Limits and consistency of non-local and graph approximations to the Eikonal equation

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In this presentation, we will talk about a non-local approximation of the following time-dependent (local) Eikonal equation with Dirichlet-type boundary conditions

$$\begin{cases} \dfrac{\partial}{\partial t} f(x,t) = -\left|\nabla f(x,t)\right| + P(x), & (x,t) \in (\Omega \setminus \Gamma)\times]0,T[, \\ f(x,t) = \psi(x), & (x,t) \in (\Gamma \times]0,T[) \cup \Omega \times \{0\}, \end{cases} \tag{$\mathcal{P}$}$$

where $\nabla f(x,t)$ denotes the (weak) gradient of f in the space variable x. More precisely, we consider the general *non-local* Eikonal equation in a time-dependent form :

$$\begin{cases} \dfrac{\partial}{\partial t} f^{\varepsilon}(u,t) = -\left|\nabla^-_{J_\varepsilon} f^{\varepsilon}(u,t)\right|_{\infty} + \tilde{P}(u), & (u,t) \in (\tilde{\Omega} \setminus \tilde{\Gamma})\times]0,T[, \\ f^{\varepsilon}(u,t) = \tilde{\psi}(u), & (u,t) \in (\tilde{\Gamma} \times]0,T[) \cup \tilde{\Omega} \times \{0\}, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where

$$\left|\nabla^-_{J_\varepsilon} f^{\varepsilon}(u,t)\right|_{\infty} = \max_{v \in \tilde{\Omega}} J_\varepsilon(u,v)(f^{\varepsilon}(u,t) - f^{\varepsilon}(v,t)),$$

and J_ε is a properly scaled kernel. $(\mathcal{P}_\varepsilon)$ covers the case of weighted graphs with n vertices as a special case by properly instantiating the sets $(\tilde{\Omega}, \tilde{\Gamma})$. Based on the theory of viscosity solutions, we prove existence and uniqueness of the viscosity solutions of both the local and non-local problems, as well as regularity properties of these solutions in time and space. We then derive error bounds between the solution to the non-local problem and that of the local one, both in continuous-time and Forward Euler time discretization. We then turn to studying continuum limits of non-local problems defined on random weighted graphs with n vertices. In particular, we establish that if the kernel scale parameter decreases at an appropriate rate as n grows, then almost surely, the solution of the problem on graphs converges uniformly to the viscosity solution of the local problem as the time step vanishes and the number vertices n grows large [1].

[1] Jalal Fadili, Nicolas Forcadel, Thi Tuyen Nguyen, and Rita Zantout. Limits and consistency of non-local and graph approximations to the eikonal equation. *arXiv preprint arXiv :2105.01977*, 2022.

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