Control of state-constrained McKean-Vlasov equations: application to portfolio selection

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Motivation

- Stochastic control problem with a large number N of similar interacting players.
- Control problem in which the probability law of one player appears.

These two problems are linked in the limit $N \to \infty$.

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- Population dynamics
- Smart charging
- Crowd motion

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Mean-Field Game (MFG) $\to$ competitive interaction $\to$ Nash equilibrium

Mean-Field Control (MFC) $\to$ cooperative interaction $\to$ social planner problem

Lasry and Lions 2006; Huang, Caines, and Malhamé 2006; Carmona and Delarue 2018

Contents

Classical stochastic control

A classical problem is to solve

$$\inf_{\alpha} \ \mathbb{E}\Big[\int_0^T f(s, X_s^\alpha, \alpha_s) \ \mathrm{d}s + g(X_T^\alpha) \Big]$$

with a controlled diffusion process (in $\mathbb{R}^d$)

$$\mathrm{d}X_t^\alpha = b(t, X_t^\alpha, \alpha_t) \ \mathrm{d}t + \sigma(t, X_t^\alpha, \alpha_t) \ \mathrm{d}W_t.$$

W_t a Brownian motion in dimension d, and α_t is a control adapted to the Brownian filtration $\mathcal{F}_t$.

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Example: Merton problem

Utility function U, wealth X_t^α, α_t amount invested in an asset:

$$\inf_{\alpha} \mathbb{E}\Big[- U(X_T^\alpha)\Big]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b\ \mathrm{d}t + \alpha_t \sigma\ \mathrm{d}W_t.$$

N players game

N symmetric players with controls $\alpha_t^i = \alpha_t(X_t^i)$

$$\inf_{\alpha} \; \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\Big[\int_0^T f\Big(s, X_s^{i,\alpha}, \alpha_s^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_s^{i,\alpha}}\Big)\,\mathrm{d}s + g\Big(X_T^{i,\alpha}, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_T^{i,\alpha}}\Big)\Big]$$

$$\mathrm{d}X_t^{i,\alpha} = b\Big(t, X_t^{i,\alpha}, \alpha_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,\alpha}}\Big)\,\mathrm{d}t + \sigma\Big(t, X_t^{i,\alpha}, \alpha_t^i, \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,\alpha}}\Big)\,\mathrm{d}W_t^i.$$

Non commuting operations

The optimization and limit procedures **do not commute**!

$$
\begin{array}{ccc}
\begin{array}{c}\text{SDE State Dynamics}\\ \text{for N players}\end{array} & \xrightarrow{\text{Optimization}} & \begin{array}{c}\text{Nash Equilibrium}\\ \text{for N players}\end{array}\\
\lim_{N\to\infty}\Big\downarrow & & \Big\downarrow\lim_{N\to\infty}\\
\text{McKean Vlasov Dynamics} & \xrightarrow{\text{Optimization}} & \begin{array}{c}\text{Mean Field Game?}\\ \text{Controlled McK-V Dynamics?}\end{array}
\end{array}
$$

Figure from Carmona, Delarue, and Lachapelle 2012

We will focus on the control of McKean-Vlasov dynamics, also called **mean-field control**

Mean-field control

In general the joint law of (X_t^α, α_t) appears in the cost and dynamics

$$\min_\alpha J(\alpha) := \mathbb{E}\Big[\int_0^T f(s, X_s^\alpha, \mathbb{P}_{(X_s^\alpha, \alpha_s)}, \alpha_s)\ \mathrm{d}s + g(X_T^\alpha, \mathbb{P}_{X_T^\alpha})\Big]$$

with McKean-Vlasov dynamics (in $\mathbb{R}^d$)

$$\mathrm{d}X_t^\alpha = b(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}t + \sigma(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}W_t.$$

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with McKean-Vlasov dynamics (in $\mathbb{R}^d$)

$$\mathrm{d}X_t^\alpha = b(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}t + \sigma(t, X_t^\alpha, \mathbb{P}_{(X_t^\alpha, \alpha_t)}, \alpha_t)\ \mathrm{d}W_t.$$

Example: Markowitz mean-variance problem in continuous time

$$\min_\alpha\ \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b\ \mathrm{d}t + \alpha_t \sigma\ \mathrm{d}W_t.$$

Contents

Constraints

What about adding **state constraints** on X_t^α? Examples: bounded state, positive state...

In the stochastic control case (no mean-field interaction), this problem is treated by Bokanowski, Picarelli, and Zidani 2015; Bokanowski, Picarelli, and Zidani 2016 with constraints

$$X_t^\alpha \in \mathcal{K} \ \forall t \in [0,T], \text{a.s.},$$

for a non empty closed set $\mathcal{K}$.

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$$X_t^\alpha \in \mathcal{K} \ \forall t \in [0,T], \text{a.s.},$$

for a non empty closed set $\mathcal{K}$. It turns out that we can go further and enforce **probabilistic constraints**:

$$\Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0 \ \forall t \in [0,T].$$

Related literature

- For MFG: state constrained to stay in a **compact** $\rightarrow$ Cannarsa, Capuani, and Cardaliaguet 2018; Graber and Mayorga 2021...
- Deterministic MFC by control of Fokker-Planck equations $\rightarrow$ Bonnet 2019
- MFC with **smooth terminal expectation** constraint $\rightarrow$ Chen and Wang 2019
- MFC cost with probabilistic constraints for a standard diffusion $\rightarrow$ Daudin 2021

To the best of our knowledge we are the first authors to consider MFC with general probabilistic constraints.

Examples of possible constraints in the form

$$\Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0 \ \forall t \in [0, T]$$

- $\mathbb{P}(X_t^\alpha \in \mathcal{K}_t) \geq p_t$
- $\mathcal{W}_2(\mathbb{P}_{X_t^\alpha}, \eta_t) \leq \delta_t$
- $\varphi(\mathbb{P}_{X_T^\alpha}) \leq 0$
- $\phi(t_i, \mathbb{P}_{X_{t_i}^\alpha}) \leq 0$ for $t_1 < \cdots < t_k$

Primal Markowitz mean-variance problem

$$\begin{aligned} \min_\alpha \ & -\mathbb{E}[X_T^\alpha] \\ & \mathrm{d}X_t^\alpha = \alpha_t b \ \mathrm{d}t + \alpha_t \sigma \ \mathrm{d}W_t \\ & \mathrm{Var}(X_T) \leq \vartheta. \end{aligned}$$

Markowitz mean-variance problem with conditional expectation constraint

$$\inf_{\alpha} \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}W_t$$
$$E[X_t^\alpha \mid X_t^\alpha \leq \theta] \geq \delta.$$

Contents

An auxiliary problem with exact penalization

$$J(\alpha) = \mathbb{E}\Big[\int_0^T f(s, X_s^\alpha, \alpha_s, \mathbb{P}_{(X_t^\alpha, \alpha_t)})\ \mathrm{d}s + g(X_T^\alpha, \mathbb{P}_{X_T^\alpha})\Big]$$

$$V^\Psi := \inf_{\alpha \in \mathcal{A}} \big\{ J(\alpha) : \Psi(t, \mathbb{P}_{X_t^\alpha}) \leq 0,\ \forall\, t \in [0, T] \big\}.$$

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Define a new state variable

$$Z_z^\alpha(\cdot) := z - \mathbb{E}\Big[\int_0^\cdot f\big(s, X_s^\alpha, \alpha_s, \mathbb{P}_{(X_s^\alpha, \alpha_s)}\big) \, \mathrm{d}s \Big] = z - \int_0^\cdot \widehat{f}\big(s, \mathbb{P}_{(X_s^\alpha, \alpha_s)}\big) \, \mathrm{d}s,$$

with $\widehat{f}(t, \nu) = \int_{\mathbb{R}^d \times A} f\big(t, x, a, \nu\big) \, \nu(\mathrm{d}x, \mathrm{d}a)$ and $\widehat{g}(\mu) = \int_{\mathbb{R}^d} g(x, \mu) \mu(\mathrm{d}x)$.

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with $\widehat{f}(t,\nu) = \int_{\mathbb{R}^d \times A} f\big(t, x, a, \nu\big) \, \nu(\mathrm{d}x, \mathrm{d}a)$ and $\widehat{g}(\mu) = \int_{\mathbb{R}^d} g(x,\mu)\mu(\mathrm{d}x)$. Thus

$$J(\alpha) = z - Z_z^\alpha(T) + \widehat{g}(\mathbb{P}_{X_T^\alpha})$$

An auxiliary problem with exact penalization

Unconstrained mean-field control problem

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \Big[\{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \sup_{s \in [0,T]} \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ \Big],$$

with the notation $\{x\}_+ = \max(x, 0)$. We see that $\mathcal{Y}^{\Psi}(z) \geq 0$.

An auxiliary problem with exact penalization

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with the notation $\{x\}_+ = \max(x, 0)$. We see that $\mathcal{Y}^{\Psi}(z) \geq 0$. We consider the infimum of the **zero level-set**

$$\mathcal{Z}^{\Psi} := \inf\{z \in \mathbb{R} \mid \mathcal{Y}^{\Psi}(z) = 0\}.$$

$\mathcal{Y}^{\Psi}$ being convex, positive and non-increasing, if $\mathcal{Z}^{\Psi} < \infty$ then $\mathcal{Y}^{\Psi}$ is decreasing on $(-\infty, \mathcal{Z}^{\Psi}]$ then $\mathcal{Y}^{\Psi}(z) = 0$ on $[\mathcal{Z}^{\Psi}, \infty)$.

Representation results

Theorem 1 (make the constraint function vary)

If the value V^{Ψ} of the control problem is finite, it verifies the bounds

$$\mathcal{Z}^{\Psi} \leq V^{\Psi} \leq \inf_{\varepsilon > 0} \mathcal{Z}^{\Psi+\varepsilon}.$$

Representation results

Theorem 1 (make the constraint function vary)

If the value V^Ψ of the control problem is finite, it verifies the bounds

$$\mathcal{Z}^\Psi \leq V^\Psi \leq \inf_{\varepsilon > 0} \mathcal{Z}^{\Psi+\varepsilon}.$$

Theorem 2

If $V^\Psi < +\infty$, then $\varepsilon \mapsto \mathcal{Z}^{\Psi+\varepsilon}$ is continuous in zero. Hence

$$\mathcal{Z}^\Psi = V^\Psi.$$

Summary

Solving

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \Big[\{ \widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T) \}_+ + \sup_{s \in [0,T]} \{ \Psi(s, \mathbb{P}_{X_s^\alpha}) \}_+ \Big],$$

and computing the infimum of the zero level set

$$\mathcal{Z}^{\Psi} := \inf\{ z \in \mathbb{R} \mid \mathcal{Y}^{\Psi}(z) = 0 \},$$

gives us the value of the control problem $V^{\Psi} = \mathcal{Z}^{\Psi}$.

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$$\mathcal{Z}^{\Psi} := \inf\{z \in \mathbb{R} \mid \mathcal{Y}^{\Psi}(z) = 0\},$$

gives us the value of the control problem $V^{\Psi} = \mathcal{Z}^{\Psi}$.

Moreover it can be seen that ε-optimal controls α^{ε} for the auxiliary problem $\mathcal{Y}^{\Psi}(V^{\Psi})$ are ε-admissible ε-optimal controls for the original problem in the sense that

$$J(X_0, \alpha^{\varepsilon}) \leq V + \varepsilon, \quad \sup_{0 \leq s \leq T} \Psi(s, \mathbb{P}_{X_s^{\alpha^{\varepsilon}}}) \leq \varepsilon.$$

An alternative standard auxiliary problem

Solving

$$\bar{\mathcal{Y}}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \left[\{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \int_0^T \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ \mathrm{d}s \right],$$

instead of

$$\mathcal{Y}^{\Psi} : z \in \mathbb{R} \mapsto \inf_{\alpha \in \mathcal{A}} \left[\{\widehat{g}(\mathbb{P}_{X_T^\alpha}) - Z_z^\alpha(T)\}_+ + \sup_{s \in [0,T]} \{\Psi(s, \mathbb{P}_{X_s^\alpha})\}_+ \right],$$

and computing the infimum of the zero level set

$$\bar{\mathcal{Z}}^{\Psi} := \inf\{z \in \mathbb{R} \mid \bar{\mathcal{Y}}^{\Psi}(z) = 0\},$$

also allows us to obtain the value and optimal control of the problem, if we assume existence of optimal controls for the auxiliary problem.

Contents

Numerical scheme

We discretize the problem in time with $t_k = k\frac{T}{N}$ and use the algorithm from Carmona and Laurière 2019 by taking a sequence of neural network $(\alpha_i^{\theta_i})_{i=1,\cdots,N} : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}^d$ with parameters θ_i to approximate the (Markovian feedback) control.

We solve the auxiliary problem in a segment K in which we assume $\mathcal{Z}^\Psi$ lies, discretized as $z_1 < \cdots < z_M$. For $i = 0, \cdots, N_T - 1$, $j = 1, \cdots, N$

$$
\begin{aligned}
X_{i+1}^j &= X_i^j + b\big(t_i, X_i^j, \alpha_i^{\theta_i}(X_i^j, z), \overline{\mu}_i\big)\Delta t_i + \sigma\big(t_i, X_i^j, \alpha_i^{\theta_i}(X_i^j, z), \overline{\mu}_i\big)\Delta W_i^j \\
Z_z^\alpha &= z - \frac{1}{N}\sum_{i=0}^{N_T-1}\sum_{l=1}^{N} f\big(t_i, X_i^l, \alpha_i^{\theta_i}(X_i^l, z), \overline{\mu}_i\big)\ \Delta t_i \\
\overline{\mu}_i &= \frac{1}{N}\sum_{j=1}^{N}\delta_{(X_i^j, \alpha_i^{\theta_i}(X_i^j, z))} \\
X_0^j &\sim \mu_0
\end{aligned}
$$

Numerical scheme

We solve by stochastic gradient descent $\inf_\theta \sum_{m=1}^{M} w_\Lambda(z_m)$ with w defined by

$$w_\Lambda(z)$$

$$:= \mathbb{E}\Big[\{\frac{1}{N}\sum_{l=1}^{N} g\Big(X^l_{N_T}, \frac{1}{N}\sum_{j=1}^{N}\delta_{X^j_{N_T}}\Big) - Z^\alpha_z\}_+ + \Lambda\sum_{i=1}^{N_T}\{\Psi\Big(t_i, \frac{1}{N}\sum_{j=1}^{N}\delta_{X^j_i}\Big)\}_+\Delta t_i\Big].$$

and for $i = 0, \cdots, N_T - 1$, $j = 1, \cdots, N$

$$X^j_{i+1} = X^j_i + b\big(t_i, X^j_i, \alpha^\theta_i(X^j_i, z), \overline{\mu}_i\big)\Delta t_i + \sigma\big(t_i, X^j_i, \alpha^\theta_i(X^j_i, z), \overline{\mu}_i\big)\Delta W^j_i$$

$$Z^\alpha_z = z - \frac{1}{N}\sum_{i=0}^{N_T-1}\sum_{l=1}^{N} f\big(t_i, X^l_i, \alpha^\theta_i(X^l_i, z), \overline{\mu}_i\big)\ \Delta t_i$$

$$\overline{\mu}_i = \frac{1}{N}\sum_{j=1}^{N}\delta_{(X^j_i, \alpha^\theta_i(X^j_i, z))}$$

$$X^j_0 \sim \mu_0$$

Numerical scheme

- Define $\alpha^* = \alpha^{\theta^*}$ with θ^* the solution to the previous minimization problem.
- Compute $V_0 = \inf\{w(z_i),\ i \in [\![1, M]\!] \mid w_\Lambda(z_i) \leq \varepsilon\}$ with $\alpha = \alpha^*$ in the dynamics for some threshold ε.
- Return the value V_0 and the optimal controls $\hat{\alpha}_i : x \mapsto \alpha_i^*(x, V_0)$ for $i = 0, \cdots, N_T - 1$.

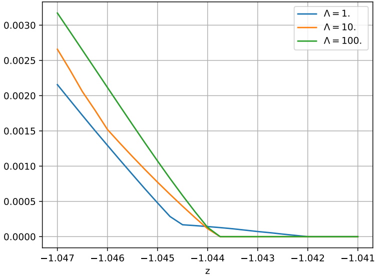
Numerical results

Markowitz mean-variance problem with conditional expectation constraint

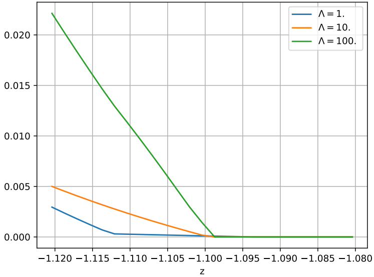
$$\min_\alpha \lambda \mathrm{Var}(X_T^\alpha) - \mathbb{E}[X_T^\alpha]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}W_t$$
$$E[X_t^\alpha \mid X_t^\alpha \leq \theta] \geq \delta.$$

Primal Markowitz mean-variance problem

$$\min_\alpha \; - \mathbb{E}[X_T^\alpha]$$
$$\mathrm{d}X_t^\alpha = \alpha_t b \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}W_t$$
$$\mathrm{Var}(X_T) \leq \vartheta.$$



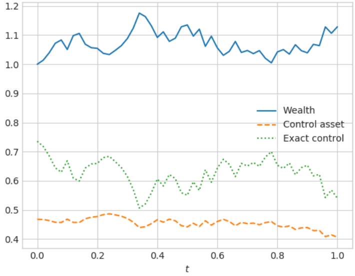
Conditional expectation constraint



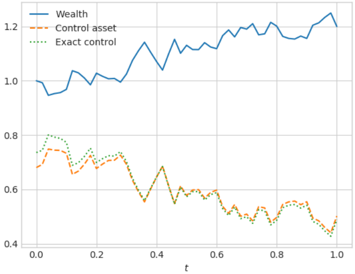
Terminal variance constraint

Auxiliary value function $\mathcal{Y}_\Lambda(z)$ for several values of Λ

Trajectories



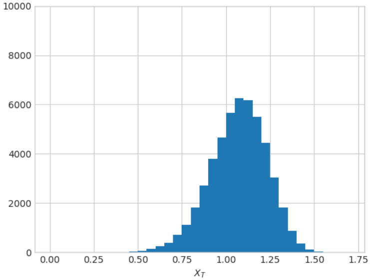
Conditional expectation constraint



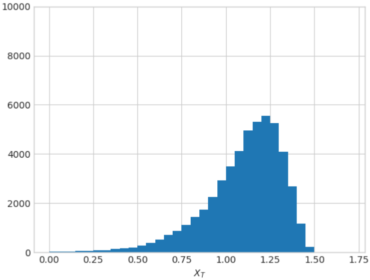
Terminal variance constraint

Sample path of the controlled process X_t^α, with the analytical optimal control (for the unconstrained case) and the computed control. On the left figure we don't have the true control but plot the unconstrained one for comparison

Histogram



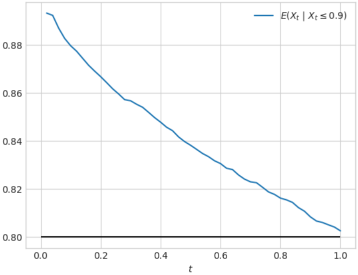
Conditional expectation constraint



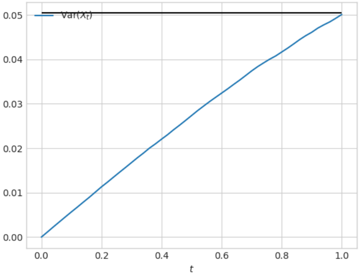
Terminal variance constraint

Histogram of X_T for 50000 samples

Constraints



Conditional expectation constraint



Terminal variance constraint

On the left: conditional expectation $E[X_t^\alpha \mid X_t^\alpha \leq 0.9]$ estimated with 50000 samples. On the right: variance $\mathrm{Var}(X_t^\alpha)$ estimated with 50000 samples.

Conclusion

We have been able to:

- **Extend the level-set approach** to the mean-field control problem.
- Prove **representation results** of the constrained problem by an unconstrained one.
- Design a machine learning numerical scheme to compute the **optimal value and control**.

Potential future research

- Carefully assess the optimality of the computed control.
- Solve more difficult cases with an explicit solution.

Bibliography I

Bokanowski, O., A. Picarelli, and H. Zidani (2015). "Dynamic Programming and Error Estimates for Stochastic Control Problems with Maximum Cost". In: *Appl Math Optim* 71, pp. 125–163.

— (2016). "State-Constrained Stochastic Optimal Control Problems via Reachability Approach". In: *SIAM Journal on Control and Optimization* 54.5, pp. 2568–2593.

Bonnet, B. (2019). "A Pontryagin Maximum Principle in Wasserstein spaces for constrained optimal control problems". In: *ESAIM: COCV* 25, p. 52.

Cannarsa, P., R. Capuani, and P. Cardaliaguet (2018). "Mean Field Games with state constraints: from mild to pointwise solutions of the PDE system". In: *arXiv:1812.11374*.

Carmona, R. and F. Delarue (2018). *Probabilistic Theory of Mean Field Games: vol. I, Mean Field FBSDEs, Control, and Games*, Springer.

Carmona, R., F. Delarue, and A. Lachapelle (2012). "Control of McKean–Vlasov dynamics versus mean field games". In: *Math Finan Econ* 7.

Bibliography II

Carmona, R. and M. Laurière (2019). “Convergence analysis of machine learning algorithms for the numerical solution of mean field control and games: II- the finite horizon case”. In: *arXiv:1908.01613, to appear in the Annals of Applied Probability.*

Chen, L. and J. Wang (2019). “Maximum principle for delayed stochastic mean-field control problem with state constraint”. In: *Advances in Difference Equations* 348.

Daudin, S. (2021). “Optimal control of the Fokker-Planck equation under state constraints in the Wasserstein space”. In: *arXiv:2109.14978.*

Graber, J. and S. Mayorga (2021). “A note on mean field games of controls with state constraints: existence of mild solutions”. In: *arXiv:2109.11655.*

Huang, M., P. Caines, and R. Malhamé (2006). “Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle”. In: *Communication in Information and Systems* 3, pp. 221–252.

Lasry, J.-M. and P.-L. Lions (2006). “Jeux à champ moyen. II – Horizon fini et contrôle optimal”. In: *Comptes Rendus. Mathématique. Académie des Sciences, Paris* 10. DOI: 10.1016/j.crma.2006.09.018.