tensor}$$

Asymptotic expansion

We seek for the asymptotic approximation of the solution in the following form:

$$\mathbf{u}^\varepsilon \approx \sum_{k=1}^{n} u_k(x)\, \mathbf{w}^k(y) + O(\varepsilon) \ , \ \ y = \frac{x}{\varepsilon} \tag{6}$$

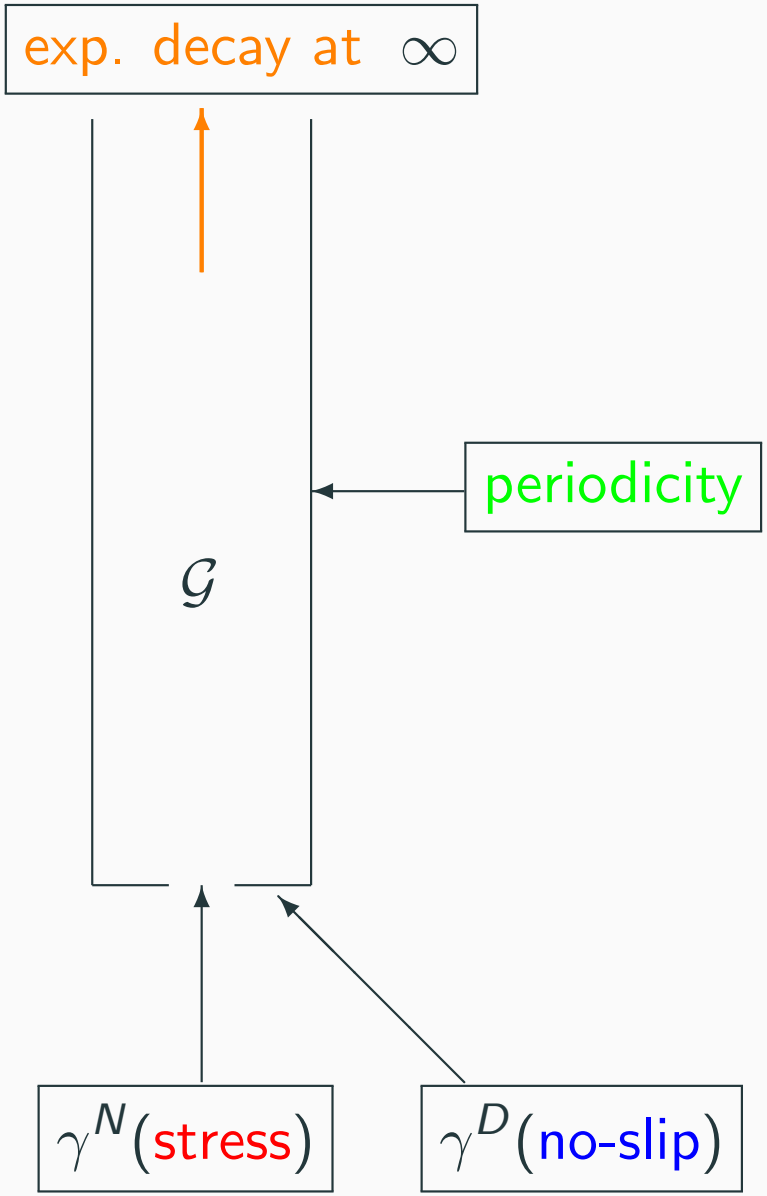
$$p^\varepsilon \approx \frac{1}{\varepsilon} \sum_{k=1}^{n} u_k(x)\, \pi^k(y) + P(x) + O(\varepsilon) \ , \tag{7}$$

$(\mathbf{w}^k, \pi^k)$, $k = 1, \cdots, n$ - the solutions of the auxiliary boundary-value problems posed in the infinite strip

$$\mathcal{G} = \langle 0, 1 \rangle^{n-1} \times \langle 0, +\infty \rangle$$

$\mathbf{u} = (u_1, \cdots, u_n)$, P – the effective velocity and pressure

Boundary layer



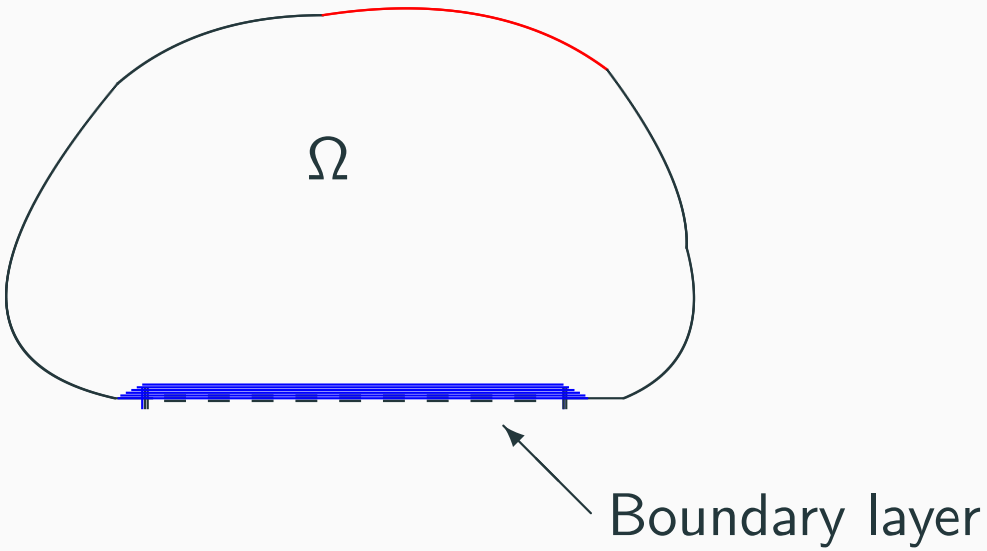
Boundary layer

Outside of the boundary layer (i.e. for large y_n) we want

$$\mathbf{w}^k \approx \mathbf{e}_k \quad , \quad \pi^k \approx 0 \ ,$$

so that

$$\mathbf{u}^\varepsilon \approx \sum_{k=1}^{n} u_k(x)\, \mathbf{e}_k = \mathbf{u} \ , \ p^\varepsilon \approx P \ .$$



Auxiliary problem

$$-\mu\,\Delta_y \mathbf{w}^k + \nabla_y \pi^k = \mathbf{0} \;\text{ in }\; \mathcal{G}\,, \tag{8}$$

$$\operatorname{div}_y \mathbf{w}^k = 0 \;\text{ in }\; \mathcal{G}\,, \;\; \mathbf{w}^k = 0 \;\text{ on }\; \gamma^D\,, \tag{9}$$

$$\boxed{\mathbf{T}(\mathbf{w}^k, \pi^k)\mathbf{e}_n = \mathbf{d}^k} \;\text{ on }\; \gamma^N\,, \tag{10}$$

$$(\mathbf{w}^k, \pi^k) \;\text{ is 1-periodic in }\; y'\,, \;\; \nabla_y \mathbf{w}^k \in L^2(\mathcal{G})\,, \tag{11}$$

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The unique solutions of the boundary layer problems (8)-(11), for any $\mathbf{d}^k \in \mathbb{R}^n$, exponentially decay to some **constants**, that are **not given in advance.** The idea is to control those constants choose by choosing vectors $\mathbf{d}^k$ in a way that

$$\boxed{\lim_{y_n \to +\infty} \mathbf{w}^k = \mathbf{e}_k}\,. \tag{12}$$

Auxiliary problem

So $\mathbf{d}^k$ are not given in advance, they are the **part of the problem.**
We can prove that a unique triple
$(\mathbf{w}^k, \pi^k, \mathbf{d}^k) \in H^1(\Omega)^n \times L^2(\Omega) \times \mathbb{R}^n$ can be found, such that
$(\mathbf{w}^k, \pi^k)$ stabilize exponentially to $(\mathbf{e}_k\ ,\ 0)$, as $y_n \to +\infty$.
Furthermore $\mathbf{d}^k \cdot \mathbf{e}_k > 0$. That is essential since $\mathbf{d}^k$ appear as the coefficients in our effective boundary condition.

As a consequence of the exponential decay, outside of the boundary layer we have

$$\sum_{k=1}^{n} u_k(x)\, \mathbf{w}^k(y) \approx \mathbf{u} = (u_1, \cdots, u_n)$$
$$\sum_{k=1}^{n} u_k(x)\, \pi^k(y) \approx 0 \ .$$

Thus the effective velocity and pressure of the fluid are $(\mathbf{u}, P)$.

The effective boundary condition

We denote by $\mathbf{s}$ the fluid stress on the porous boundary

$$\mathbf{s} = \mathbf{T}(\mathbf{u}, P)\,\mathbf{n}$$
$$\mathbf{T}(\mathbf{u}, P) = -2\mu\mathbf{D}(\mathbf{u}) + P\,\mathbf{I} \quad .$$

Let $\mathbf{L}$ be the positive definite symmetric matrix defined from the auxiliary problem (8)-(11) by

$$[\mathbf{L}]_{kj} = \left[d_j^k\right] \quad .$$

The effective boundary condition then reads

$$\boxed{\mathbf{u} = \mathbf{K}_\varepsilon(\mathbf{s} - P_0\,\mathbf{n}) \quad \text{on } \Gamma} \quad , \tag{13}$$

where

$$\mathbf{K}_\varepsilon = \varepsilon\mathbf{L}^{-1}$$

is the (positive definite and symmetric) permeability tensor of the porous wall.

The effective boundary condition

Vector $P_0\,\mathbf{n}$ is the exterior stress on the boundary. Thus, it is a generalized **Darcy law** on the boundary, saying that the velocity on the boundary is proportional to the difference between the inner and the outer stress on the boundary.

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What is the relation with Beavers-Joseph law?

The effective boundary condition

In 2D case, the permeability tensor $\mathbf{K}_\varepsilon$ turns out to be diagonal.

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In 2D case, the permeability tensor $\mathbf{K}_\varepsilon$ turns out to be diagonal. In 3D case, if the geometry of the pores is isotropic (e.g. circular, rectangular or elliptic holes), then the permeability tensor $\mathbf{K}_\varepsilon$ becomes block-diagonal

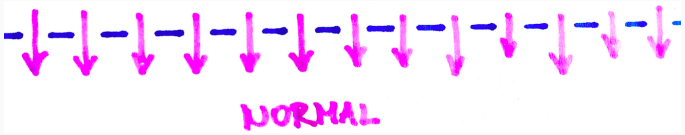
$$\mathbf{K}_\varepsilon = \varepsilon \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} . \tag{14}$$

Beavers-Joseph law

In that case, the effective boundary condition (13) splits in two parts. For the normal velocity, we get the (scalar) Darcy law on the boundary

$$\boxed{u_n = k_\varepsilon \left(P_0 - \mathbf{s}_n\right)} \,. \tag{15}$$

$k_\varepsilon = \mathbf{n} \cdot \mathbf{K}_\varepsilon \mathbf{n} > 0$ – the normal permeability of the boundary.
$\mathbf{s}_n$ – the normal component of the stress vector $\mathbf{s}$.



Beavers-Joseph law

As for the tangential part, we get the effective boundary condition claiming that the slip velocity is proportional to the shear rate

$$\boxed{\varepsilon\, \mathbf{s}_\tau = \mathbf{M}\, \mathbf{u}_\tau} \ . \tag{16}$$

$$\text{In 2D} \quad \mathbf{M} = \frac{1}{k_{11}} \ . \qquad \text{In 3D} \quad \mathbf{M} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \ .$$

Vectors $\mathbf{u}_\tau$ and $\mathbf{s}_\tau$ are the tangential parts of the velocity and the stress, respectively

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Vectors $\mathbf{u}_\tau$ and $\mathbf{s}_\tau$ are the tangential parts of the velocity and the stress, respectively

Thus the shear stress in the fluid is proportional to the slip velocity, which is exactly the **Beavers-Joseph** law.

Effective equations

$$
\begin{aligned}
&-\mu\Delta\mathbf{u} + \nabla P = \mathbf{0} \quad \text{in} \quad \Omega\,, \\
&\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega\,, \\
&\boxed{\mathbf{u} = \mathbf{K}_\varepsilon\,(\mathbf{s} - P_0\,\mathbf{n})} \quad \text{on} \quad \Gamma\,, \\
&\mathbf{u} = \mathbf{0} \quad \text{on} \quad S\,, \\
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\end{aligned}
}
\tag{17}
$$

Its weak formulation reads:

$$W = \{\mathbf{v} \in H^1(\Omega)^n\,;\ \operatorname{div}\mathbf{v} = 0\,,\ \mathbf{v} = 0 \text{ on } \partial\Omega\backslash\Gamma\ \}$$

Find $\mathbf{u} \in \mathbf{g} + W$ such that

$$\mu \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{w} + \int_\Gamma \mathbf{K}_\varepsilon{}^{-1}\mathbf{u} \cdot \mathbf{w} = -\int_\Gamma P_0\, w_n \ \ ,\ \forall\, \mathbf{w} = (w_i) \in W \ \ .$$

Effective equations

The bilinear form on the left-hand side is obviously coercive, so that the problem admits a unique solution. If the Stokes system is replaced by the Navier-Stokes, the existence and uniqueness still hold, for data that is not too large.

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We recall that the effective boundary condition reads

$$\boxed{\mathbf{u} = \mathbf{K}_\varepsilon \left(\mathbf{s} - P_0\, \mathbf{n}\right)} \quad ,$$

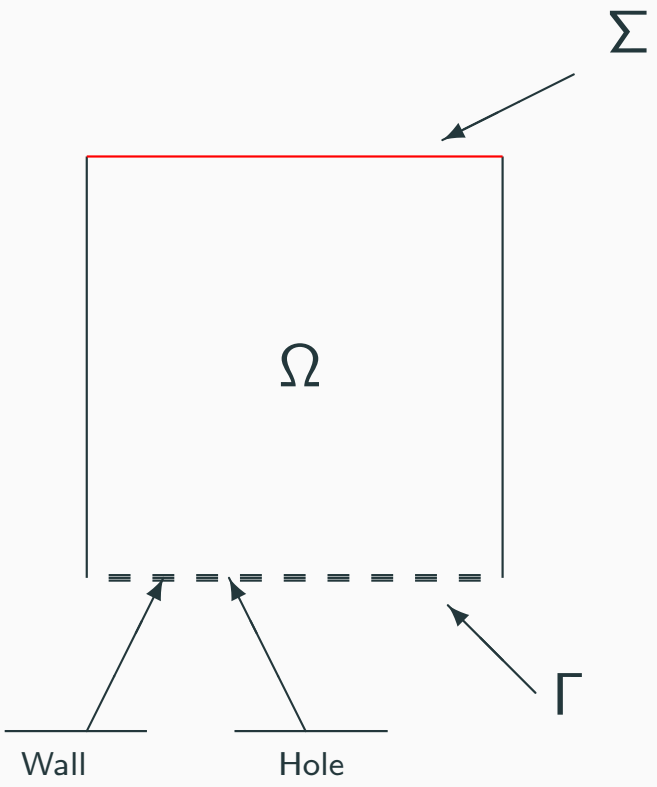
Even though $\mathbf{K}_\varepsilon = O(\varepsilon)$ is small, the Darcy condition cannot be approximated by the no-slip condition $\boxed{\mathbf{u} = \mathbf{0}}$.

First of all, such problem is not well-posed (unless $\int_\Sigma \mathbf{g} \cdot \mathbf{n} = 0$). Secondly the stress in the boundary layer $\mathbf{s} \sim O\left(\frac{1}{\varepsilon}\right)$ so that

$$\mathbf{K}_\varepsilon \left(\mathbf{s} - P_0\, \mathbf{n}\right) \not\approx \mathbf{0}$$

Rigorous justification

Rectangular domain



Rigorous justification

Theorem (Justification for rectangular domain)
For any $\delta > 0$ and $\Omega_\delta = \Omega \cap \{0 < x_3 < \delta\}$ there exists a constant $C > 0$ such that

$$|\mathbf{u}^\varepsilon - \mathbf{u}|_{L^2(\Omega_\delta)} + |p^\varepsilon - P|_{L^2(\Omega_\delta)} \leq C\,\varepsilon \ . \tag{18}$$

Furthermore, for $t \in [1, 2]$

$$|\mathbf{u}^\varepsilon - \mathbf{u}|_{L^t(\Omega)} \leq C\varepsilon^{\frac{1}{t}}$$

$$\left| p^\varepsilon - \left(P + \varepsilon^{-1} \sum_{k=1}^{2} \pi^k(\mathbf{x}/\varepsilon)\, u_k \right) \right|_{L^t(\Omega)} \leq C\varepsilon^{\frac{1}{t}} \ , \ 1 \leq t \leq 2 \, .$$

Finally,

$$p^\varepsilon - P \rightharpoonup 0 \ \text{ weakly in } \ (H^1(\Omega))' \ . \tag{19}$$

References

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- E.Marušić-Paloka, I.Pažanin, The effective boundary condition on a porous wall, International journal of engineering science, 173 (2022), 103638.