A spatio-temporal advection-diffusion model describing human behaviors during a catastrophic event

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Joint work with

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Introduction

☞ During a disaster situation (tsunamis, earthquake,...), several human behaviors can be observed. Three main categories can be identified:

- Alert

- Panic

- Control

☞ The aim of the ANR Com2sica project is to model this situation as a compartmental SIR type model.

☞ A first attempt in this way is an ODE model.

Alert-Panic-Control (APC) diagram

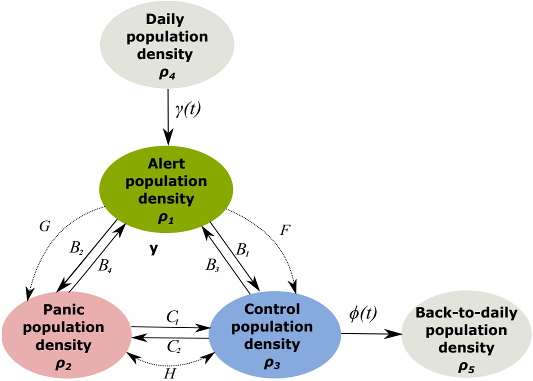

Figure: *The transfer diagram of the APC model.*

V. Lanza, E. Dubos-Paillard, R. Charrier, N. Verdière, D. Provitolo, O. Navarro, M.A. Aziz-Alaoui, *Spatio-Temporal Dynamics of Human Behaviors During Disasters: A Mathematical and Geographical Approach*, In Complex Systems, Smart Territories and Mobility Springer, Cham, (2021) 201–218.

The temporal APC model

- $\rho_i : \mathbb{R}^+ \longrightarrow \mathbb{R}$ for $i = 1, \cdots, 5$.

$$\begin{cases} \dfrac{d}{dt}\rho_1 = & \underbrace{-(b_1 + b_2 + \delta_1)\rho_1 + \gamma(t)\rho_4 + b_3\rho_3 + b_4\rho_2}_{Intrinsic\ transitions} \underbrace{-\mathscr{F}(\rho_1, \rho_3) - \mathscr{G}(\rho_1, \rho_2)}_{Imitation\ terms}, \\ \dfrac{d}{dt}\rho_2 = & \underbrace{-(b_4 + c_1 + \delta_2)\rho_2 + b_2\rho_1 + c_2\rho_3}_{Intrinsic\ transitions}, \underbrace{+\mathscr{G}(\rho_1, \rho_2) - \mathscr{H}(\rho_2, \rho_3)}_{Imitation\ terms}, \\ \dfrac{d}{dt}\rho_3 = & \underbrace{-(b_3 + c_2 + \delta_3)\rho_3 + b_1\rho_1 + c_1\rho_2 - \varphi(t)\rho_3}_{Intrinsic\ transitions} \underbrace{+\mathscr{F}(\rho_1, \rho_3) + \mathscr{H}(\rho_2, \rho_3)}_{Imitation\ terms}, \\ \dfrac{d}{dt}\rho_4 = & \underbrace{-\gamma(t)\rho_4}_{Intrinsic\ transitions}, \\ \dfrac{d}{dt}\rho_5 = & \underbrace{+\varphi(t)\rho_3}_{Intrinsic\ transitions}, \quad t \geq 0 \end{cases} \tag{1}$$

☞ The initial condition $(\rho_1(0), \rho_2(0), \rho_3(0), \rho_4(0), \rho_5(0))^T = (0, 0, 0, 1, 0)^T$, since the population is supposed to be in a daily behavior before the onset of the disaster.

Alert-Panic-Control (APC) diagram

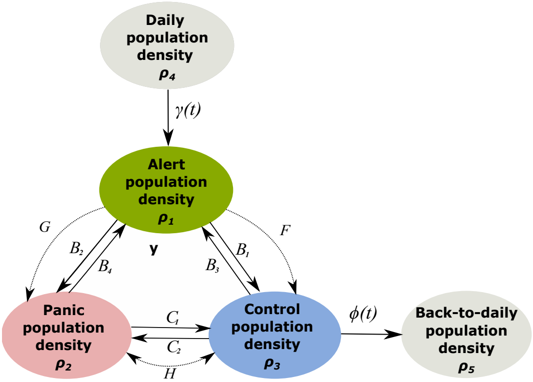

Figure: *The transfer diagram of the APC model.*

☞ The role played by the spatial configuration and its constraints is not negligible in the dynamics of human reactions.

Outline

1. A spatio-temporal advection-diffusion PDE model describing human behaviors during a catastrophic event
2. Well-posedness and positivity
3. Numerical simulations

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- **Advection terms:** $-\nabla \cdot (\rho_i \vec{v}_i(\rho))$, $i \in \{2, 3\}$ with

$$\vec{v}_i(\rho) = V_i(\rho)\nu(x) \quad \text{where} \quad V_i(\rho) = V_{i,max}\left(1 - \frac{\sum_{i=1}^{5} \rho_i}{\rho_{max}}\right),$$

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- **Boundary conditions:** Let $q_i(\rho) := -d_i \nabla \rho_i + \rho_i \vec{v_i}(\rho)$ be the flux of ρ_i, then we set

$$q_i(\rho) \cdot n = q^i_{out}(\rho) \cdot n, \quad i = 1, \cdots, 5,$$

where

$$q^i_{out}(\rho) := V_{i,out}\, \rho_i \quad \text{where} \quad V_{i,out} \geq 0.$$

The spatio-temporal APC model

We obtain the following APC model:

$$
\begin{cases}
\partial_t \rho_1 = & d_1 \Delta \rho_1 - (b_1 + b_2 + \delta_1)\rho_1 + \gamma(t)\rho_4 + b_3\rho_3 + b_4\rho_2 \\
& -\mathscr{F}(\rho_1,\rho_3) - \mathscr{G}(\rho_1,\rho_2) \qquad \text{in } \Omega,\ t \geq 0, \\
\partial_t \rho_2 = & d_2 \Delta \rho_2 - (b_4 + c_1 + \delta_2)\rho_2 + b_2\rho_1 + c_2\rho_3 \\
& -\nabla \cdot (\rho_2 \vec{v_2}(\rho)) + \mathscr{G}(\rho_1,\rho_2) - \mathscr{H}(\rho_2,\rho_3) \qquad \text{in } \Omega,\ t \geq 0, \\
\partial_t \rho_3 = & d_3 \Delta \rho_3 - (b_3 + c_2 + \delta_3)\rho_3 + b_1\rho_1 + c_1\rho_2 - \varphi(t)\rho_3 \\
& -\nabla \cdot (\rho_3 \vec{v_3}(\rho)) + \mathscr{F}(\rho_1,\rho_3) + \mathscr{H}(\rho_2,\rho_3) \qquad \text{in } \Omega,\ t > 0, \\
\partial_t \rho_4 = & d_4 \Delta \rho_4 - \gamma(t)\rho_4 \qquad \text{in } \Omega,\ t \geq 0 \\
\partial_t \rho_5 = & d_5 \Delta \rho_5 + \varphi(t)\rho_3 \qquad \text{in } \Omega,\ t \geq 0
\end{cases}
$$

Boundary conditions

$$(-d_i \nabla \rho_i + \rho_i \vec{v}_i(\rho)) \cdot n = V_{i,out}\, \rho_i \cdot n, \quad i = 1, \cdots, 5, \quad \text{on } \partial\Omega,\ t \geq 0,$$

Initial condition

$$\rho(0,x) = (0,0,0,\theta(x),0)^T := \rho_0(x) \quad \forall x \in \Omega.$$

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2. Well-posedness and positivity
3. Numerical simulations

- We consider the APC model setting in the Banach space

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- Let $\mathscr{A} : D(\mathscr{A}) \to X$ be defined by

$$\mathscr{A} := \operatorname{diag}(d_1\Delta, d_2\Delta, d_3\Delta, d_4\Delta, d_4\Delta, d_5\Delta) \quad \text{and} \quad D(\mathscr{A}) = W^{2,p}(\Omega)^5.$$

Functional framework and linear operators

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- The boundary operator $\mathscr{L} : D(\mathscr{A}) \to \partial X$ is given by

$$(\mathscr{L}\rho)_i := d_i \partial_n \rho_i.$$

Nonlinearities

- Let $\alpha \in (1/p + 1/2, 1)$. The nonlinearities are defined on the Banach space:

$$X_\alpha := \{\varphi \in W^{2\alpha,p}(\Omega)^5 \mid d_i \partial_n \varphi_{i|\partial\Omega} = 0\}$$

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- The nonlinear operator $\mathscr{K} : [0,\infty) \times X_\alpha \longrightarrow X$ is defined by

$$\begin{cases} \mathscr{K}_1(t,\rho_1,\nabla\rho) = -(b_1+b_2+\delta_1)\rho_1 + \gamma(t)\rho_4 + b_3\rho_3 + b_4\rho_2 \\ \qquad\qquad\qquad - \mathscr{F}(\rho_1,\rho_3) - \mathscr{G}(\rho_1,\rho_2), \\ \mathscr{K}_2(t,\rho_2,\nabla\rho) = -(b_4+c_1+\delta_2)\rho_2 + b_2\rho_1 + c_2\rho_3 - \nabla\cdot(\rho_2\vec{v_2}(\rho)) \\ \qquad\qquad\qquad + \mathscr{G}(\rho_1,\rho_2) - \mathscr{H}(\rho_2,\rho_3), \\ \mathscr{K}_3(t,\rho_3,\nabla\rho) = -(b_3+c_2+\delta_3)\rho_3 + b_1\rho_1 + c_1\rho_2 - \nabla\cdot(\rho_3\vec{v_3}(\rho)) \\ \qquad\qquad\qquad + \mathscr{F}(\rho_1,\rho_3) + \mathscr{H}(\rho_2,\rho_3) - \phi(t)\rho_3, \\ \mathscr{K}_4(t,\rho_4,\nabla\rho) = -\gamma(t)\rho 4, \\ \mathscr{K}_5(t,\rho_5,\nabla\rho) = \phi(t)\rho_3, \end{cases}$$

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- The nonlinear boundary term $\mathscr{M} : X_\alpha \longrightarrow \partial X$ is given by

$$\mathscr{M}(\rho)_i := (-\rho_i\vec{v}_{i,\mathrm{out}} + \rho_i\vec{v_i}(\rho))\cdot n.$$

Abstract formulation

With these notations, the model writes as follows:

$$\begin{cases} u'(t) = \mathscr{A}u(t) + \mathscr{K}(t, u(t)), & t \geq 0, \\ \mathscr{L}u(t) = \mathscr{M}(u(t)), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{2}$$

where $u(t) := (\rho_1(t,\cdot), \rho_2(t,\cdot), \rho_3(t,\cdot), \rho_4(t,\cdot), \rho_5(t,\cdot))^T$,
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Theorem 1 (Local existence and positivity)

For each $u_0 \in X_\alpha$ there exist a maximal time $T(u_0) > 0$ and a unique maximal solution $u(\cdot, u_0) \in C([0, T(u_0)), X_\alpha) \cap C^1((0, T(u_0)), X)$ of equation (2).
Moreover, if $u_0 \geq 0$ a.e. on Ω, then for any $t \in (0, T(u_0))$, $u(t, x) \geq 0$ a.e. on Ω.

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☞ In order to highlight the behavior of the populations during the catastrophic event, we suppose that people cannot return to an everyday behavior.

☞ We present different scenarios of evacuation.

☞ We set $\partial\Omega = \Gamma \cup \Gamma_{out}$ where Γ designate the wall (where the flux is zero) and Γ_{out} is the escape region (where the flux is different from zero),

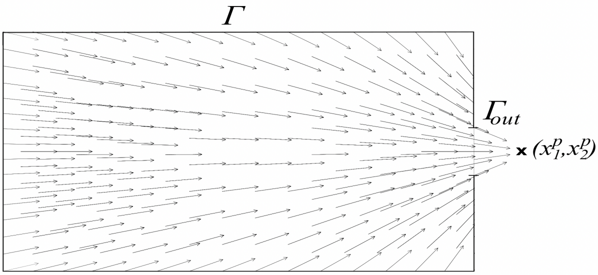
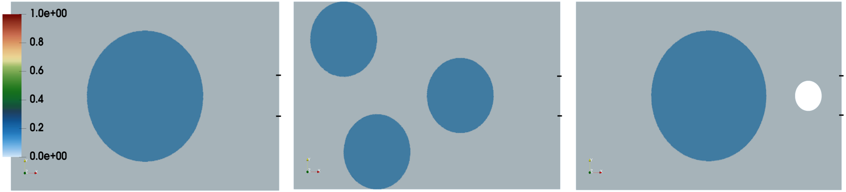


Figure: *The direction vector $\nu(x_1, x_2)$.*

Different initial conditions at each scenario



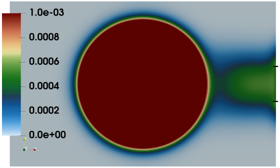
(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

- (a) The population is concentrated in a single group in the center of the domain;
- (b) The population is subdivided into three groups;
- (c) An obstacle is located between the exit and the population, which is concentrated in a single group within the domain.

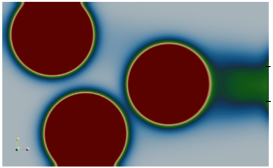
See the attached videos!

Advection and pedestrians transported by the crowd phenomenon.

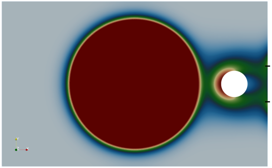
☞ Even if in our model there is no advection terms in the equation describing the evolution of the population in alert and daily, the alert population nevertheless undergoes a phenomenon of advection.



Alert (with new scale) at final time: Scenario 1



Alert at final time: Scenario 2



Alert at final time: Scenario 3

MERCI!

Simulations: Table of values

	Parameters
Diffusion	$d_1 = 0.001$
	$d_2 = 0.05$
	$d_3 = 0.01$
	$d_4 = 0.01$
Advection	$V_{2,max} = 0.3$
	$V_{3,max} = 0.2$
Speed at the boundary	$V_1 = 0.2$
	$V_2 = 0.1$
	$V_3 = 0.3$
	$V_4 = 0.2$

	Parameters
Imitation	$\alpha_{a \to c} = 0.6$
	$\alpha_{a \to p} = 0.7$
	$\alpha_{p \to c} = 0.6$
	$\alpha_{c \to p} = 0.7$
Intrinsic transitions	$c_1 = 0.1$
	$c_2 = 0.4$
	$b_1 = 0.1$
	$b_2 = 0.2$
	$b_3 = 0.001$
	$b_4 = 0.001$